Lecture Notes from March 30, 2023

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Last time

- Positivity
- States

Warm up: *Let $\mathcal{A} = M_n(\mathbb{C})$, $X \in \mathcal{A}$, $X = X^*$, then for $\lambda \in \sigma(x)$ there exists $\varphi \in \mathcal{S}(\mathcal{A})$ such that $\varphi(x) = \lambda$.*

We know that $\lambda \in \sigma(x)$ in $M_n(\mathbb{C})$ means that λ is an eigenvalue and there exists $v \in \mathbb{C}^n$ such that $\|v\| = 1$ and $Xv = \lambda v$. Now define $\varphi : \mathcal{A} \to \mathbb{C}; \quad A \mapsto \langle Av, v\rangle$. We have seen that $\varphi \in \mathcal{S}(\mathcal{A})$ since $\varphi(1) = 1$, $\|\varphi\| = 1$; and $\varphi(X) = \langle Xv, v\rangle = \lambda$.

1.47 Theorem. *Let $\mathcal{A}$ be a C^*-algebra with unit. Let $x \in \mathcal{A}, x = x^*$.*

- *If $\varphi \in \mathcal{S}(\mathcal{A})$, then $\varphi(x) \in \mathbb{R}$ and for each $\lambda \in \sigma(x)$, there exists $\varphi \in \mathcal{S}(\mathcal{A})$ such that $\varphi(x) = \lambda$.*
- *An element $x = x^*$ is positive if, and only if $\varphi(x) \geq 0$ for each $\varphi \in \mathcal{S}(\mathcal{A})$.*

Proof.
- Let $x = x_+ - x_-$ where $x_\pm \geq 0$ and $x_+x_- = x_-x_+ = 0$. By the lemma characteriing states, we have $\varphi(x_\pm \geq 0)$, so $\varphi(x) = \varphi(x_+) - \varphi(x_-) \implies \varphi(x) \in \mathbb{R}$. Next, consider $\lambda \in \sigma(x)$. Let $\mathcal{A}_x$ be the abelian C^*-algebra generated by $\{1, x\}$, then $\sigma_{\mathcal{A}_x}(x) = \sigma_{\mathcal{A}}(x)$. Using Gelfand for the $\mathcal{A}_x \cong C(\sigma(x))$, we have an isometric $*$-isomorphism $\mathcal{G} : \mathcal{A}_x \to C(\sigma(x))$. For $\lambda \in \sigma(x)$, we have $\delta_\lambda : C(\sigma(x)) \to \mathbb{C}$, $f \mapsto f(\lambda)$ is a state on $C(\sigma(x))$, since $\delta_\lambda(1) = 1 = \|\delta_\lambda\|$ (1 is the constant function 1), and by isomorphism we get $\nu_\lambda = \delta_\lambda \circ \mathcal{G}$. $\nu_\lambda(1) = \delta_\lambda(1) = 1$ and by isometry $\|\nu_\lambda\| = \|\delta_\lambda\| = 1$ hence ν_λ is a state on $\mathcal{A}_x$. ν_λ can be extended to a state ν on $\mathcal{A}$ by *Hahn-Banach*[1] such that the extension satisfies $\|\nu\| = \nu(1) = 1$ since $1 \in \mathcal{A}_x$. Thus,$\nu(x) = \nu_\lambda(x) = \delta_\lambda(\mathrm{id}) = \lambda$ giving us the result.
- If $x = x^*$, then w already know that for $\varphi \in \mathcal{S}(\mathcal{A})$, $\varphi(x) \geq 0$. Conversely, if $x - x^* \in \mathcal{A}$ such that for any $\varphi \in \mathcal{S}(\mathcal{A})$, $\varphi(x) \geq 0$, then if $\lambda \in \sigma(x)$, $\lambda \geq 0$ there exists φ by the first part such that $\varphi(x) = \lambda \geq 0$ and so $\sigma(x) \subset [0, \infty)$, hence $x \geq 0$.

□

1.48 Definition. We give a few definitions relating cones and positive elements in a real vector space.

1. (Convex cone) A subset C of a real vector space V is called a convex cone if it is convex and $\mathbb{R}^+C \subset C$.

2. (Dual cone) If V is a topological vector space (such that vector space operations are continuous) and V' is the dual of V, then for a cone C in V, define the dual cone to C as

$$C' := \{\alpha \in V' : \alpha(C) \subset \mathbb{R}^+\}.$$

3. (Predual cone) If $W \subset V'$ is a subset, then

$$(W)^{‘} := \{v \in V : (\forall w \in W)\ w(v) \geq 0\}$$

   is called the predual cone of W.

1.49 Remark. If V is a real topological space, $W \subset V'$, then $W^{‘}$ is closed since $W^{‘} = \bigcap_{w \in W} w^{-1}[0, \infty)$ is the intersection of closed sets, where each $w^{-1}[0, \infty)$ is closed since it is a preimage of a closed set under a continuous map.

1.50 Theorem. *If V is a real topological vector space and $C \subset V$ a closed convex cone, then*

$$C = (C')^{‘} \text{ and } (C')^{\perp} = C \cap (-C).$$

Proof. By definition of C', $C \subset (C')^{‘}$. To show the reverse inclusion, let $x \notin C$. We need to show that $x \notin (C')^{‘}$. Using *separation properties and geometric Hahn-Banach*[2], there exists a linear functional $\alpha \in V'$ with $\inf\limits_{c \in C} \alpha(c) > \alpha(x)$. From scaling properties of C and linearity of α, $\inf\limits_{c \in C} \alpha(c) = 0$, $\alpha(x) < 0$, so $x \notin (C')^{‘}$. We then also get

$$\begin{aligned} C \cap (-C) &= (C')^{‘} \cap -(C')^{‘} \\ &= \{\alpha \in C : \alpha(a) = 0 \forall a \in C'\} \\ &= (C')^{\perp}. \end{aligned}$$

□

1.51 Lemma. *Let $\mathcal{A}$ be a C^*-algebra with unit, then $\mathcal{A}^+$ forms a closed convex cone in $\mathcal{A}$.*

Proof. Recall $\mathcal{A}^+ = \{a \in \mathcal{A} : (\forall \varphi \in \mathcal{S}(\mathcal{A})\ \varphi(a) \geq 0\}$, so $\mathcal{A}^+ = \mathcal{S}(\mathcal{A})^{‘}$. □

1.52 Remark. Recall the two theorems used in the proofs above.

1. Hahn-Banach theorem: (Corollary 6.5 from John B. Conway - A Course in Functional Analysis) If X is a normed space over $\mathbb{C}$, $\mathcal{M}$ is a linear manifold in X, and $f : \mathcal{M} \to \mathbb{C}$ is a bounded linear functional, then there is an $F \in X'$ such that $F|_{\mathcal{M}} = f$ and $\|F\| = \|f\|$.

   We apply this theorem with $\mathcal{M} = \mathcal{A}_x$ a subspace of $\mathcal{A}$.

2. (Theorem 2.4.7. from Gert K. Pedersen - Analysis Now) Separation properties and geometric Hahn-Banach: Let A and B be disjoint, nonempty, convex subsets of a topological vector space X. If A is open, there is a $\alpha \in X'$ and a $t \in \mathbb{R}$ such that $\mathrm{Re}\alpha(x) < t < \mathrm{Re}\alpha(y)$, for every $x \in A$ and $y \in B$.

   We apply theorem with $C^c = A$ which is open since C is closed and $B = \{x\}$.

2 The Gelfand–Naimark–Segal (GNS) construction

We start with an observation. Let $\mathcal{A}$ be a C^*-algebra with unit, $(\pi, \mathcal{H}$ a representation of $\mathcal{A}$, then we recall π is a contraction. For each $v \in \mathcal{H}$, $\|v\| = 1$, we get $\varphi_v(a) = \langle \pi(a)v, v\rangle$ a state, because $\varphi_v(1) = \langle \pi(1)v, v\rangle = \|v\|^2 = 1$, and $\varphi_v(a^*a) = \|\pi(a)v\|^2 \geq 0$. The next goal is to find, for any $\mathcal{A}$ and φ, a representation $(\pi, \mathcal{K})$ such that there exists $v \in \mathcal{H}$ and $\varphi(a) = \langle \pi(a)v, v\rangle$.