Lecture Notes from February 02, 2023

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Last time

- Functional calculus and its limitations

Warm up/Recap: Topologies on $\mathbb{B}(\mathcal{H})$ Definitions:

- Norm topology: Topology induced by the supremum norm on $\mathbb{B}(\mathcal{H})$.
- Strong operator topology, denoted SOT: coarsest topology such that $\forall\ v \in \mathcal{H}$, the map $T_v : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$; $A \mapsto Av$ is continuous.
- Weak operator topology, denoted WOT: coarsest topology such that $\forall v, w \in \mathcal{H}$, $\lambda_{v,w}(A) = \langle Av, w \rangle$ is continuous.

Characterization using sequences: (Let $A \subset \mathbb{B}(\mathcal{H})$)

- Norm topology: $X \in \bar{A} \iff$ *there exist* $\{x_n\}$ *sequence in* A *such that* $x_n \to x$
- Strong operator topology: We know $X \in \bar{A}^{SOT} \iff$ for any strongly open set U and $X \in U$, we have $U \cap X \neq \phi$. Using basis of topology, for each open U and $X \in U$ we can find a finite set of vectors $\{v_1, v_2, \cdots, v_n\}$ in $\mathcal{H}$ and $\epsilon > 0$,

$$V := \{Y \in \mathbb{B}(\mathcal{H}) : \|(Y - X)v_j\| < \epsilon, \forall j \in \{1, 2, \cdots, m\}\}$$

  with $X \in V$ and $V \subset U$. Hence,

  $X \in \bar{A}^{SOT} \iff \forall \{v_1, v_2, \cdots, v_n\} \subset \mathcal{H}, \epsilon > 0$, *we can find* $Y \in X$ *such that* $\|(Y - X)v_j\| < \epsilon > 0$ *for all* $j \in \{1, 2, \cdots, m\}$.
- Weak operator topology, denoted WOT: $X \in \bar{A}^{WOT} \iff$ *there exists* X_n *sequence in* A *such that* $\forall v, w \in \mathcal{H}$, $\langle X_n v, w \rangle \to \langle Av, w \rangle$ *in* $\mathbb{C}$.

We return to functional calculus and hope to use weaker topologies to get more functions of operators.

1.47 Definition. Let $\mathcal{H}$ be a Hilbert space, $E \subset \mathbb{B}(\mathcal{H})$, then we define the commutant of E

$$E' := \{A \in \mathbb{B}(\mathcal{H}) : AB = BA,\ \forall B \in E\}$$

We now have a lemma on the properties of the commutant, (which are strikingly similar to properties of a closed space and its perp).

1.48 Lemma. *For sets* $E, F \subset \mathbb{B}(\mathcal{H})$*, we have*

1. $E \subset F' \iff F \subset E'$
2. $E \subset E''$
3. $E \subset F \implies F' \subset E'$
4. $E' = E'''$
5. $E = E'' \iff$ *there is* $F \subset \mathbb{B}(\mathcal{H})$ *such that* $E = F'$

Proof. 1. If $E \subset F'$ then for $A \in E$, $B \in F$ $AB = BA$ and thus by symmetry $F \subset E'$. Swapping E and F givesthe converse.

2. If in part 1. we choose $F = E'$ then $F \subset E'$, i.e., $E' \subset E'$ implies $E \subset F'$, i.e., $E \subset E''$

3. If $E \subset F$ let $A \in F'$ then $AB = BA$ for all $B \in F$ and for all $C \in E$ $AC = CA$ since $C \in E \subset F$ hence $A \in E'$ and we have $F' \subset E'$

4. From part 2. $E' \subset (E')''$ and combinig parts 2. and 3. we get $E \subset E'' \implies E''' \subset E'$. Hence $E' = E'''$.

5. If $E = F'$ then by part 4. $F' = F''' \implies E = E''$. Conversely, if $E = E'' = (E')'$, then setting $F = E'$ gives $E = E'' = (E')' = F'$.

□

1.49 Lemma. *Let* $E \subset \mathbb{B}(\mathcal{H})$*, then*

1. E' *is a closed subalgebra of* $\mathbb{B}(\mathcal{H})$*.*
2. *If* E *is commutative, then so is* E''*.*
3. *If* E *is invariant under taking adjoints, then so is* E'*.*

Proof. 1. $\forall A \in E, \{A\}' = \{B \in B(H) = AB = BA\}$ a is closed by continuity of multiplication, so $E' = \bigcap_{A \in E} \{A\}'$ is closed. E is a subalgebra since if $C, D \in E''$, then $AC = CA$ and $AB = BA$ $\forall A \in E$

- for any scalar k $A(kB) = k(AB) = kBA$ hence $kB \in E$.
- $A(B + C) = AB + AC = BA + BC = (B + C)A$ hence $B + C \in E$.
- $A(BC) = BAC = BCA = (BC)A$ hence $BC \in E$.

2. If E is commutative, then $E \subset E'$; and by the properties $E'' \subset E'$, so by reversing and taking commutant using properties, $E'' \subset E'''$. This implies E'' is commutative.

3. This follows from taking adjoint of products as follows. We have $A \in E \iff A^* \in E$. If $B \in E'$, $AB = BA$ for all $A \in E$ and taking adjoints $B^*A^* = A^*B^*$ for all $A^* \in E$. Since E is invariant under adjoints we have $B \in E'$ giving us the result.

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