# MATH 7321 Lecture Notes

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**Last Time:**

- Gelfand Representation for Commutative $C^*$-Algebras.
- Properties of Spectrum.

## 1 Functional Calculus for Operators

**Corollary 1.** *If $\mathcal{H}$ is a (complex) Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ be normal. Then there is an isometric embedding $\Phi : C(\sigma(A)) \longrightarrow \mathcal{B}(\mathcal{H})$, with $\Phi(id_{\sigma(A)}) = A$.*

*Proof.* This follows from choosing $\Phi = \mathcal{G}^{-1}$ in the preceding theorem applied to the $C^*$-algebra generated by 1 and $A$. □

**Remark 2.** *This allows us to assign an operator $\Phi(f) \equiv f(A)$ to each continuous function $f$ and by isomorphism property, for $f, g \in C(\sigma(A))$*

$$f(A)g(A) = (fg)(A) ,$$

*as well as $(f(A))^* = \overline{f}(A)$.*
*Moreover, $\sigma(f(A)) = f(\sigma(A))$, because*

$$\begin{aligned} \sigma(f(A)) &= \sigma(\Phi(f)) , \\ &\overset{iso}{\approx} \sigma_{C(\sigma(A))}(f) , \\ &= f(\sigma(A)). \end{aligned}$$

**Note:** Here $\overline{f(\sigma(A))} = f(\sigma(A))$ as $f$ is continuous and $\sigma(A)$ is compact.

**Warm up:** Find a $C^*$-algebra $\mathcal{A}$ and $a \in \mathcal{A}$ such that $\sigma_{\mathcal{A}}(a) = [0, 1]$ .

**Example 3.** *$\mathcal{A} = C([0, 1])$ and $a \in \mathcal{A}$ is defined by $a(x) = x$.*

- *Can we find $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$?*

# 2 Limitations of Functional Calculus

To see what limitations the functional calculus has, we consider an example:

**Example 4.** ***Summability Properties:*** *Let Hilbert space $\mathcal{H}$ has orthonormal basis $(e_j)_{j\in J}$ and let $x : J \longrightarrow \mathbb{C}$, $j \longrightarrow x_j$ be bounded. We claim*

$$Av = \sum_{j\in J} x_j \langle v, e_j\rangle e_j,$$

*defines a normal operator and $\sigma(A) = \overline{x(J)}$. We note that, the series converges by summability properties of orthonormal basis and $\|A\| = \|x\|_\infty$. By*

$$A^*v = \sum_{j\in J} \overline{x}_j \langle v, e_j\rangle e_j \ ,$$

*we see that, $A$ is normal. From orthonormal property $Ae_j = x_je_j$, gives eigen-values/vectors. Hence, $\overline{x(J)} \subseteq \sigma(A)$.*
*Conversely, if $\lambda \notin \overline{x(J)}$, then*

$$(A - \lambda 1)v = \sum_{j\in J} (x_j - \lambda)\langle v, e_j\rangle e_j,$$

*and hence,*

$$(A - \lambda 1)^{-1} = \sum_{j\in J} (x_j - \lambda)^{-1}\langle v, e_j\rangle e_j,$$

*defines a bounded operator $(A - \lambda 1)^{-1}$. This implies that, $\lambda \notin \sigma A$. So, $\sigma(A) = \overline{x(J)}$. Moreover, we can prove that if $f \in C(\sigma(A))$, then*

$$f(A)v = \sum_{j\in J} f(x_j)\langle v, e_j\rangle e_j \ .$$

*This can be shown first for polynomials and then by taking limits in $C(\sigma(A))$. We can define a functional calculus beyond the range of Gelfand's representation theorem in this case.*
*If $E \subset C$ is closed, then we define*

$$P_E v = \sum_{x_j \in E} \langle v, e_j \rangle e_j$$

*as the spectral projection associated with $E$. We see that*

$$\begin{aligned} P_E^2 v &= P_E \left( \sum_{x_i \in E} \langle v, e_i \rangle e_i \right) \\ &= \sum_{x_j \in E} \langle \sum_{x_i \in E} \langle v, e_i \rangle e_i, e_j \rangle e_j \\ &= \sum_{x_j \in E} \left( \sum_{x_i \in E} \langle v, e_i \rangle \langle e_i, e_j \rangle \right) e_j \\ &= \sum_{x_j \in E} \langle v, e_j \rangle e_j \ , \qquad \textit{(because } (e_j)_{j \in J} \textit{ is orthonormal)} \\ &= P_E \end{aligned}$$

*Thus $P_E^2 = P_E$. This defines an orthogonal projection. Also, we see that $P_E$ commutes with $A$, which proved as follows:*

$$\begin{aligned} (P_E A) v &= P_E(Av) \\ &= P_E \left( \sum_{j \in J} x_j \langle v, e_j \rangle e_j \right) \\ &= \sum_{x_i \in E} \left\langle \sum_{j \in J} x_j \langle v, e_j \rangle e_j, e_i \right\rangle e_i \\ &= \sum_{x_i \in E} \sum_{j \in J} x_j \langle v, e_j \rangle \langle e_j, e_i \rangle e_i \\ &= \sum_{x_i \in E} x_i \langle v, e_i \rangle e_i \ . \end{aligned}$$

*and*

$$\begin{aligned}(AP_E)v &= A(P_E v)\\ &= A\left(\sum_{x_j\in E}\langle v,e_j\rangle e_j\right)\\ &= \sum_{i\in J} x_i\left\langle \sum_{x_j\in E}\langle v,e_j\rangle e_j, e_i\right\rangle e_i\\ &= \sum_{x_j\in E}\sum_{i\in J} x_i\langle v,e_j\rangle\langle e_j,e_i\rangle e_i\\ &= \sum_{x_j\in E} x_j\langle v,e_j\rangle e_j\ .\end{aligned}$$

*Thus,* $(P_E A)v = (AP_E)v$. *Also, we have*

$$\sigma(A|_{P_E(\mathcal{H})}) = \overline{E\cap x(J)} \subset E\ .$$

*Hence, this operator* $A|_{P_E(\mathcal{H})}$ *is the "the piece" of* $A$ *for which the spectrum is in* $E$. *It would be nice to have* $P_E = f(A)$ *for some* $f \in C(\sigma(A))$. *Then we would have*

$$f(x) = \begin{cases} 1, & x_j \in E\\ 0, & x_j \notin E\ .\end{cases}$$

*So,* $f$ *is* $\{0,1\}$*- valued. Thus,* $\sigma(A) = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$ *splits* $\sigma(A)$ *in two closed and open subsets. If* $\sigma(A)\cap E$ *is not closed and open then, we can not find such a function* $f$ *in* $C(\sigma(A))$.

Example: Take $J = \mathbb{N}$, and $x_j$ to denumerate $\mathbb{Q}\cap[0,1]$. Then we can not find $P_E$ other than $P_E = O$ or $P_E = I$ corresponding to $f \in C([0,1])$. Because, if $P_E \neq O$ then there is $0 \neq y \in E$ such that $P_E v = y$ for some $v \in \mathcal{H}$. This implies that

$$\sum_{x_j\in E}\langle v,e_j\rangle e_j = y$$

$$\sum_{x_j\in E}\langle v,e_j\rangle\langle e_j,e_i\rangle = \langle y,e_i\rangle,\quad i\in J$$

$$\langle v,e_i\rangle = \langle y,e_i\rangle,\quad i\in J,\quad \textit{as } (e_j)_{j\in J} \textit{ is orthonormal},$$

$$\implies y = v.$$

So, $P_E v = v$ for all $v$ , and this implies $P_E = I$.
Thus, $P_E = O$ or $P_E = I$.