Positive Functionals and States
Manpreet Singh
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Last Time

- Reproducing Kernel spaces and properties.

Warm up

Question: If $A \in M_n(\mathbb{C})$ then AA^* and A^*A have the same eigenvalues except possibly for $\lambda = 0$.

Solution: Let $A^*Ax = \lambda x$ with $\lambda \neq 0$, $x \neq 0$. Let $y = Ax$ then by $x \neq 0$ and $||Ax||^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle = \langle x, x \rangle = \lambda ||x||^2 \neq 0$. We get $y = Ax \neq 0$. Note that $AA^*y = AA^*Ax = \lambda Ax = \lambda y$. Thus, y is an eigen vector corresponding to eigen value λ for AA^*. Therefore, $\sigma(A^*A) \setminus \{0\} \subset \sigma(AA^*) \setminus \{0\}$. By switching the roles between A and A^*, we get the reverse inclusion.

Postive Functionals and States

0.1 Definition. A normal element of a C^*-algebra $\mathcal{A}$ is called positive if $\sigma(a) \subset \mathbb{R}^+$. We write $a \geq 0$ and let $\mathcal{A}^+$ denote the positive elements in $\mathcal{A}$.

0.2 Example. If X is compact, $\mathcal{A} = C(X)$, then for $f \in \mathcal{A}$, $\sigma(f) = f(X)$, so $f \geq 0$ iff f has values in $[0, \infty)$.

0.3 Lemma. *If $\mathcal{A}$ is a complex algebra with unit and $x, y \in \mathcal{A}$, then $\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}$.*

Proof. We want to show if $\lambda \neq 0$, $\lambda 1 - xy$ has an inverse if and only if $\lambda 1 - yx$ does. Without loss of generality, let $\lambda = 1$, otherwise divide both $\lambda 1 - xy$ and $\lambda 1 - yx$ by λ. Let $1 - xy$ be invertible and $z = (1 - xy)^{-1}$. Define $a := 1 + yzx$ then left multiplying and right multiplying both sides of $(1 - xy)xy = xy(1 - xy)$ by z gives $xyz = zxy$. Also $xy = 1 - z^{-1}$, so we get $xyz = zxy = z(1 - z^{-1}) = z - zz^{-1} = z - 1$. Now

$$a(1 - yx) = (1 + yzx)(1 - yx)$$

$$a(1 - yx) = 1 + yzx - yx - yzxyx$$

$$a(1 - yx) = 1 + yzx - yx - y(z - 1)x$$

$$a(1-yx) = 1 + yzx - yx - yzx + yx$$

$$a(1-yx) = 1$$

So, $a = (1-yx)^{-1}$. By exchanging roles of $xandy$, proves our claim. □

We use this lemma to deduce properties of positive elements in C^*-algebras.

0.4 Lemma. *Let $\mathcal{A}$ be a C^*-algebra with unit and $a \in \mathcal{A}$, then*

1. *if $a \geq 0$, then there is a unique $b \in \mathcal{A}^+$ with $b^2 = a$.*
2. *if a is Hermitian, then there is a unique pair $a_+, a_- \in \mathcal{A}^+$ such that $a = a_+ - a_-$ and $a_+ a_- = a_- a_+ = 0$.*
3. *if $a, b \in \mathcal{A}^+$ then $a + b \in \mathcal{A}^+$.*
4. *if $-aa^* \geq 0$ then $a = 0$.*

Proof. 1. We recall Gelfand's representation, mapping $a \mapsto \hat{a}$, with $\mathcal{P} \in \tau$, $\hat{a}(\mathcal{P}) \equiv \mathcal{P}(a)$. By $a \geq 0$, we get $\hat{a} : \tau \to \mathbb{R}^+$. Taking $\hat{b} = \sqrt{\hat{a}} : \mathcal{P} \mapsto \sqrt{\mathcal{P}(a)}$ then gives $b^2 = a$ and $b \geq 0$. For the uniqueness, let $c \geq 0$, $c^2 = a$, then $ca = c^3 = ac$. So $\{1, c, a\}$ generates a commutative C^*- subalgebra $\mathcal{A}^{'} \subset \mathcal{A}$. We have $c \geq 0$ in $\mathcal{A}^{'}$, but also $b \in \mathcal{A}^{'}$ and $b \geq 0$ and both satisfy $c^2 = b^2 = a$. Using Gelfand's representation for $\mathcal{A}^{'}$, then shows $\hat{c} = \hat{b} = \sqrt{\hat{a}}$. □