MATH 7321 Lecture Notes

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Last Time:

- States and convex cones.
- The GNS construction.

Warm up: Examples of the GNS construction.
We know that given a state ϕ on $\mathcal{A}$, which is a positive linear functional on $\mathcal{A}$ with $\|\phi\| = 1$, the GNS construction associates to ϕ a Hilbert space representation π_ϕ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\phi$ together with a cyclic vector v in $\mathcal{H}_\phi$ such that $\phi(a) = \langle \pi_\phi(a)v, v\rangle$ for all $a \in \mathcal{A}$. We have following examples for GNS construction.
(a) Take $\mathcal{A} = M_n(\mathbb{C})$, $\phi(a) = \frac{1}{n}tr[a]$, then $\mathcal{H}_\phi = M_n(\mathbb{C})$, $\langle a, b\rangle = tr[ab^*]$ and $\pi_\phi(s)a = sa$. The cyclic vector is $\frac{1}{\sqrt{n}}I$, because

$$\frac{1}{n}\langle \pi(s)1, 1\rangle = \frac{1}{n}\langle s, 1\rangle = \frac{1}{n}tr[s] = \phi(s) \ .$$

(b) If $\mathcal{H}$ is a complex Hilbert space, $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then for $v \in \mathcal{H}$, with $\|v\| = 1$ and $\phi(a) = \langle av, v\rangle$ for all $a \in \mathcal{A}$.
It is easy to verify that ϕ is a positive linear functional on $\mathcal{A}$ with $\|\phi\| = 1$. So we can apply the GNS construction to obtain a Hilbert space representation π_ϕ of $\mathcal{A}$ on Hilbert space $\mathcal{H}_\phi$.

Then, the GNS construction gives $\pi_\phi(s) = s$. This follows from the fact that $\phi(a) = \langle av, v\rangle$ for all $a \in \mathcal{A}$, which implies that $\phi(a) = \langle \pi_\phi(a)v, v\rangle = \langle av, v\rangle$. Since v is a unit vector in $\mathcal{H}$, this implies that $\pi_\phi(a)v = av$ for all $a \in \mathcal{A}$. Thus, $\pi_\phi(a) = a$ on $\mathcal{H}$.

Hence, in this case, the GNS construction does not give rise to a new Hilbert space or a new representation of $\mathcal{A}$, but simply recovers the original Hilbert space $\mathcal{H}$ and the identity representation of $\mathcal{A}$ on $\mathcal{H}$.

Next, We examine consequences of the GNS construction.

Cyclic Representations vs GNS Construction

Theorem 1. *A representation $(\pi, \mathcal{H})$ of a C^*-algebra $\mathcal{A}$ is cyclic if and only if it is equivalent to $(\pi_\phi, \mathcal{H}_\phi)$, with ϕ a state on $\mathcal{A}$.*

Proof. Let ϕ be a state and $(\pi_\phi, \mathcal{H}_\phi)$ the associated GNS representation. We need to show ϕ is cyclic.

Indeed $\phi = K_1$, $\pi_\phi(s)K_1 = K_{s^*}$. So, $\pi_\phi(\mathcal{A})\phi = \mathcal{H}^0$, which is dense in $\mathcal{H}$.(This is already given in the proof of previous Theorem in the notes from April 4, 2023)

Conversely, let $(\rho, \mathcal{H})$ be cyclic. So, there is $v \in \mathcal{H}$, $\|v\| = 1$, which gives

$$\phi(s) = \langle \rho(s)v, v\rangle, \quad \phi \in \rho(\mathcal{A}) \ .$$

We want to show $(\rho, \mathcal{H})$ and $(\pi_\phi, \mathcal{H}_\phi)$ are equivalent.

Consider, the map $\Phi : \mathcal{H} \longrightarrow \mathcal{A}^{'}$, defined by

$$\Phi(w)(s) = \langle \rho(s)w, v\rangle,$$

then Φ is injective.

If $\phi(w) = 0$, then $\langle \rho(s)w, v\rangle = \langle w, \rho(s^*)v\rangle = 0$, for each $s \in \mathcal{A}$.

Hence, $w \in (\rho(\mathcal{A})v)^\perp = \{0\}$.

We also note that $\Phi \circ \rho(s) = \pi(s) \circ \Phi$, because for $w \in \mathcal{H}$and $x \in \mathcal{A}$, we have

$$
\begin{aligned}
(\pi(s)\Phi(w))(x) &= \Phi(w)(xs) \\
&= \langle \rho(xs)w, v\rangle \\
&= \langle \rho(x)\rho(s)w, v\rangle \\
&= \Phi(\rho(s)w)(x) \ .
\end{aligned}
$$

Moreover, Φ is unitary. To see this we observe that $\Phi(v) = \phi$ and by the intertwining relationship

$$
\Phi(\rho(\mathcal{A})v) = \pi_\phi(\mathcal{A})\phi = \mathcal{H}^0 \ .
$$

For $s, t \in \mathcal{A}$, we have

$$
\begin{aligned}
\langle \Phi(\rho(s)v), \Phi(\rho(t)v)\rangle &= \langle \pi_\phi(S)\phi, \pi_\phi(t)\phi\rangle \\
&= \langle \pi_\phi(t^*)\pi_\phi(s)\phi, \phi\rangle \\
&= \langle \pi_\phi(t^*s)\phi, \phi\rangle \\
&= \phi(t^*s) \\
&= \langle \rho(t^*s)v, v\rangle \\
&= \langle \rho(s)v, \rho(t^*v) \ .
\end{aligned}
$$

Thus, $\Phi|_{\rho((A))v} : \rho(\mathcal{A})v \longrightarrow \mathcal{H}^0$ is an isometry, which extends to $\tilde{\phi} : \mathcal{H} \longrightarrow \mathcal{H}_\phi$. Since $range(\Phi)$ is contained/dense in $\mathcal{H}$.
By continuity, $\tilde{\phi}$ is an isometry and its range is $\mathcal{H}$. Since, an isometry between two Hilbert spaces is also a unitary operator, so $\tilde{\phi}$ is a unitary operator between $\mathcal{H}$ and $\mathcal{H}_\phi$.
Also, we know that, in C^*-algebras two representations are equivalent if there exists a unitary operator between the Hilbert spaces on which the representations act, which intertwines the representations. Thus, representations $(\rho, \mathcal{H})$ and $(\pi_\pi, \mathcal{H}_\phi)$ are unitarily equivalent. □