MATH 7321 Lecture Notes

Note-taker: Kumari Teena

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Last Time:

- $L^\infty(\mu)$ as a C^*-algebra.
- $L^\infty(\mu)$ and associated multiplication operation on $L^\infty(\mu)$.

Warm up: Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Show that $T = S^*S$ for some $S \in \mathcal{B}(\mathcal{H})$ if and only if $\sigma(T) \subset [0, \infty)$.

Proof. Assume $T = S^*S$ and let $\lambda \in \sigma(T)$.
So, $T - \lambda 1$ is not invertible which means there is a sequence $(v_n)_{n=1}^\infty$ such that $\|v_n\| = 1$ for each $n \in \mathbb{N}$ and $\|(T - \lambda 1)v_n\| \longrightarrow 0$.
Hence, $\langle (T - \lambda 1)v_n, v_n\rangle \longrightarrow 0$ and by $T = S^*S$

$$\langle Sv_n, Sv_n\rangle - \lambda\|v_n\|^2 \longrightarrow 0,$$

$$\|Sv_n\|^2 - \lambda\|v_n\|^2 \longrightarrow 0.$$

Thus, $\lambda \geq 0$ because $\|Sv_n\|^2 \geq 0$ and $\|v_n\|^2 \geq 0$.
Conversely, if $\sigma(T) \subset [0, \infty)$. We know there is a $*-$isomorphism between $C^*(T) = A_T = \overline{span\{T^n(T^*)^m : \ n, m \geq 0\}}$ and $C(\sigma(T))$.
Using $\Phi = \mathcal{G}^{-1}$, we get $S = \Phi(f)$ for $f(x) = \sqrt{x}$ such that $S^2 = T$ and $S^* = \Phi(\overline{f}) = \Phi(f) = S$, (since f is real valued). Therefore, $T = S^*S$ where $S \in \mathcal{B}(\mathcal{H})$. □

Define $\mathcal{M} = \{\mathcal{M}_\phi : \ \phi \in L^\infty(\mu)\}$.

Proposition 1. *$\mathcal{M}$ and $L^\infty(\mu)$ are isometrically isomorphic as C^*-algebras.*

Proof. By properties listed, $\phi \longrightarrow \mathcal{M}_{phi}$ is a $*-$algebra homomorphism which is contractive.

(To show that the map is an isometry , prove the range of the map is closed sub-algebra of $\mathcal{B}(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H})$ is complete and since the map is an isomorphism. So, they are isometrically the same.)

We show $\|\phi\|_\infty = \|\mathcal{M}_\phi\|$.

Take $\lambda \in ess - range(\phi)$, and let $\psi = \phi - \lambda$. Then, by $0 \in ess - range(\phi)$, for each $\epsilon > 0$ we have

$$E = \{x \in X : \ |\psi(x)| < \epsilon\}$$

has non-zero measure and

$$\begin{aligned}
\|\mathcal{M}_\psi \chi_E\|^2 &= \|\psi \chi_E\|^2 \\
&= \int_E |\psi(x)|^2 d\mu \\
&< \epsilon^2 \mu(E), \qquad as\ |\psi(x)| < \epsilon\ on\ E \\
\implies \quad \|\mathcal{M}_\psi \left(\frac{\chi_E}{\mu(E)} \right) \|^2 &< \epsilon^2.
\end{aligned}$$

Thus, $\mathcal{M}_\psi$ is not boundedly invertible, hence neither $\mathcal{M}_{\phi-\lambda}$, and so $\lambda \in \sigma(\mathcal{M}_\phi)$.

We know that the spectral value satisfies $|\lambda| \leq r(\mathcal{M}_\phi) \leq \|\mathcal{M}_\phi\|$. Hence $\|\phi\|_\infty \leq \|\mathcal{M}_\phi\|$

The other side inequality can be obtained as follows: we know that the operator $\mathcal{M}_\phi$ is multiplication operator defined by

$$\mathcal{M}_\phi(f) = \phi f, \ \ \forall \ \ f \in L^2(\mu).$$

Then

$$\begin{aligned}
\|\mathcal{M}_\phi(f)\|^2 &= \int |\phi(x) f(x)|^2 d\mu(x), \\
&\leq \int |\phi(x)|^2 |f(x)|^2 d\mu(x), \\
&\leq \|\phi\|_\infty^2 \int |f(x)|^2 d\mu(x), \\
&= \|\phi\|_\infty^2 \|f\|^2, \\
\implies \quad \|\mathcal{M}_\phi\| &= \sup_{\|f\| \leq 1} \frac{\|\mathcal{M}_\phi(f)\|}{\|f\|} \leq \|\phi\|_\infty.
\end{aligned}$$

Thus, this implies that $\|\phi\|_\infty = \|\mathcal{M}_\phi\|$.

We want to establish $L^\infty(\mu) = (\mathcal{M}_{C(X)})''$. □

Definition 2. *An abelian algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$ is maximal abelian if it not a proper sub-algebra of a larger abelian algebra of operators on $\mathcal{H}$.*

Proposition 3. *The C^*-algebra $\mathcal{M}$ is maximal abelian.*

Proof. By μ being a probability measure , $L^\infty(\mu) \subset L^2(\mu)$.
We show that if $T \in \mathcal{B}(\mathcal{H})$ commutes with $\mathcal{M}$, that is $T \in \mathcal{M}'$, then $T \in \mathcal{M}$.
So, there is $\psi \in L^\infty(\mu)$ such that $T = \mathcal{M}_\psi$.
If there is such a ψ, it must be $\psi = T1$ where $1 \in L^2(\mu)$. We know for any $\phi \in L^\infty(\mu)$

$$T_\phi = T\mathcal{M}_\phi 1 = \mathcal{M}_\phi T1 = \mathcal{M}_{\phi\psi},$$

and

$$\|\psi\phi\|_2 = \|T\phi\|_2 \leq \|T\|\|\phi\|_2.$$

So, $\|\psi\|_\infty \leq \|T\|$.
Because if $\alpha > \|T\|$, setting $E = \psi^{-1}((\alpha, \infty))$, then we get $\mu(E) = \|\chi_E\|^2 = 0$. Otherwise,

$$\begin{aligned}
\|T\chi_E\| &= \|\psi\chi_E\|^2 \\
&= \int |\psi|^2 \chi_E d\mu \\
&\geq \alpha^2 \int \chi_E d\mu \qquad as \ \ |\psi|^2 \geq \alpha^2 \\
&= \alpha^2 \mu(E) = \alpha^2 \|\chi_E\|^2 \quad \ \forall \ \ \alpha \geq \|T\|.
\end{aligned}$$

Now taking $\alpha_n \downarrow \|T\|$, e.g. $\alpha_n = \|T\| + \frac{1}{n}$, gives $\|\psi\|_\infty \leq \|T\|$.
By span$\{\chi_E : \ E \ is \ measurable\}$ dense in $L^2(\mu)$ and T bounded/ continuous, we have $Tf = \psi f$ for each $f \in L^2(\mu)$. Hence, $T = \mathcal{M}_\psi$. □

From this we observe a consequence for the spectrum.

Corollary 4. *If $\psi \in L^\infty(\mu)$, then $ess - range(\phi) = \sigma_{\mathcal{M}}(\mathcal{M}_\phi) = \sigma(\mathcal{M}_\phi)$, where $\mathcal{M}_\phi$ is operator in $\mathcal{B}(\mathcal{H})$.*

Proof. If $\mathcal{M}_\phi$ is as given for some $\phi \in L^\infty(\mu)$ and $\lambda \in \rho(\mathcal{M}_\phi)$ i.e. $(\mathcal{M}_\phi - \lambda id)^{-1}$ exists, then $\mathcal{M}_\phi$ commutes with $(\mathcal{M}_\phi - \lambda)^{-1}$. Since by previous proposition, $\mathcal{M}$ is maximal abelian, so $(\mathcal{M}_\phi - \lambda)^{-1} \in \mathcal{M}$. Thus, $\sigma_{\mathcal{M}}(\mathcal{M}_\phi) = \sigma(\mathcal{M}_\phi)$. Together with $ess - range(\phi) = \sigma_{\mathcal{M}}(\mathcal{M}_\phi)$, we get the identity. □