Lecture Notes from March 7, 2023

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Last time

- Maximality and cyclic vectors.
- Reproducing kernels.

We continue with the proof of the lemma from last time.

2.2 Lemma. *Let M be a set and $\mathcal{H} \subset \mathbb{C}^M$ a Hilbert space such that for each $m \in M$,*

$$\Lambda_m : f \mapsto f(m)$$

is a continuous linear functional, then there exists a function $K : M \times M \to \mathbb{C}$ with the following properties:

(1) For $y \in M$, $K_y \equiv K(\cdot, y) \in \mathcal{H}$ gives for all $f \in \mathcal{H}$,

$$\langle f, K_y \rangle = f(y)\,.$$

(2) For every finite $\{x_1, \cdots, x_n\} \subset M$, the matrix $(K(x_j, x_l))_{j,l=1}^n$ is positive semidefinite.

(3) The set $\{K_y : y \in M\}$ is total in $\mathcal{H}$, i.e. has dense linear span in $\mathcal{H}$.

(4) If $\{e_j\}_{j \in J}$ is an orthonormal basis and $x, y \in M$, we have that

$$K(x, y) = \sum_{j \in J} e_j(x)\overline{e_j(y)}.$$

Proof. We had already shown (1) and (2). The property (3) follows from $f \perp \{K_m : m \in M\}$ implying $f(m) = 0$ for each $m \in M$, so the orthogonal complement of $\{K_m : m \in M\}$ is $\{0\}$, hence the span of the set of kernel functions is dense. Finally, we obtain (4) from the properties of the orthonormal basis,

$$K_y = \sum_{j \in J} \langle K_y, e_j \rangle e_j$$

and by taking the inner product of both sides with K_x. $\square$

This lemma shows that having a Hilbert space of functions for which all point evaluation functionals gives a positive kernel. The converse holds, too, so each positive kernel K determines a Hilbert space with bounded point evaluation functionals.

2.3 Theorem. *Let* K *be a positive kernel on* M*, then we can equip* $\mathcal{H}^0 = \mathrm{span}\{K_y : y \in M\}$ *with an inner product*

$$\langle \sum_j c_j K_{y_j}, \sum_k d_k K_{y_k} \rangle = \sum_{j,k} c_j \overline{d_k} K(y_k, y_j)\,.$$

For this space, we have $f(y) = \langle f, K_y \rangle$ *for each* $f \in \mathcal{H}^0$ *and* $y \in M$*. The completion of* $\mathcal{H}^0$ *is a Hilbert space of functions on* M *with continuous point evaluation functionals and* K *is the corresponding kernel according to the lemma.*

Proof. We first need to show that the inner product is well defined. To this end, we note that for $f = \sum_j c_j K_{y_j}$ and $h = \sum_k d_k K_{y_k}$, the value

$$\sum_{j,k} c_j \overline{d_k} K(y_k, y_j) = \sum_{j,k} c_j \overline{d_k} K_{y_j}(y_k) = \sum_k \overline{d_k} f(y_k)$$

depends only on f, not on the particular way f was written as a linear combination of kernel functions. Similarly, the above value depends only on h, not on the explicit expression in terms of the kernel functions. This permits us to write

$$\langle f, h \rangle = \sum_{j,k} c_j \overline{d_k} K(y_k, y_j)\,.$$

By the positivity of the kernel, this defines a positive semidefinite sesquilinear form on $\mathcal{H}^0$. From the definition, setting $h = K_y$, we get $\langle f, K_y \rangle = f(y)$. If $\langle f, f \rangle = 0$, then for each $y \in M$, $|f(y)|^2 \leq K(y, y)\langle f, f \rangle = 0$, so f is the zero function. This shows that the sequilinear form is positive definite.

Let $\mathcal{H}_K$ be the completion of this inner product space, then the set $\{K_y : y \in M\}$ is total, because its span is $\mathcal{H}^0$. Thus, we can identify each $f \in \mathcal{H}_K$ with a function $f : M \to \mathbb{C}$, $f(y) = \langle f, K_y \rangle$, and the point evaluation functionals are obtained as inner products with the kernel functions. $\square$

Next, we investigate how the choice of the kernel reflects in properties of the associated Hilbert space.

2.4 Lemma. *Let* K *and* L *be positive kernels on* M*, then* $K + L$ *is a positive kernel and*

$$\mathcal{H}_{K+L} = \mathcal{H}_K + \mathcal{H}_L \equiv \{f + h : f \in \mathcal{H}_K, h \in \mathcal{H}_L\}\,.$$

Proof. Since the sum of two positive semidefinite matrices is positive semidefinite, $Q = K + L$ is a psotive kernel on M. Consider $\mathcal{X} = \mathcal{H}_K \oplus \mathcal{H}_L$ and define

$$\Phi : \mathcal{X} \to \mathcal{H}_K + \mathcal{H}_L, (f, h) \mapsto f + h\,.$$

The kernel of Φ consists of $(f, h) \in \mathcal{X} : f(y) = h(y) = 0$, which can be rewritten as

$$\ker \Phi = \{(f, g) : \langle (f, h), (K_y, L_y) \rangle = 0 : y \in M\}.$$

In particular, the kernel is closed and its perp is given by the span of $\{(K_y, L_y) : y \in M\}$. From the decomposition of the direct sum into $\ker \Phi$ and the orthogonal complement, $\ker \Phi^\perp$ is realized as a reproducing kernel on M, whose reproducing kernel is given by

$$(x, y) \mapsto \langle (K_y, L_y), (K_x, L_x) \rangle = \langle K_y, K_x \rangle + \langle L_y, L_x \rangle = K(x, y) + L(x, y).$$

This shows $\mathcal{H}_K + \mathcal{H}_L = \mathcal{H}_{K+L}$.

$\square$