Lecture Notes from February 28, 2023

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Last Time

- Towards extended functional calculus.

Finishing the theorem from last time-

Warm-up Let (K, ρ, μ) be a Borel measure space for a compact set $K \subset \mathbb{C}$. Let $\mathcal{H} = \mathcal{L}^2(\mu)$, then

$$\mathcal{M}_{C(K)} \equiv \{M_\phi : \phi \in C(K)\}$$

forms a commutative C^*-algebra on $\mathcal{H}$

Q1. Does $\mathcal{M}_{C(K)}$ have a cyclic vector?

The answer is Yes. Take $f = 1$(the characteristic function on K. Then $M_\phi(f) = M_\phi(1) = \phi$ for any $\phi \in C(K)$ and by $C(K)$ being a dense linear subspace in $L^2(\mu)$, we have a cyclic vector.

Q1. If $\phi \in C(K)$, find conditions such that for $T = M_\phi$, $\Delta_T = \overline{\text{span}\{T^n(T^*)^m; n, m \geq 0\}}^{\|\cdot\|}$ has a cyclic vector.

If we can show that $\text{span}\{\phi^n\overline{\phi}^m\}$ is dense in $\mathcal{L}^2(\mu)$, then we have a cyclic vector given by 1(Consider the case when $\phi = 1$,then since we know that $\overline{M_{C(K)}} = \mathcal{L}^2(\mu)$. We also have that $M_\phi(f) = f$ for any $f \in \mathcal{L}^2(\mu)$. Since K is compact and using Stone-Weierstrass Theroem, since $M_\phi \in C(K)$ is a separating subset of $C(K)$. Then the complex unital *-algebra generated by M_ϕ namely Δ_T is dense in $C(K)$. Now since $C(K)$ is dense in $\mathcal{L}^2(\mu)$, we say that Δ_T is a dense linear supspace in $\mathcal{L}^2(\mu)$ and has a cyclic vector 1).

2.5 Remark.

$$\begin{array}{ccc} \mathcal{A}_T & \xrightarrow{\mathcal{G}} & C(\Gamma) = C(\sigma(T)) \\ \downarrow & & \downarrow \\ \{T\}'' & \xrightarrow{\mathcal{G}'} & \mathcal{L}^\infty(\mu) \end{array}$$

The above diagram commutes where μ is a Borel measure of $\sigma(T)$ and $\mathcal{G}$ is an isometry in terms of C^*-algebra and the map from $\mathcal{A}_T$ to $\{T\}''$ is an embedding.

2.6 Theorem. *Let* T*be a normal operator in* $\mathcal{H}$*,* $\mathcal{A}_T$ *has cyclic vector, then there is a positive regular Borel measure* ν *on* $\mathbb{C}$*, supp*$(\nu) = \sigma(T)$*, isometric isomorphism* γ *from* $\mathcal{H}$ *to* $\mathcal{L}^2(\nu)$ *such that* $\mathcal{G}'(A) = \gamma A \gamma^{-1}$ *is a* $*$*-isometric isomorphism from* $\mathcal{A}_T''$ *to* $\mathcal{L}^\infty(\nu)$*, and* $\mathcal{G}'$ *is a* $*$*- isomorphic isomorphism from* $\mathcal{A}_T''$ *to* $\mathcal{L}^\infty(\nu)$*.*

Moreover, if there is a measure ν_1 on $\mathbb{C}$, $\mathcal{G}_1'$ a $*$-isomorphism from $\mathcal{A}_T''$ to $\mathcal{L}^\infty(\nu_1)$ extending $\mathcal{G}$, then $\nu_1 \sim \nu$, and

$$\mathcal{L}^\infty(\nu_1) = \mathcal{L}^\infty(\nu)$$

and $\mathcal{G}_1' = \mathcal{G}'$

Proof. We had taken f cyclic to $\mathcal{A}_T$. Now Consider the map $\psi : C(\sigma(T)) \to \mathbb{C}$ defined as

$$\psi(\phi) \equiv \langle \phi(T)f, f\rangle$$

We have seen that there is a regular Borel measure ν on $\mathbb{C}$ such that for all $\phi \in C(\mathbb{C})$,

$$\psi(\phi) \equiv \langle \phi(T)f, f\rangle \int \phi d\nu$$

and taking $\phi = 1$ gives that ν is a probability measure.
Next we show that $\sigma(T)$ is the support of ν.
If it were not, we would find a relatively open set $\emptyset \neq U \subset \sigma(T)(U = A \cap \mathbb{C}$, where A is open in $\mathbb{C})$ such that $\nu(U) = 0$. Using Urysohn's Lemma, taking $\phi \in C(\sigma(T))$ such that $\phi \geq 0, \phi(x) = 1$ for some $x \in U$ and $\phi|_{U^c} = 0$ gives a contradiction because then

$$\begin{aligned}\langle \sqrt{Q(T)}f, \sqrt{Q(T)}f\rangle &= \int \phi d\nu \\ &= 0\end{aligned}$$

So $Q(T)$ annihilates f but $h \in \mathcal{A}_T$ and f is cyclic, then f is separating (Lemma from Last time). So we would get $\phi(T) = 0$ but $\phi \neq 0$ as $\mathcal{G}$ acts between continuous functions with the sup-norm and the bounded operator. Therefore, $\sigma(T) = supp(\nu)$. We now define

$$\Phi : \mathcal{A}_T f \mapsto C(\sigma(T))$$

by

$$\phi(T)f \mapsto \phi$$

Then Φ is an isometry if we equip $\mathcal{A}_T f$ with the norm $\|\phi(T)f\| = \|\phi\|_{\mathcal{L}^2(\nu)}$. By definition,

$$\begin{aligned}\|\phi(T)f\|^2 &= \langle (\phi(T))^*\phi(T)f, f\rangle \\ &= \langle |\phi(T)|^2 f, f\rangle \\ &= \int_\sigma (T)|\phi|^2 d\nu \\ &= \|\phi\|_{\mathcal{L}^2}\end{aligned}$$

The norm on $\mathcal{A}_T f$ coincides with the norm on $\mathcal{H}$. So, we have Φ is a isometry that extends to a map defined on $\mathcal{H}$.

We also note that we can get any continuous functions being dense in $\mathcal{L}^2(\nu)$ and included in $\mathrm{Ran}(\Phi)$, the range of the extended isometry is all of $\mathcal{L}^\in(\nu)$, so Φ is unitary.

Next, we define $\mathcal{G}'$ from T'' to $\mathcal{B}(\mathcal{L}^2(\nu))$ by

$$\mathcal{G}'(A) = \Phi A \Phi^{-1}$$

Next we show that $\mathcal{G}'|_{\mathcal{A}_T} = \mathcal{G}$(i.e., $\mathcal{G}'$ extends the Gelfand transform) .
For any $\phi \in C(\sigma(T))$, $g \in C(\sigma(T))$,

$$\begin{aligned}\mathcal{G}'(\phi(T))g &= (\Phi\phi(T)\Phi^{-1})g \\ &= \Phi(\phi(T))g(T)f \\ &= \Phi(\phi g(T))f \\ &= \phi g \\ &= M_\phi g\end{aligned}$$

i.e., any continuous function of T corresponds to a multiplication operator. It is true for all $g \in C(\sigma(T))$ and then by continuity (and boundedness) of $\mathcal{G}'(\phi(T))$, this identity holds for each $g \in \mathcal{L}^2(\nu)$. Now using $\mathcal{A}_T'' = \overline{\mathcal{A}_T}^{WOT}$(By Bicommutant theorem) and $\overline{C(\sigma(T))}^{W^*} = \mathcal{L}^\infty(\nu)$(as $\overline{C(\sigma(T))}^{W^*} = \overline{C(\sigma(T))}^{WOT}$), we get that $\mathcal{G}'$ is a $*$-isometric isomorphism between $\mathcal{A}_T''$ and $\mathcal{L}^\infty(\nu)$ □