Lecture Notes from January 24, 2023

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1.1 Last week

- "Highlights" from last term:
  - representation theory
  - spectral theory
  - the Gelfand transform
- Connectedness and Banach algebras.

Recall: For a Banach algebra $\mathcal{A}$ with unit, $G(A)$ is the group of invertible elements of $\mathcal{A}$ and G_0 is the connected component of the identity. Then since G_0 is a normal subgroup of $G(A)$, we defined the quotient group $\Lambda_{\mathcal{A}} = G(A)/G_0$, and called it the *index group* of $\mathcal{A}$.

1.2 Warm-up

Let X be a compact Hausdorff space, and $\mathcal{A} = C(X)$. Then

$$G(A) = \{f \in \mathcal{A} : f(x) \neq 0 \text{ for all } x \in X\}.$$

Because $\mathcal{A}$ is abelian, we deduce that $G_0 = e^{\mathcal{A}}$. Then if $f \in G_0$, we have $f = e^g$ for some continuous function g on X. Moreover we can define a function $g_\lambda(x) = e^{\lambda g(x)}$ that interpolates continuously between $g_0(x) = 1$ and $g_1(x) = f$; thus, 1 and f are *homotopic*. Conversely, if f is homotopic to 1, then $f \in G_0$. This shows that G_0 is concretely given by the class of functions homotopic to any constant non-zero function.

Similarly, $f_1 \sim f_2$ (f_1 and f_2 are in the same connected component) if they are homotopic. Namely, f_1 and f_2 belonging to the same coset of $\Lambda_{\mathcal{A}}$ is equivalent to f_1 and f_2 being homotopic.

See 2.16 on page 35 in ([Dou91]) for more on this.

Next, we revisit the definition of the spectrum using two familiar examples.

1.3 Examples

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $e_n(z) = z^n$ for $n \in \mathbb{Z}$, $\mathcal{P} = \text{Span}\{e_n\}_{n=0}^N$, and $\mathcal{T} = \text{Span}\{e_n\}_{n=-N}^N$. We call $\mathcal{P}$ and $\mathcal{T}$ the spaces of (analytic) polynomials and trigonometric polynomials, respectively. Then in the case of the trig. polynomials, the Banach algebra obtained from closure in the supremum norm is $\mathcal{A} \equiv \overline{\mathcal{T}}^{\infty} = C(\mathbb{T})$ and the spectrum is $\Gamma_{\mathcal{A}} = \mathbb{T}$.

To see why this is true, note that $\chi \in \Gamma_{\mathcal{A}} \Rightarrow \chi(e_1) = z$ and $\chi(e_n) = z^n$, and these stay bounded if and only if $|z| \leq 1$. Moreover, $\chi(e_{-n}) = z^{-n}$ stays bounded if and only if $|z| \geq 1$, so it must be that $|z| = 1$. Hence, $\chi(e_1) = z \in \mathbb{T}$, and for $\tau \in \mathcal{T}$, $\chi(\tau) = \tau(z)$. We conclude that the elements in $\Gamma_{\mathcal{A}}$ are the "point evaluations", $\chi = \delta_z$, and $\mathcal{T}$ is dense in $C(\mathbb{T})$ by the Stone-Weierstraß theorem.

Now write $\mathcal{A} = \overline{\mathcal{P}}^{\infty}$. We (only) get continuous homomorphisms for $|z| \leq 1$, and they are point evaluations, $\chi(p) = p(z)$, as for $C(\mathbb{T})$. Hence, we can consider $\Gamma_{\mathcal{A}}$ as point evaluations of analytic polynomials on $\mathbb{D} \equiv \{z \in \mathbb{C} : |z| \leq 1\}$.

See 2.50 on page 47 in ([Dou91]) for more on this.

This identification of $\Gamma_{\mathcal{A}}$ as $\mathbb{D}$ is useful for invertibility:

$$f \in C(X) \text{ is invertible} \quad \iff \quad f(x) \neq 0 \text{ for all } x \in X$$

Thus $f(z) = z$ is NOT invertible in $\overline{\mathcal{P}}^{\infty}$ because $f(0) = 0$ and δ_0 is in the spectrum!

Takeaway: If someone gives you a commutative algebra, look at its spectrum to learn about invertibility.

As another brief example, note that the spectrum of the (noncommutative) algebra $\mathcal{A}$ of $n \times n$ matrices with complex entries is $\Gamma_{\mathcal{A}} = \emptyset$. I.e., there are no nontrivial continuous homomorphisms on $\mathcal{A}$. Thus, we may not learn anything from looking at the spectrum.

1.4 Commutative C^*-Algebras and the Gelfand Transform

We start with a lemma.

1.6 Lemma. *Let $\mathcal{A}$ be a C^*-algebra, and $a \in \mathcal{A}$ Hermitian. Then $\sigma(a) \subset \mathbb{R}$.*

Proof. Without loss of generality, suppose $\mathcal{A}$ has a unit; otherwise, we adjoin one. Let $\alpha + i\beta \in \sigma(a)$ with $\alpha, \beta \in \mathbb{R}$. We show that $\beta = 0$.

For all $\lambda \in \mathbb{R}$, $\alpha + i(\beta + \lambda) \in \sigma(a + i\lambda 1)$, and by comparing the norm with its spectral radius, we have $|\alpha + i(\beta + \lambda) \leq \|a + i\lambda 1\| \iff \alpha^2 + (\beta + \lambda)^2 \leq \|a + i\lambda\|^2$, hence

$$\begin{aligned}
\alpha^2 + (\beta + \lambda)^2 &= \|a + \lambda 1\| \|a - \lambda 1\| \quad (a \in \mathcal{A} \text{ is Hermitian}) \\
&= \|(a + \lambda 1)(a - \lambda 1)\| \quad (\mathcal{A} \text{ is a } C^*\text{-alg.}) \\
&= \|a^2 + \lambda^2 1\| \\
&\leq \|a^2\| + \lambda^2.
\end{aligned}$$

Thus we must have $\alpha^2 + \beta^2 + 2\beta\lambda \leq \|a\|^2$, and for this to hold for each λ requires $\beta = 0$.

□

1.5 Preview for Next Class: The Gelfand Representation and Functional Calculus

We write two versions of the *Stone-Weierstraß theorem*, which are stated differently than they usually appear in the literature. For examples of such generalized versions of Stone-Weierstraß, see page 121 in ([Rud91]) and page 43 in ([Dou91]).

1.7 Theorem. *(i) Let X be compact and $\mathcal{A} \subset C(X)$ a closed $*$-subalgebra that separates points. Then $\mathcal{A} = C(X)$, or there is $x_0 \in X$ such that $\mathcal{A} = \{f \in C(X) : f(x_0) = 0\}$.*

(ii) Let X be locally compact, $\mathcal{A} \subset C_0(X)$ a closed $*$-subalgebra that separates points and has no common root. Then $\mathcal{A}$ is dense in $C_0(X)$.

References

[1] Ronald G. Douglas. *Banach Algebra Techniques in Operator Theory*. Vol. 179. Graduate Texts in Mathematics. Springer, 1991. ISBN: 0387983775.

[2] Walter Rudin. *Functional Analysis*. 2nd ed. International Series in Pure and Applied Mathematics. McGraw-Hill, 1991. ISBN: 0070542368.