Lecture Notes from 2 March 2023

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Last time

- Construction of $\mathcal{G}'$

Warm-up: Find/characterize all maximal Abelian C^*-algebras in $M_n(\mathbb{C}) = B(\mathbb{C}^n)$.

Idea: normal matrices can be diagonalized.

1.2 Proposition. *Any maximal Abelian sub-algebra $\mathcal{M} \subset M_n(\mathbb{C})$ is given by $\mathcal{M} = \{U^*DU : D$ is diagonal$\}$ for a fixed unitary U.*

Proof. To see that any such $\mathcal{M}$ is maximal Abelian, assume the contrary. That is $\mathcal{M} \subset \mathcal{A}$, $A \in \mathcal{A}$, $A \notin \mathcal{M}$, and $\mathcal{A}$ is an Abelian C^*-algebra of $M_n(\mathbb{C})$. If such an element A exists, we can assume that $(UAU^*)_{j,j} = 0$ for all $1 \leq j \leq n$. Otherwise, A could be recovered from $\mathcal{M}$. This means, $(UAU^*)_{i,j} \neq 0$ for some $i \neq j \in \{1, ..., n\}$. Write $P_j = U_j U_j^*$, then $P_i A P_j \neq 0$.

However, $P_i, P_j \in \mathcal{M}$. Since $\mathcal{A}$ is Abelian, we see that $P_i A P_j = P_i P_j A = 0A = 0$ which is a clear contradiction. $\square$

In conclusion, $\mathcal{M}$ looks like $C(X)$ where $X = \{1, \cdots, n\}$,

$$\mathcal{M} \cong \left\{ \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} : d_i \in \mathbb{C}, \forall i \in X \right\}$$

but, of course, continuity is not very interesting in the discrete topology. Anyway, onward to completing the proof from last time!

Proof. (continued) For the uniqueness of ν, up to mutual absolute continuity, if there exists

$$\mathcal{G}_1' : \mathcal{A}_T'' \to L^\infty(\nu),$$

with ν_1 a regular Borel measure having support on $\sigma(T)$, and $\mathcal{G}_1'$ an isometric $*$-isomorphism, then $\mathcal{G}' \circ (\mathcal{G}_1')^{-1}$ is an isometric $*$-isomorphism from $L^\infty(\nu)$ to itself, and when restricted to $\sigma(T)$, it is the identity. We conclude by our preceding theorem (see previous day's notes) that $\nu \sim \nu_1$, and $L^\infty(\nu) = L^\infty(\nu_1)$. $\square$

Remark: It can be shown if $\mathcal{A}$ is an Abelian C^*-algebra on a separable Hilbert space, then $\mathcal{A} \subset B$, where B is a maximal Abelian C^*-algebra, which has a cyclic vector which is separating for $\mathcal{A}$.

As a consequence, if $\mathcal{H}$ is separable, and $T \in B(\mathcal{H})$ is normal, then the conclusion of the preceeding theorem holds. In Douglas Farenick's "Fundamentals of Functional Analysis", this fact comes in the form of the Hahn Decomposition Theorem on page 112, and gives explicitly the separation via the positive and negative supports of the signed measure.

2 The next chapter

We turn to contemplating the following question:

Given a C^*-algebra $\mathcal{A}$, can we find some Hilbert space $\mathcal{H}$ and a $*$-isometry between $\mathcal{A}$ and $B(\mathcal{H})$?

2.1 Positivity and the Gelfand-Neumark-Segal Theorem

We start by studying reproducing kernels, a topic of keen interest to Data Scientists and Mathematicians alike. The motivation to study such objects will become clear in the next few lectures, but as a preview, this class of functions is in one-to-one correspondence with Hilbert spaces in which elements can be properly interpreted as functions; more excitingly, the Hilbert space structure can be recovered in its entirety from just its reproducing kernel, hence the name. This provides a way to encode a Hilbert space as a function, and then specify a new Hilbert space through transformations of that function. That being said, let's learn how to encode the structure of such a Hilbert space into a single function.

2.1 Lemma. *Let M be a set and $\mathcal{H} \subset \mathbb{C}^M$ a hilbert space such that $\forall m \in M$,*

$$\Lambda_m : f \mapsto f(m)$$

is a continuous linear functional. Then there exists a function $K : M \times M \to \mathbb{C}$ with the following properties:

1. *For $y \in M$, $K_y \equiv K(\cdot, y) \in \mathcal{H}$ gives for all $f \in \mathcal{H}$,*

$$\langle f, K_y \rangle = f(y)$$

2. *For every finite $\{x_1, \cdots, x_n\} \subset M$, the matrix $(K(x_j, x_l))_{j,l=1}^n$ is positive semidefinite.*

3. *The set $\{K_y : y \in M\}$ is total in $\mathcal{H}$, i.e. has dense linear span in $\mathcal{H}$.*

4. *If $\{e_j\}_{j \in J}$ is an orthonormal basis and $x, y \in M$, we have that*

$$K(x, y) = \sum_{j \in J} e_j(x) \overline{e_j(y)}.$$

Proof. 1. From continuity of each Λ_m, the Riesz Representation Theorem gives a K_m such that $f(m) = \Lambda_m(f) = \langle f, K_m \rangle$ for each $f \in \mathcal{H}$. We write $K_y(x) \equiv K(x, y)$ when the notation is convenient.

2. By $K(y, x) = K_x(y) = \langle K_x, K_y \rangle = \overline{\langle K_y, K_x \rangle} = \overline{K(x, y)}$, we know that for any finite $\{x_1, \cdots, x_n\} \subset M$ the matrix $(K(x_i, x_j))_{i,j=1}^n = (\overline{K(x_j, x_i)})_{i,j=1}^n$ is Hermitian. And we have, for $c \in \mathbb{C}^n$,

$$\begin{aligned} \sum_{j,l=1}^{n} \overline{c_j} c_l K(x_j, x_l) &= \sum_{j,l=1}^{n} \overline{c_j} c_l \left\langle K_{x_l}, K_{x_j} \right\rangle \\ &= \sum_{j,l=1}^{n} \left\langle c_l K_{x_l}, c_j K_{x_j} \right\rangle \\ &= \left\langle \sum_{l=1}^{n} c_l K_{x_l}, \sum_{j=1}^{n} c_j K_{x_j} \right\rangle \\ &= \| \sum_{j=1}^{n} c_j K_{x_j} \|^2 \\ &\geq 0 \end{aligned}$$

So the matrix is positive semidefinite.

The proof will conclude in next Tuesday's notes. □

But a few remarks regarding the properties of K, the reproducing kernel constructed from the Hilbert space in the preceding theorem, before we conclude:

It is properties 1. and 3. which allow us to completely recover the Hilbert space and its structure from the reproducing kernel; specifically, property 1. will enable us to recover the inner product, and property 3. will allow us to retrieve any element as the limit of a sequence of partial evaluations of K. Property 2. helps compare reproducing kernels and transformations of such. And property 4. provides a way to explicitly write down the function after choosing an orthonormal basis since, in the proof, we acquire it from the Riesz Representation Theorem, which does not give explicit instructions on how to find the dual element in question, only that it exists and that it is unique.