Lecture Notes from February 7th, 2023

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0 The Von-Neumann Theorem

Last time, we discovered similarities between the commutant and orthogonality of subspaces. This similarity becomes more manifest if we define orthogonality among members of an algebra

0.0.1 Definition. For *-algebra $\mathcal{A}$ and $a, b \in \mathcal{A}$, $a \perp b \Leftrightarrow ab - ba = 0$

With this definition of orthogonality, the commutant is identical to the orthogonal complement of a subset.

0.0.2 Definition. Let E $\subset$ A for *-algebra A. Then $E^{\perp} = \{$ b $\in$ A : ab - ba $= 0 \} = E^{'}$

0.0.3 Theorem. *For * - subalgebra $\mathcal{A} \subset B(H)$, H a Hilbert Space which does not have non-trivial invariant subspaces, we have $\overline{\mathcal{A}}^{s} = \overline{\mathcal{A}}^{w} = \mathcal{A}''$*

Proof. We first observe $\mathcal{A}^{''}$ is weakly closed because for v,w $\in$ H, B $\in \mathcal{A}^{'}$, $\lambda_{B,v,w}$: A $\to \langle (AB - BA)v, w\rangle = \langle A(Bv), w\rangle - \langle Av, B^{*}w\rangle$ is weakly continuous in $\mathcal{A}$, hence $(\mathcal{A}')' = \bigcap_{B,v,w} \mathcal{A}^{-1}_{B,v,w}(0)$ is weakly closed. From $\mathcal{A} \subset \mathcal{A}''$ we get the first inclusion $\overline{\mathcal{A}}^{s} \subset \overline{\mathcal{A}}^{w} \subset \mathcal{A}''$. To get the remaining inclusion, $\mathcal{A}'' \subset \overline{\mathcal{A}}^{s}$, we wish to show that for any B $\in \mathcal{A}''$, for any $\{v_1, v_2, ..., v_m\} \in H$ there exists a sequence $(A_n)_{n=1}^{\infty} \in \mathcal{A}$ s.t. for each $j \in \{1, 2, ..., m\}$ $A_n v_j \to Bv_j$. This is simply the sequential definition of closure induced by the definition of closure in the strong topology: $X \in \overline{\mathcal{A}}^{s} iff \forall \epsilon > 0 \exists Y \in \mathcal{A} s.t. |Vert(X - Y)v_j| < \epsilon \forall j \in 1, 2, ..., m$. We first show that for a fixed $v \in H$, $Bv = \overline{Av}$. Let E $=\overline{\mathcal{A}v}$. By continuity of multiplication, E is a $\mathcal{A}$ invariant closed subspace, hence P_E, the projection onto E, commutes with $\mathcal{A}^{'}$, hence $P_E \in \mathcal{A}$. From the assumption that $B \in \mathcal{A}''$, we have $BP_E = P_E B$ and thus B has E as an invariant subspace. Since $\mathcal{A}$ has no non-trivial invariant subspaces, E = H or E = 0, and in both cases $v \in E$. Let us define $K = H^m$,

then for $a \in \mathcal{A}$ we define $\tilde{a} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \mapsto \begin{pmatrix} av_1 \\ av_2 \\ \vdots \\ av_m \end{pmatrix}$ on K. This is a non-degenerate representation of $\mathcal{A}$ on K, as if $\tilde{a}v = 0$ for each $\tilde{a}$, then $av_j = 0$ for each v_j hence $v_j = 0$ for each $j \in \{1, 2, ..., m\}$. Now we show $\tilde{\mathcal{A}}'' \subset \tilde{\mathcal{A}}''$. Let $c \in \tilde{\mathcal{A}}^{'}$. We write v_j as the projection on the jth copy of H in K, $V_j \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = v_j$ for $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \in K$. With this notation, we have $V_j\tilde{a} = aV_j$ for each

$j \in 1, 2, ..., m$ and we get $V_j c V_l^* a = V_j c \tilde{a} V_l^* = V_j \tilde{a} c V_l^* = a V_j c V_l^*$. Therefore, $V_j c V_l^* \in \mathcal{A}'$. If $b \in \mathcal{A}''$ then b commutes with $\{V_j c V_l^*\}_{j,l=1}^m$ and for $w \in K, cw = \begin{pmatrix} V_1 cw \\ V_2 cw \\ \vdots \\ V_m cw \end{pmatrix} = \sum_{l=1}^m \begin{pmatrix} v_1 c V_l^* w_l \\ v_2 c V_l^* w_l \\ \vdots \\ v_m c V_l^* w_l \end{pmatrix}$.

So, $\tilde{b} cw = \sum_{l=1}^m \begin{pmatrix} v_1 c V_l^* b w_l \\ v_2 c V_l^* b w_l \\ \vdots \\ v_m c V_l^* b w_l \end{pmatrix} = c \tilde{b} w$. Thus if $b \in \mathcal{A}''$ then $\tilde{b} \in \tilde{\mathcal{A}}''$. Finally we show that $\mathcal{A}$ is dense in the strong topology $\mathcal{A}''$. Let $\tilde{b} \in \mathcal{A}''$, $v_1, v_2, ..., v_m \in H$. By $\tilde{\mathcal{A}}'' \subset \tilde{\mathcal{A}}''$ we have $\tilde{b} \in \tilde{\mathcal{A}}'' \in B(K)$. Then for $b \in \mathcal{A}'', bv \in \overline{\mathcal{A}'' v}$, we get $\tilde{b} v = \sum_{l=1}^m \begin{pmatrix} bv_1 \\ bv_2 \\ \vdots \\ bv_m \end{pmatrix} \in \overline{\tilde{A} K}$. Thus we have a sequence $(a_n)_{n=1}^\infty$ st $\tilde{a_n} v \mapsto bv$ for each $j \in \{1, 2, ..., m\}$

□

0.0.4 Definition. A sub-algebra $A \in B(H)$ with $A = A''$ is called a Von-Neumann Algebra on H.