Lecture Notes from February 21, 2023

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Last time

- $L^\infty(\mu)$ as a C^*-algebra and as operators on $L^2(\mu)$.
- Spectrum of $L^\infty(\mu)$

Warm up:

Let $\mathcal{A}$ be a C^*-algebra, $\chi : \mathcal{A} \to \mathbb{C}$ a character. Show if $a \geq 0$, $\chi(a) \geq 0$.

If $a \geq 0$, by definition, $\exists b \in \mathcal{A}$ such that $b^*b = a$. Thus,

$$\chi(a) = \chi(b^*b) = \chi(b^*)\chi(b) = \overline{\chi(b)}\chi(b) \geq 0.$$

Recall the Riesz Representation Theorem.

Definition: $\Phi : C_c(X) \to \mathbb{C}$ is called positive if $f \geq 0 \implies \Phi(f) \geq 0$.

1.6 Theorem. *Let X be a locally compact space. Then, for every positive functional $\Phi : C_c(X) \to \mathbb{C}$, there exists a unique Borel measure μ such that*

1. $\Phi(f) = \int_X f d\mu$,
2. $\mu(K) < \infty \quad \forall K \subset X$, *where K is a compact set,*
3. *For each measurable set E, $\mu(E) = \inf\{\mu(U) : E \subset U, \quad U$ open$\}$,*
4. *If E is measurable and $\mu(E) < \infty$, then $\mu(E) = \sup\{\mu(K) : K \subset E, \quad K$ compact$\}$.*

The last two items are regularity conditions on the measure that ensure uniqueness, and a measure that satisfies property 2 - 4 is called a Radon measure.

Note: This version of the Riesz Representation Theorem is Theorem 2.14, starting on page 40, in "Real and Complex Analysis" by Walter Rudin. Rudin then goes on, with this theorem in hand, to define Lebesque measure.

Our next goal is to extend $\mathcal{G}$, the Gelfand transform, so we can develop a functional calculus of measurable functions.

The situation will be as follows. If $T \in B(\mathcal{H})$, $T^*T = TT^*$, and $\mathcal{H}$ separable, we want to show that $\exists \mu$ on $\mathbb{C}$ and a map
$\mathcal{G}' : \{T\}'' \to L^\infty(\mu)$ such that the following diagram (in which the vertical down arrows indicate inclusions) commutes.

$$\begin{array}{ccc} \overline{span\{T^n(T^*)^m : m, n \geq 0\}}^{\|\cdot\|} & \xrightarrow{\mathcal{G}} & C(\sigma(T)) \\ \downarrow & & \downarrow \\ \{T\}'' & \xrightarrow{\mathcal{G}'} & L^\infty(\mu) \end{array}$$

To achieve this goal we will collect some ingredients.

The first two being the weak-$*$ and weak operator topologies.

1.7 Proposition. *Let* $(X, \mathcal{S}, \mu)$ *be a probability space. Then, the WOT on* $L^\infty(\mu)$ *(viewed as operators on* $L^2(\mu)$*) and the weak-$*$ topology on* $L^\infty(\mu)$ *(viewed as the dual of* $L^1(\mu)$*) are identical.*

Proof. The weak operator topology is the coarsest on $L^\infty(\mu)$ such that all maps given by $g, h \in L^2(\mu)$, $\Lambda_{g,h} : L^\infty(\mu) \to \mathbb{C}$, $f \mapsto \Lambda_{g,h}(f) = \int fg\overline{h}d\mu$ are continuous.

The weak-$*$ topology is the coarsest on $L^\infty(\mu)$ such that all maps given by $q \in L^1(\mu)$, $\Phi_q : L^\infty(\mu) \to \mathbb{C}$, $f \mapsto \int fgd\mu$ are continuous.

We now see that if Φ_q is continuous for each $q \in L^1(\mu)$ and $g, h \in L^\infty(\mu)$ are given, then $g\overline{h} \in L^1(\mu)$. Hence, $\Phi_{g\overline{h}}$ is continuous. But, $\Phi_{g\overline{h}} = \Lambda_{gh}$.

Conversely, suppose Λ_{gh} is continuous and we are given $q \in L^1(\mu)$ Then, let $g(x) = \frac{q(x)}{\sqrt{|q(x)|}}$ for $q \neq 0$ and 0 for $q = 0$ and let $h(x) = \sqrt{|q(x)|}$. Thus, g, h are in $L^2(\mu)$ and $\Lambda_{g,h}(f) = \Phi_q(f)$, so Φ_q is continuous. Therefore, the WOT and the weak-$*$ topology are the same on $L^\infty(\mu)$. $\square$

We still need some regularity. Recall continuous functions are not dense in $L^\infty(\mu)$ with respect to the norm topology.

1.8 Theorem. *If* X *is a compact Hausdorff space and* μ *a finite (positive) Borel measure then* $C(X)$ *is weak-$*$ dense in* $L^\infty(\mu)$.

Proof. We show this is true in the unit ball. By scaling, the density argument then extends to all of $L^\infty(\mu)$. We know simple functions are dense in the unit balls of $L^\infty(\mu)$. Explicitly, in the real case, if $f \in B_1(0) \subset L^\infty(\mu)$, for a fixed $n \in \mathbb{N}$ let $A_i = f^{-1}([\frac{i-1}{n}, \frac{i}{n}))$ and let $B_i = f^{-1}((\frac{-i}{n}, \frac{1-i}{n}])$. Define $\phi_n = \sum_{i=1}^n \frac{i}{n}\chi_{A_i} - \sum_{i=1}^n \frac{i}{n}\chi_{B_i}$. Then, $\|f - \phi_n\|_\infty \leq \frac{1}{n}$.

Thus, we only need to show we can get arbitrarily close to a given simple function $\psi \in B_1(0)$. Since ψ is simple, $\psi = \sum_1^n \alpha_i \chi_{E_i}$ with $|\alpha_i| < 1$, $\{E_i\}_1^n$ pairwise disjoint measurable subsets of X and $\cup_1^n E_i = X$. Using Tietze's Extension Theorem, we can find, for any choice of compact sets $K_i \subset E_i$ a function $\psi \in C(X)$ such that $\|\psi\|_\infty \leq 1$, $\phi|_{K_i} = \psi|_{K_i}$.

Next, given $f \in L^1(\mu)$, for any such choice of $\{K_i\}_1^n$ and ϕ, we get

$$
\begin{aligned}
|\int_X f(\psi - \phi)d\mu| &\leq \int_X |f||\psi - \phi|d\mu \\
&= \sum_1^n \int_{E_i \setminus K_i} |f||\psi - \phi|d\mu \\
&\leq 2\sum_1^n \int_{E_i \setminus K_i} |f|d\mu.
\end{aligned}
$$

Using the regularity of $|f|d\mu$, we can choose K_i such that $\int_{E_i \setminus K_i} |f|d\mu < \frac{\epsilon}{2n}$.
Hence, $|\int_X f(\psi - \phi)d\mu| < \epsilon$. ☐

It might be tempting to mistake the previous proof as being a result of circular reasoning with respect to the choice of K_i. However, the choice of K_i resulting in the correct ϵ bound is made after the function f is fixed. Then, TET gives us a ψ, and the proof is sound.

Next we want to see how to imbed $C(\sigma(T)) \subset L^\infty(\mu)$. That is, how should we choose the Borel measure μ?

1.9 Definition. Two measures ν_1 and ν_2 on a $\sigma-$algebra $\mathcal{S}$ are mutually absolutely continuous if for $E \in \mathcal{S}$ $\nu_1(E) = 0 \iff \nu_2(E) = 0$.

1.10 Theorem. *If* ν_1 *and* ν_2 *are finite (positive) regular Borel measures on a compact metric space* X *and there exists a *-isometric isomorphism* $\Phi : L^\infty(\nu_1) \to L^\infty(\nu_2)$ *for which* $\Phi(f) = f$ *for all* $f \in C(X)$ *then* Φ *is the identity,* $L^\infty(\nu_1) = L^\infty(\nu_2)$*, and* $\nu_1 \sim \nu_2$.