Lecture Notes from January 26, 2023

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Last time

- Index group for $C(X)$
- Spectrum and Gelfand map for $C(\mathbb{T})$ vs disc algebra
- Commutative C^*-algebras and spectral theory

1.5 Theorem. *Let $\mathcal{A}$ be a commutative C^*-algebra then $\mathcal{G} : \mathcal{A} \to C(\Gamma_{\mathcal{A}})$ is a C^*-isomorphism. In particular, $\|\hat{x}\|_\infty = \|x\|$ and $(\hat{x})^* = \widehat{x^*}$.*

Proof. If $a \in \mathcal{A}$ is Hermitian, then $\|\hat{a}\|_\infty = r(a) = \|a\|$(since the spectral is real) by a being normal and properties of the Gelfand map. From the preceeding lemma, $\hat{a}(\Gamma_{\mathcal{A}}) \subset \sigma(a)$ and we see that $\hat{a}$ is real valued. We conclude for $x = b + ic$ with b, c Hermitian,

$$\begin{aligned} (\hat{x}) &= \widehat{b - ic} \\ &= \hat{b} - i\hat{c} \\ &= (\hat{b} + i\hat{c})^* \\ &= \hat{x}^* \end{aligned}$$

This shows that the Gelfand transform is an isometric inventive homomorphism because

$$\begin{aligned} \|\hat{x}\|_\infty^2 &= \|\hat{x} * \hat{x}\|_\infty \\ &= \|\widehat{x * x}\|_\infty \text{ (Hermitian)} \\ &= \|x * x\|_\infty \\ &= \|x\|^2 \end{aligned}$$

It remains to show that $\mathcal{G}$ is onto. From $\mathcal{G}$ being an isomerty $\mathcal{G}(\mathcal{A}) \subset C(\Gamma_{\mathcal{A}})$ is a complete subalgebra invariant under conjugation(REASON: any cauchy sequence in $\mathcal{G}(\mathcal{A})$ is also a cauchy sequence in $C(\Gamma_{\mathcal{A}})$ which is complete.Hence the sequence converges in $C(\Gamma_{\mathcal{A}})$ But $\mathcal{G}$ being an isomerty, the limit is in $\mathcal{G}(\mathcal{A})$).

By $\Gamma_{\mathcal{A}} \in \mathcal{A}'$, $\mathcal{G}(\mathcal{A})$ separates points. To see this let $\chi \neq \chi'$ be two distinct points in $\Gamma_{\mathcal{A}}$. Then there must exist $a \in \mathcal{A}$ such that $\chi(a) \neq \chi'(a)$. Finally, since $0 \notin \Gamma_{\mathcal{A}}$, there is no common root for $\mathcal{G}(\mathcal{A})$. Now using the Stone-Weierstrass theorem gives us that $\mathcal{G}(\mathcal{A}) = C(\Gamma_{\mathcal{A}})$ □

We can also use Gelfand's representation to study commutative C^*-subalgebras of C^*-algebras.

1.6 Theorem. *Let a be a normal element of a C^*-algebra with unit and $C^*(a)$ the $*$-algebra generated by 1 and a. Then*

$$C^*(a) \cong C(\sigma(a))$$

where the isomorphism maps a to $\mathrm{id}_{\sigma(a)}$.

Proof. From $C^*(a) = \overline{span\{a^n(a^m)^* : n, m \geq 0\}}$, it follows from continuity of multiplocation and $a^*a = aa^*$ that $C^*(a)$ is commutative. By Gelfand's representation theorem $C^*(a) \cong C(\Gamma_{C^*(a)})$.

We have $\hat{1}(x) = 1$ and

$$\hat{a} : \Gamma_{C^*(a)} \to \sigma_{C^*(a)}(a)$$
$$\hat{a}(\chi) = \chi(a)$$

We show that $\hat{a}$ is a homeomorphism(for which we show that continuous maps has continuous inverse).
Since $C(\Gamma_{C^*(a)})$ separates points of $\Gamma_{C^*(a)})$ and the C^*-algebra is generated by 1 and a, $\hat{a}$ must already separate points(REASON:if not then a,a^*,$a^n(a^m)^*$ and hence $\overline{span\{a^n(a^m)^* : n, m \geq 0\}}$ will not separate points which is not true because $C^*(a)$ does so). Thus $\hat{a}$ is 1-1.
Now using that $\Gamma_{C^*(a)}$, is compact , $\hat{a}$ being continuous sends $\Gamma_{C^*(a)}$ to a compact set, hence any closed subset of $\Gamma_{C^*(a)}$ is mapped to closed set. Since complements are preserved hence any open subset of $\Gamma_{C^*(a)}$ is mapped to open set. So we get that $\hat{a}$ has continuous inverse.
It remains to show that $\sigma(a) = \sigma_{C^*(a)}(a)$. By the inclusion of C^*-algebras,

$$\sigma(a) \subset \sigma_{C^*(a)}(a)$$

To see the reverse inclusion, we assume $\lambda \in \sigma_{C^*(a)}(a) \setminus \sigma(a)$ then there is

$$b = (a - \lambda 1)^{-1} \in \mathcal{A}$$

Let $m > \|b\|$ and choose $f \in C(\sigma_{C^*(a)}(a))$ with $f(\lambda) = m$ and $|f(z)(z - \lambda)| \leq 1$ for all $z \in \sigma_{C^*(a)}(a)$. To find such a function, we specialize to $ran(f) \subset [0, m]$ and let $f|_{B_{1/m}(\lambda)} = 0$
Now using the inverse of Gelfand's map, which is given by

$$\mathcal{G}^{-1} : C(\sigma_{C^*(a)}(a)) \to A$$

then we get with with $g(z) = f(z)(z - \lambda)$.

$$\begin{aligned} m &= \|f\|_\infty \\ &= \|\mathcal{G}^{-1}(f)\| \\ &= \|\mathcal{G}^{-1}(f)(a - \lambda 1)b\| \\ &= \|\mathcal{G}^{-1}(g)b\| \\ &\leq \|\mathcal{G}^{-1}(g)\|\|b\| \\ &= \|g\|_\infty\|b\| \\ &\leq \|b\| \end{aligned}$$

contradicting our choice of m. Hence such a λ does not exist. □

1.7 Corollary. *Let $\mathcal{A}$ be a C^*-algebra with unit, $\mathcal{B}$ be another C^*-algebra with unit and $a \in \mathcal{B}$ be normal then*

$$\sigma_{\mathcal{B}}(a) = \sigma_{\mathcal{A}}(a)$$

.

Proof. By inclusion of C^*-algebras,

$$\sigma_{\mathcal{A}}(a) \subset \sigma_{\mathcal{B}}(a) \subset \sigma_{C^*(a)}(a)$$

and combined with $\sigma_{\mathcal{A}}(a) = \sigma_{C^*(a)}(a)$ from the proof of previous theorem , we get that the equality $\sigma_{\mathcal{B}}(a) = \sigma_{\mathcal{A}}(a)$ holds.

$\square$