Lecture Notes from February 14, 2023

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1.1 Last week

- "Highlights" from last week:
  - $A'' = \overline{A}^S = \overline{A}^W$
  - Schur's Lemma

1.2 Warm-up

Let $(\pi, \mathcal{H})$ be an irreducible map of an involutive semigroup S, what is $\overline{\text{Span } \pi(S)}^s$?

Proof. Since Span $\pi(S)$ is an algebra:

$$\begin{aligned} \overline{\text{Span } \pi(S)}^S &= (\text{Span } \pi(S))'' \\ &= (\mathbb{C}\mathbf{1})' = \mathcal{B}(\mathcal{H}). \end{aligned}$$

□

1.6 Remark. $L^\infty(\mu)$ forms a C^*-algebra with respect to "pointwise" multiplication. We can interpret $L^\infty(\mu)$ as a space of functions defined up to sets of measure zero.

1.7 Proposition. *Let* $f \in L^\infty(\mu)$*, then ess-ran*(f) *is compact in* $\mathbb{C}$ *and* $\sigma_{L^\infty(\mu)}(f) =$ *ess-ran*(f).

Proof. Suppose $\lambda \notin$ ess-ran(f), then there is $\epsilon > 0$ such that

$$\mu\Big(\{x \in \mathcal{X} : |f(x) - \lambda| < \epsilon\}\Big) = 0.$$

For any such λ, ϵ' sufficiently small, we get that $\lambda' \in B_{\epsilon'}(\mathcal{A})$, $\lambda' \notin$ ess-ran(f). Hence, the set of such λ is open, so ess-ran(f) is closed.
Note also if $|\lambda| > ||f||_\infty$ then $\lambda \notin$ ess-ran(f), so ess-ran(f) is compact.

Next, we show that there exists $\lambda \in \mathbb{C}$, $|\lambda| = ||f||_\infty$, $\lambda \in$ ess-ran(f).
Let $m = ||f||_\infty$, and assume

$$\partial B_m(\circ) \cap \text{ess-ran}(f) = \emptyset.$$

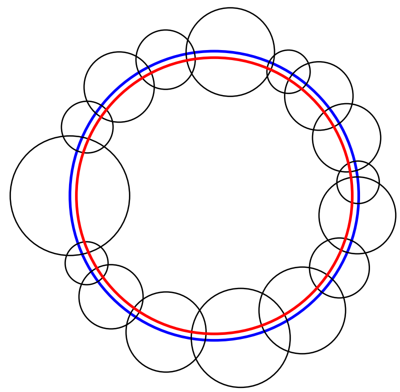
Since for each λ with $|\lambda| = m$, by $(\text{ess-ran}(f))^c$ being open, there is $\epsilon(\lambda) > 0$ such that

$$B_{\epsilon(\lambda)}(\lambda) \cap \text{ess-ran}(f) = \emptyset.$$

Using that $\partial B_m(\circ)$ is compact , there is a set $\lambda_1, \lambda_2 \ldots \lambda_n, |\lambda_j| = m$ for each j such that

$$\partial B_m(\circ) \subset \cup B_{\epsilon(\lambda_j)}(\lambda_j)$$

and hence



so there is $\epsilon' > 0$ such that

$$\{\lambda : m - \epsilon' < |\lambda| < m + \epsilon'\}$$

is also covered by this union , hence $||f||_\infty \leq m - \epsilon'$, a contradiction to our assumption.
Next , we show if $\lambda \in$ ess-ran(f), then $\lambda \in \sigma_{L^\infty(\mu)}(f)$. Assume $\lambda \notin \sigma_{L^\infty(\mu)}(f)$, so $f - \lambda$ has an inverse in $L^\infty(\mu)$, so for some $L > 0$,

$$\mu\Big(\{x : |f(x) - \lambda|^{-1} > L\}\Big) = 0$$

and setting $\epsilon = \frac{1}{L}$, we get

$$\mu\Big(\{x : |f(x) - \lambda| < \epsilon\}\Big) = 0$$

then by definition, $\lambda \in$ ess-ran(f).
Conversely, if $\lambda \in$ ess-ran(f), then $f - \lambda$ does not have an inverse in $L^\infty(\mu)$, so for each $\epsilon > 0$,

$$\{x : |f(x) - \lambda| < \epsilon\}$$

has non-zero measure thus $\lambda \in$ ess-ran(f). □

As a consequence of Gelfand's representation theorem, we have :

1.8 Theorem. *Let* Γ *be the space of non-trivial homomorphisms from* $L^\infty(\mu)$ *to* $\mathbb{C}$*, then there is an isometric* $*$*-isomorphism between* $L^\infty(\mu)$ *and* $C(\Gamma)$.

We would like to replace $C(\Gamma)$ with a class of functions on the measure space. To motivate this, we consider another example.

1.9 Example. Let $(\mathcal{X}, \mu)$ be a probability space, $\mathcal{H} = L^2(\mu)$ and consider $L^\infty(\mu)$ as a space of multiplication operators on $L^2(\mu)$, i.e. for $\phi \in L^\infty(\mu)$, $f \in L^2(\mu)$,

$$M_\phi f = \phi f.$$

Let's study properties of M_ϕ.

1.10 Proposition. *For $\phi \in L^\infty(\mu)$, M_ϕ as defined,*

(i) $\|M_\phi\| \leq \|\phi\|_\infty$

(ii) for any polynomial P, $M_{P(\phi)} = P(M_\phi)$

(iii) $M_\phi^* = M_{\overline{\phi}}$

(iv) if ϕ is invertible in $L^\infty(\mu)$ then M_ϕ is invertible in $B(\mathcal{H})$ and $M_{\phi^{-1}} = M_\phi^{-1}$

(v) $\mathcal{A} = \{M_\phi : \phi \in L^\infty(\mu)\}$ *is a Banach algebra with respect to operator on $\mathcal{H}$.*