Lecture Notes from February 9, 2023

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Last time: Von Neumann's bicommutant theorem. Note that we did not assume a unital C^*-algebra, but this is resolved by non-degeneracy of the group action.
Notation: We write $\mathbb{1}$ for the identity operator to distinguish it from the scalar identity $1 \in \mathbb{C}$.

We begin with Schur's theorem.

1.6 Theorem. *A representation $(\pi, \mathcal{H})$ of an involutive semigroup is irreducible if and only if $(\pi(S))' = \mathbb{C}\mathbb{1}$.*

Proof. Assume that π is not irreducible. Then there exists a non-trivial closed subspace E such the orthogonal projection onto E, P_E, commutes with $\pi(S)$. (We discussed and proved this fact last term during Week 5. The reader can find the notes here.) Thus $(\pi(S))' = \mathbb{C}\mathbb{1}$ implies P_E is a multiple of the identity, so E is either $\{0\}$ or $\mathcal{H}$ itself. We conclude π acts irreducibily.

Conversely, assume that π is irreducible and let $A \in (\pi(S))'$. Then since $\pi(S) = \pi(S^*)$, we also have $A^* \in (\pi(S))'$. Thus,

$$A = B + iC \quad \text{with } B = B^*,\ C = C^*, \text{ and } B, C \in (\pi(S))'.$$

Then since B and C are Hermitian, their spectrum are subsets of the real line, and moreover nonempty. We show that there is only one element; to this end, assume that $\sigma(B) \supset \{\lambda, \mu\}$. Then since λ and μ are distinct and the space is Hausdorff, we can separate them by ε-neighborhoods, $\mathcal{N}(\lambda)$ and $\mathcal{N}(\mu)$. Then using Urysohn, we can define $f, h \in C(\sigma(B))$ with $f(\lambda) = 1,\ h(\mu) = 1$ and $f|_{\sigma(B)\setminus\mathcal{N}(\lambda)} = h|_{\sigma(B)\setminus\mathcal{N}(\mu)} = 0$. Hence, the product fh is zero on all of $\sigma(B)$ and by the functional calculus, $f(B)h(B) = 0$.

Since $(\pi(S))'$ is closed and contains all polynomials in B, we have $h(B) \in (\pi(S))'$. By irreducibility of π, invariant subspaces are either $\mathcal{H}$ or $\emptyset$. Then since $\overline{h(B)\mathcal{H}}$ is an invariant subspace, we deduce that $\overline{h(B)\mathcal{H}} = \mathcal{H}$. On the other hand, from $f(B)h(B) = 0$ we get $f(B) = 0$, a contradiction to $f \not\equiv 0$. Consequently, $\sigma(B) = \{\lambda\}$ for some $\lambda \in \mathbb{R}$.

Thus, $C(\sigma(B))$ is one-dimensional and analogously the same is true for C instead of B. This implies $(\pi(S))' = \mathbb{C}\mathbb{1}$. $\square$

Now we deduce consequences for representation theory of abelian involutive semigroups.

1.7 Corollary. *If S is an abelian involutive semigroup, and $(\pi, \mathcal{H})$ is an irreducible representation of S, then $\dim \mathcal{H} = 1$.*

Proof. By S abelian, we have $\pi(S)' \supset \pi(S)$. By irreducibility, $\pi(S)' = \mathbb{C}\mathbb{1} \Rightarrow \pi(S) \subset \mathbb{C}\mathbb{1}$. If $\dim \mathcal{H} > 1$, then we can choose a one-dimensional subspace E of $\mathcal{H}$ such that the orthogonal projection onto E, P_E has the property $P_E\pi(S) = \pi(S)P_E$. Then $P_E \in \pi(S)'$, but P_E is not a multiple of the identity! Hence, $\dim \mathcal{H} = 1$. □

We continue to explore C^* algebras and representation theory in the infinite-dimensional setting.

1.8 Proposition. *Let $\mathcal{A}$ be a Banach algebra, $b \in \mathcal{A}$, and $\mathcal{B}$ the closed subalgebra generated by*

$$\{\mathbb{1}, b, (b - \lambda\mathbb{1})^{-1} : \lambda \in \rho(b)\}.$$

Then $\mathcal{B}$ is commutative and $\rho_{\mathcal{B}}(b) = \rho_{\mathcal{A}}(b)$.

Proof. $\mathcal{B}$ is commutative because

$$(b - \lambda\mathbb{1})b = b(b - \lambda\mathbb{1}),$$

and if $\lambda \in \rho(b)$, then

$$b(b - \lambda\mathbb{1})^{-1} = (b - \lambda\mathbb{1})^{-1}b.$$

Hence, by continuity of multiplication, the closure of the algebra generated by taking products of elements $\mathbb{1}$, b, and $(b - \lambda\mathbb{1})^{-1}$, is also commutative.

By $\mathcal{B} \subset \mathcal{A}$, we know $\rho_{\mathcal{B}}(b) \subset \rho_{\mathcal{A}}(b)$. On the other hand, $\lambda \in \rho_{\mathcal{A}}(b) \Rightarrow (b - \lambda\mathbb{1})^{-1} \in \mathcal{B}$, so $\lambda \in \rho_{\mathcal{B}}(b)$. We conclude $\rho_{\mathcal{B}}(b) = \rho_{\mathcal{A}}(b)$. □

1.9 Question. How do we get a functional calculus for a bigger class of functions than "just" $C(\sigma(T))$, for $T \in B(\mathcal{H})$, $TT^* = T^*T$? Namely, we saw that spectrum of the C^*-algebra of continuous functions on $[0, 1]$ were point evaluations, so what makes sense in the context of functions that are not defined pointwise?

1.10 Answer. To extend the functional calculus from continuous functions to larger classes of functions, we use tools from measure theory and find that measurable functions are the natural generalization we need.

As a first example, we introduce the space $L^\infty(\mu)$: Let (X, μ) be a measure space and $f : X \to \mathbb{C}$ be measurable. We define the *essential range* of f to be

$$\text{ess-ran}(f) = \{\lambda \in \mathbb{C} : (\forall \epsilon > 0)\ \mu(\{x \in X : |f(x) - \lambda\mathbb{1}| < \epsilon\}) > 0\}$$

and

$$\|f\|_\infty = \sup\{|\lambda| : \lambda \in \text{ess-ran}(f)\}.$$

We say $f \in L^\infty(\mu)$ if $\|f\|_\infty < \infty$. Moreover, the completion of the space of measurable functions with respect to $\|\cdot\|_\infty$ is called $L^\infty(\mu)$

1.11 Remark. L^∞ forms a C^*-algebra with respect to the "usual" multiplication and conjugation.

In the coming lectures, we will use this example and viewpoint as a starting point to answer the question of extending the functional calculus.