Lecture Notes from March 9, 2023

taken by Bernhard Bodmann

1.1 Last time

- Reproducing kernel Hilbert spaces ,
- Relationships between reproducing kernels and associated Hilbert spaces.

1.2 Warm up

For a matrix $A \in M_n(\mathbb{C})$, show that all the non-zero the eigenvalues of AA^* and of A^*A form the same set.

To see this, we note that if $AA^*x = \lambda x$, then defining $y = A^*x$, if $\|y\|^2 = \|A^*x\|^2 = \langle AA^*x, x\rangle = \lambda\|x\|^2 \neq 0$, we obtain $A^*Ay = A^*AA^*x = \lambda A^*x = \lambda y$, so y is an eigenvector of A^*A corresponding to eigenvalue $\lambda \neq 0$. Switching the roles of A and A^* gives the converse.

1.3 Inclusions of reproducing kernel spaces

We continue the relationship between kernel functions and spaces.

1.2 Lemma. *Let L and K be two positive kernels on M, then the following holds:*

(i) The inclusion $\mathcal{H}_K \subset \mathcal{H}_K$ is equivalent to the existence of $C > 0$ such that $CL - K$is a positive kernel.

(ii) If $\mathcal{H}_K \subset \mathcal{H}_L$, then the canonical inclusion map $i : \mathcal{H}_K \to \mathcal{H}_L$, $i(f) = f$ is a continuous linear map.

Proof. First let us assume $Q = CL - K$ is a positive kernel. Then $Q + K = CL$ and with the preceding lemma, we get

$$\mathcal{H}_L = \mathcal{H}_{CL} = \mathcal{H}_K + \mathcal{H}_Q,$$

so $\mathcal{H}_K \subset \mathcal{H}_L$. Conversely, assume $\mathcal{H}_K \subset \mathcal{H}_L$, and let I be the canonical embedding. We show it is a continuous linear map. To this end, we use the closed graph theorem. Let us assume $f_n \to f$ in $\mathcal{H}_K$ and assume $i(f_n) = f_n \to h$ in $\mathcal{H}_L$. We want to show $h = i(f) = f$. For $x \in M$, we use continuity of the inner products to get

$$f(x) = \langle f, K_x\rangle = \lim_n \langle f_n, K_x\rangle = \lim_n f_n(x) = \lim_n \langle i(f_n), L_x\rangle = \lim_n \langle f_n, L_x\rangle = h(x)\,.$$

By the closed graph theorem, i is bounded. We return to the second part of (i). Since i is continuous, so is ii^* Moreover, we have

$$i^* L_x = K_x$$

for each $x \in M$, because $\langle f, i^* L_x \rangle = \langle i(f), L_x \rangle = f(x) = \langle f, K_x \rangle$ for each $f \in \mathcal{H}_K$, $x \in M$. Choosing $f = K_y$ gives

$$\langle ii^* L_y, L_x \rangle = \langle i K_y, L_x \rangle = K_y(x) = K(x, y) \,.$$

Setting $C = \|i\|^2$ and let the map $D = C1 - ii^*$, then for each $v \in \mathcal{H}_L$,

$$\langle Dv, v \rangle = C\|v\|^2 - \|i^* v\|^2 \geq 0 \,.$$

Next, we define $Q(x, y) = \langle DL_y, L_x \rangle$ and observe that by definition of D, $Q = CL - K$. By the positivity of D, we can then verify that Q is a positive kernel. □

2 States and positivity

Next, we prepare the GNS representation. To this end, we need the concept of states.

2.1 Definition. A Hermitian element a in a C^*-algebra $\mathcal{A}$ is called positive, if $\sigma(a) \subset \mathbb{R}^+$. We write $a \geq 0$ and also denote the set of positive elements of $\mathcal{A}$ by $\mathcal{A}^+$.

2.2 Example. Let $\mathcal{A} = C(X)$ for compact X, then $\sigma(f) = f(X)$ and $f \geq 0$ if and only f has values in $\mathbb{R}^+$.