Lecture Notes from February 23, 2023

taken by Cristian Meraz

Last time:

- Riesz Representation theorem.
- The weak operator topology versus the weak $*$-topology for $L^\infty(\mu)$.
- $C(X)$ is weak $*$-dense in $L^\infty(\mu)$.

We recall the definitions of cyclic and separating vectors in a Hilbert space.

1.6 Definition. Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{A}$ subalgebra of $B(\mathcal{H})$. Then $v \in \mathcal{H}$ is *cyclic* if $\overline{\mathcal{A}v} = \mathcal{H}$, and *separating* if $Av = 0$ implies $A = 0$ for normal A.

Thus, cyclic vectors exhibit a "spanning"-type property, meanwhile separating vectors are special in that they provide an "easy" test for triviality of normal operators inside a subalgebra.

Warm-up: Let $T^*T = TT^*$, $T \in B(\mathbb{C}^n)$.

1.7 Question. When does $\mathcal{A}_T \coloneqq \text{Span}\{T^\ell (T^*)^m : \ell, m \geq 0\}$ have a cyclic vector?

1.8 Answer. The short answer is: $\mathcal{A}_T$ has a cyclic vector if $|\sigma(T)| = n$.

In particular, since normal matrices are diagonalizable, we can think of the case when $T = I_n$, the identity matrix. Thus, $\mathcal{A}_T$ is one-dimensional. It follows that

$$\exists v \in \mathbb{C}^n \text{ s.th. } \overline{\mathcal{A}_T v} = \mathbb{C}^n \quad \iff \quad n = 1.$$

Otherwise, for $n \geq 2$, the space $\overline{\mathcal{A}_T v}$ is still one-dimensional and thus has no cyclic vectors: The identity matrix fails so bad because it has n repeated eigenvalues!

Hence, reiterating the short answer: For a normal matrix T acting on $\mathbb{C}^n$, $\mathcal{A}_T$ has a cyclic vector if T has no repeated eigenvalues.

Aside: In physics, this is known as non-degeneracy. Degenerate roots of polynomials in elementary algebra are multiple roots, so we have a connection here whenever the polynomial in question is the characteristic polynomial of T: We want no degenerate roots.

We begin the class with a lemma.

1.9 Lemma. *In a compact metric space X, pointwise limits of decreasing sequences of continuous, nonnegative functions contain all characteristic functions of closed (compact) subsets.*

Proof. Let ρ be the metric on X. Then for $x \in X,\ K \subset X$, let

$$d(x, K) := \inf\{\rho(x, y) : y \in K\},$$

denote the distance between x and K. Let K be compact. Then if $\{\varphi_n\}_{n=1}^{\infty} \subset C(X)$ is defined by

$$\varphi_n(x) = \max\{0, 1 - nd(x, K)\},$$

we see that

$$\lim_{n\to\infty} \varphi_n(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \in X \setminus K. \end{cases}$$

That is, $\varphi_n \to \chi_K$ pointwise, and this proves the lemma.

□

The following theorem can be found as Theorem 4.55 on page 94 in Douglas ([**Douglas**]).

1.10 Theorem. *Let* $(X, \mathcal{S})$ *be a compact metric space with Borel* σ*-algebra* $\mathcal{S}$*, and let* λ_1, λ_2 *be finite regular Borel measures on this measurable space. If* Φ *is a* $*$*-isometric isomorphism between* $L^\infty(\lambda_1)$ *and* $L^\infty(\lambda_2)$ *such that* $\Phi(f) = f$ *for all* $f \in C(X)$*, then* $\lambda_1 \sim \lambda_2$ *and* Φ *is the identity.*

Proof. By Φ a $*$-isomorphism, $\Phi(f) \geq 0$ if $f \geq 0$. Since Φ is the identity on $C(X)$, if a sequence $(\varphi_n)_{n=1}^{\infty}$ in $C(X)$ is decreasing and converges pointwise (everywhere), so does $\Phi(\varphi_n) = \varphi_n$. Consequently, Φ is also the identity on all functions that are pointwise limits of decreasing sequences of continuous functions, i.e., by the preceding lemma, characteristic functions of compact subsets.

Next, consider a measurable set E. By regularity of λ_1, there exists a sequence of compact sets $(K'_n)_{n=1}^{\infty}$ such that $K'_n \subset E,\ K'_n \subset K'_{n'}$ if $n' \geq n$, and $\lambda_1(E \setminus K'_n) \to 0$. Similarly, there exists a sequence of compact sets $(K''_n)_{n=1}^{\infty}$ such that $K''_n \subset E,\ K''_n \subset K''_{n'}$ if $n' \geq n$, and $\lambda_1(E \setminus K''_n) \to 0$. By compactness of $K_n = K'_n \cup K''_n$, we have $\lambda_1(E \setminus K_n) \to 0$ <u>and</u> $\lambda_2(E \setminus K_n) \to 0$. Also, χ_{K_n} are increasing, and by Monotone Convergence theorem, are converging to some f. From $\Phi(\chi_{K_n}) = \chi_{K_n}$, $\lambda_{1,2}(\{x : f(x) \neq \chi_{K_n}(x)\}) = 0$, so $f = \chi_E$ in $L^\infty(\lambda_1)$ and in $L^\infty(\lambda_2)$. This implies by linearity that Φ is the identity on simple functions. By simple functions being dense in $L^\infty(\lambda_{1,2})$, Φ is the identity operator.

□

Next, we construct a measure for the extension $\mathcal{G}'$ of the Gelfand transform. First, a lemma.

1.11 Lemma. *If* $\mathcal{H}$ *is a Hilbert space,* $\mathcal{A}$ *commutative subalgebra of* $B(\mathcal{H})$*,* v *cyclic, then* v *is separating.*

Proof. Consider $B \in \mathcal{A},\ Bv = 0$. By commutativity, for all $A \in \mathcal{A}$,

$$BAv = ABv = 0.$$

Then $Av \in \ker B$. By density of $\mathcal{A}v$, $\ker B = \mathcal{H}$, which implies $B = 0$. □

1.12 Theorem. *Let* T *be a normal operator on* $\mathcal{H}$*, and suppose that* $\mathcal{A}_T$ *(as previously defined in the warm-up) has a cyclic vector. Then there exists a positive Borel measure,* ν*, on* $\mathbb{C}$ *(or* $\sigma(T)$*, if one considers the "trace" topology) with support* $\sigma(T)$*, an isometric isomorphism* Φ *between* $\mathcal{H}$ *and* $L^2(\nu)$ *such that* $\mathcal{G}'(A) = \Phi A \Phi^{-1}$ *is an isometric* $*$*-isomorphism between* $\mathcal{A}_T''$ *and* $L^\infty(\nu)$*, and* $\mathcal{G}'$ *extends* $\mathcal{G}$ *from* $\mathcal{A}_T$ *to* $L^\infty(\nu)$*. Finally, if* ν_1 *is a measure on* $\mathbb{C}$*,* $\mathcal{G}_1'$ *a* $*$*-isometric isomorphism from* $\mathcal{A}_T''$ *to* $L^\infty(\nu_1)$*, then* $\nu_1 \sim \nu$*,* $L^\infty(\nu_1) = L^\infty(\nu)$*, and* $\mathcal{G}_1' = \mathcal{G}'$*.*

Proof. Let f be a cyclic vector for $\mathcal{A}_T$, with $\|f\| = 1$. We define a functional ψ on $C(\sigma(T))$ by

$$\psi(\varphi) = \langle \varphi(T)f, f \rangle.$$

This is a positive (bounded) linear functional, so by Riesz Representation Theorem, there exists a regular Borel measure ν on $\sigma(T)$ such that for every $\varphi \in C(\sigma(T))$,

$$\langle \varphi(T)f, f \rangle = \int_{\sigma(T)} \varphi \, d\nu,$$

and by choosing $\varphi = 1$, we have $\|f\|^2 = 1 = \int_{\sigma(T)} d\nu$, which implies that ν is a probability measure.

We leave the rest of the proof to next class. $\square$