Lecture Notes from March 23, 2023

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Last time

- Positivity
- Spectrum
- Square Roots

Warm up: If $B \in M_n(\mathbb{C})$, $A = BB^*$, then $A \geq 0$.

Proof. Let $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$ with $x \neq 0$ such that $Ax = \lambda x$. If we can show $\lambda \in [0, \infty)$ then we are done. We know $\lambda \in \mathbb{R}$ because A is Hermitian. We can simply compute:

$$\lambda ||x||^2 = \langle Ax, x \rangle = \langle BB^*x, x \rangle = \langle B^*x, B^*x \rangle = ||B^*x||^2 \geq 0.$$

Thus, $\lambda \geq 0$. $\square$

We first finish proof of the lemma from last time.

1.6 Lemma. *Let $\mathcal{A}$ be a C^*-algebra with unit and $a \in \mathcal{A}$. Then the following hold.*

1. *If $a \geq 0$, $!b \in \mathcal{A}^+$ with $b^2 = a$.*
2. *If $a^* = a$ then $\exists!$ pair $a_+, a_- \in \mathcal{A}$ such that $a = a_+ - a_-$ and $a_+a_- = a_-a_+ = 0$.*
3. *If $a, b \in \mathcal{A}^+$, then $a + b \in \mathcal{A}^+$.*
4. *If $-aa^* \geq 0$ then $a = 0$.*

1) The first statement was proved in the last notes.

2) If $a^* = a$ then $\exists!$ pair $a_+, a_- \in \mathcal{A}$ such that $a = a_+ - a_-$ and $a_+a_- = a_-a_+ = 0$.

Proof. Let $\mathcal{A}_a$ be the commutative subalgebra of $\mathcal{A}$ generated by $\{1, a\}$. Recall the Gelfand representation $a \mapsto \hat{a}$ with $p \in \Gamma$, $\hat{a}(p) = p(a)$, which we may apply to $\mathcal{A}_a$. Hence, may define $\hat{a}_+(p) = max\{a(p), 0\}$ and $\hat{a}_-(p) = max\{-a(p), 0\}$. It is clear by construction that the corresponding a_+ and a_- satisfy the statement.

To prove uniqueness, suppose there exists y_+, y_- such that $a = y_+ - y_-$, $y_+, y_- \geq 0$, and $y_+y_- = y_-y_+ = 0$. Then, a commutes with y_+ and y_-, so $\{1, a, y_+, y_-\}$ generate $\mathcal{A}' \subset \mathcal{A}$. By Gelfand, $\mathcal{A}' \cong C(\Gamma_{\mathcal{A}'})$. For $p \in \Gamma_{\mathcal{A}'}$ with $\hat{a}(p) = 0$ we have $\hat{y}_+(p) = \hat{y}_-(p) = 0$. This follows from $0 = \hat{a}(p) = \hat{y}_+(p) - \hat{y}_-(p)$, so $\hat{y}_+(p) = \hat{y}_-(p)$. Hence, $\hat{y}_+(p)\hat{y}_-(p) = 0$, so $\hat{y}_+^2(p) = 0$, so $\hat{y}_+ = 0$. Likewise, $\hat{y}_-(p) = 0$. If $\hat{a} > 0$, then $\hat{y}_+(p) > 0$, $\hat{y}_-(p) = 0$, so $\hat{a}(p) = \hat{y}_+(p)$. Likewise, if $\hat{a} < 0$, then $\hat{y}_-(p) > 0$, $\hat{y}_+(p) = 0$ so $\hat{a}(p) = \hat{y}_-(p)$. Hence, $\hat{y}_+ = \hat{a}_+$ and $\hat{y}_- = \hat{a}_-$, so $y_+ = a_+$, and $y_- = a_-$. □

3) If $a, b \in \mathcal{A}^+$, then $a + b \in \mathcal{A}^+$.

Proof. Let $c = a + b$, $a, b \geq 0$, c Hermitian and $||a|| = \alpha$, $||b|| = \beta$. Since $\sigma(a) \subset [0, \alpha]$, $\sigma(\alpha 1 - a) \subset [0, \alpha]$, and $||\alpha 1 - a|| = r(\alpha 1 - a) \leq \alpha$. Likewise, $||\beta 1 - b|| \leq \beta$. Thus,

$$||(\alpha + \beta)1 - (a + b)|| \leq ||\alpha 1 - a|| + ||\beta 1 - b|| \leq \alpha + \beta.$$

If $\lambda \in \sigma(c)$, then $|\alpha + \beta - \lambda| \leq \alpha + \beta$ by the inequality between the norm and spectral radius for $(\alpha + \beta)1 - c$. Conjoining the results, if $\lambda \in \sigma(c)$, then $\lambda \leq \alpha + \beta$ and $\alpha + \beta - \lambda \leq \alpha + \beta$. Hence, $\lambda \geq 0$ so $c \geq 0$. □

4) If $-aa^* \geq 0$ then $a = 0$.

Proof. By a March 21 lemma, $\sigma(-a^*a) \setminus \{0\} = \sigma(-aa^*) \setminus \{0\} \subset \mathbb{R}$. Thus, $-aa^* \geq 0$ implies $-a^*a \geq 0$. Let $a = b + ic$, $b = b^*$, $c = c^*$, then $aa^* + a^*a = 2(b^2 + c^2) \geq 0$, by (3). Now, by assumption, $a^*a = (aa^* + a^*a) - aa^* \geq 0$ and $-a^*a \geq 0$. This implies $\sigma(a^*a) = \{0\}$. Finally, by the equality between norm and spectral radius of hermitian elements, $||a||^2 = ||aa^*|| = r(a^*a) = 0$, so $a = 0$. □

With the lemma now proved, we return to the warm-up question so that we may generalize it to the C^*−algebra context.

1.7 Theorem. *If $b \in \mathcal{A}$, a C^*-algebra with unit, then $a = b^*b \geq 0$.*

Proof. We know $a = a^*$, so by the lemma, $a = a_+ - a_-$. We compute,

$$(ba_-)^*(ba_-) = a_-^* b^* b a_- = a_-^*(a_+ - a_-)a_- = a_-^* a_+ a_- - a_-^* a_-^2 = -a_-^3.$$

Thus, $-(ba_-)^*(ba_-) \geq 0$, so $ba_- = 0$ and $a_-^3 = 0$. We conclude $a_- = 0$ so $a \geq 0$ as desired. □

Positivity and Involutive Semigroups

1.8 Definition. (a)Let S be an involutive semigroup. A function $\phi : S \to \mathbb{C}$ is called positive definite if $K_p(s,t) = \phi(st^*)$ is a positive kernel. (b)Let $\mathcal{A}$ be an algebra with involution. A linear functional $f : \mathcal{A} \to \mathbb{C}$ is called positive-definite in the sense of (a) with $S = \mathcal{A}$.

1.9 Lemma. *Let f be a linear functional on an algebra with involution $\mathcal{A}$, then the following are equivalent:*

- *The functional f is positive.*
- *The kernel $K_f : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ defines a positive semidefinite sesquilinear form.*
- *$f(aa^*) \geq 0$ for each $a \in \mathcal{A}$.*