Lecture Notes from March 28, 2023

taken by Joseph Walker

0 Warm Up

Let $W \in M_n(\mathbb{C})$, $W \geq 0$, $\mathrm{tr}[W] = 1$ and define $f : M_n(\mathbb{C}) \to \mathbb{C}$, $f(x) = \mathrm{tr}[XW]$. We will show that f is positive and $|f(X)| \leq \|x\|$, then $\|f\| = 1$.

By the spectral theorem, $W = \sum_{j=1}^{n} \lambda_j v_j v_j^*$, $\lambda \geq 0$, $\sum_{j=1}^{n} \lambda_j = 1$. We want to show $f(X^*X) \geq 0$ for any $X \in M_n(\mathbb{C})$. Evaluating the trace in the eigenbasis of W gives

$$\begin{aligned} f(XX^*) &= \mathrm{tr}[XX^*W] \\ &= \sum_{j=1}^{n} \langle XX^*Wv_j, v_j \rangle \\ &= \sum_{j=1}^{n} \lambda_j \langle XX^*v_j, v_j \rangle \\ &= \sum_{j=1}^{n} \lambda_j |X^*v_j|^2 \geq 0 \end{aligned}$$

To show the second proposition, we have $|f(x)| = |\sum_{j=1}^{n} \lambda_j \langle Xv_j, v_j \rangle| \leq \sum_{j=1}^{n} \lambda_j |\langle Xv_j, v_j \rangle| \leq \|X\|$ by Cauchy-Schwartz

1 Properties of Positive Linear Functionals on C*-Algebras

1.0.1 Proposition. *Le $\mathcal{A}$ be a C^*-algebra with unit, f be a positive linear functional on $\mathcal{A}$, then*

(i) $f(a^*) = \overline{f(a)}$

(ii) $|f(ab^*)| \leq f(aa^*)f(bb^*)$

(iii) $|f(x)| \leq f(1)\|x\|$

(iv) f *is continuous with* $\|f\| = f(1)$

Proof. (i) f is positive, so $K_f(a, b) = f(ab^*)$ is a positive semidefinite sesquilinear form. Using Hermitian properties, we have $f(a^*) = K_f(1, a) = \overline{K_f(a, 1)} = \overline{f(a)}$

(ii) Applying Cauchy-Schwarz to K_f yields (ii).

(iii) With $a = 1, b = x$ we get $|f(x)|^2 = |K_f(1,x)|^2 \leq K_f(1,1)K_f(x,x) = f(1)f(xx^*)$. It remains to show $f(xx^*) \leq f(1)\|x\|^2$.

Using the equality between the spectral radius and operator norm for Hermitian elements, let $t > \|x\|^2 = r(xx^*)$, then $\sigma(t1 - xx^*) = t - \sigma(xx^*) \subset \mathbb{R}^+$ and hence $t1 - xx^* \geq 0$. We use the square-root lemma, and get $u \in \mathcal{A}^+$ with $t1 - xx^* = u^2 \geq 0$. Thus $f(uu^*) = f(u^2) = f(t1 - xx^*) = tf(1) - f(xx^*) \geq 0$, so $f(xx^*) \leq tf(1)$ and taking inf over all $t > \|x\|^2$ gives $f(xx^*) \leq f(1)\|x\|^2$. Combining inequalities gives $|f(x)|^2 \leq f(1)^2\|x\|^2$ and taking the square-root on both sides gives the inequality.

(iv) We know from (iii) that $\|f\| \leq f(1)$. Using that $\|1\| = 1$, we have $|f(1)| = f(1)$, hence $sup_{\|x\|\leq 1}|f(x)| \geq f(1)$. The complimentary inequality shows $\|f\| = 1$.

□

2 States

2.0.1 Definition. Let $\mathcal{A}$ be a C^* algebra with unit. A functional $\phi \in \mathcal{A}'$ with $\phi(1) = \|\phi\| = 1$ is called a state. The set of states is denoted by $\zeta(\mathcal{A})$.

2.1 Properties of States

2.1.2 Lemma. *The set $\zeta(\mathcal{A})$ is a convex weak-*-compact subset of $\mathcal{A}'$*

Proof. The convexity of $\zeta(\mathcal{A})$ is due to the linearity of each $\phi \in \zeta(\mathcal{A})$ and the requirement $\phi(1) = 1$. The set $\overline{B_1} = \{\alpha \in \mathcal{A}' : \|\alpha\| \leq 1\}$ is weak-*-compact by Banach-Alaoglu [Rudin]. Also, $\zeta(\mathcal{A}) = \overline{B_1} \cap \{\alpha : \alpha(1) = 1\}$ is a weak-*-closed subset of $\overline{B_1}$ hence $\zeta(\mathcal{A}))$ is weak-*-compact.

□

2.1.3 Lemma. *If $\phi \in \mathcal{A}'$, $\mathcal{A}$ a C*-Algebra, with unit, then TFAE:*

(i) ϕ *is positive and* $\|\phi\| = 1$

(ii) $\phi \in \zeta(\mathcal{A})$

Proof. Assume (i), by positivity $1 = \|\phi\| = \phi(1)$, hence $\phi \in \zeta(\mathcal{A})$. Assume (ii), so $\phi \in \zeta(\mathcal{A})$. We need to show $\phi(aa^*) \geq 0$ for each $a \in \mathcal{A}$. Let $\alpha = \|aa^*\| = \|a\|^2$. By $aa^* \geq 0, \sigma(aa^*) \subset [0, \alpha]$, hence $\|\alpha 1 - \alpha\alpha^*\| = r(\alpha 1 - \alpha\alpha^*) \leq \alpha$. We observe $\alpha - \phi(aa^*) = \phi(\alpha 1 - aa^*) \leq \|\phi\|\|\alpha 1 - aa^*\| \leq \alpha$, as $\|\phi\| = 1$ and $\|\alpha 1 - aa^*\| \leq \alpha$ Thus, ϕ is positive.

□

2.2 Examples of States

2.2.4 Examples. (i) Take $\mathcal{A} = C(X)$, X compact, then $\zeta(\mathcal{A})$ is given by $\phi f \mapsto \int f dx$ for some Borel probability measure.

(ii) Take $\mathcal{A} = B(H)$, $v \in H$, $\|v\| = 1$. Then $\phi : A \mapsto \langle Av, v\rangle$ is a state, because $\phi(1) = 1$. We can also consider $(v_n)_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} \|v_n\|^2 = 1$. Then $\phi(1) = \sum_{n=1}^{\infty} \langle v, v\rangle = \sum_{n=1}^{\infty} \|v_n\|^2 = 1$, hence ϕ is a state.

References

[Rudin] Rudin, Walter (1991). Functional Analysis. International Series in Pure and Applied Mathematics. Vol. 8 (Second ed.) Theorem 3.15