# Lecture Notes from April 11, 2023

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## 1.1 Last week

- "Highlights" from last week:
  - More on consequences of GNS representation
  - Cyclic representations vs GNS construction

## 1.2 Warm-up

Skip the warm-up and come back later...

## 1.3 This week

**1.6 Theorem.** *Each* $C^*$*- algebra* $\mathcal{A}$ *is isomorphic to a closed *-subalgebra of* $\mathcal{B}(\mathcal{H})$ *for some* $\mathcal{H}$*.*

*Proof.* We can assume $\mathcal{A}$ has a unit, otherwise we embed $\mathcal{A}$ in $\mathcal{A} \bigoplus 1$. For each $\varphi \in \Phi(\mathcal{A})$, we associate $(\pi_\varphi, \mathcal{H}_\varphi)$. Next, we consider

$$\pi = \bigoplus_{\varphi \in \Phi(\mathcal{A})} \pi_\varphi,$$

with our summability convention. For each $s$,

$$||\pi(s)|| = \sup\{||\pi_\varphi(s)|| : \varphi \in \Phi(\mathcal{A})\} \leq ||s||.$$

So $\pi$ is a representation. We want to prove $\pi$ is an isometry. Consider $s \in \mathcal{A}$, then $s^*s$ is Hermitian. Thus,

$$||s^*s|| = r(s^*s) = \max\{|\lambda| : \lambda \in \sigma(s^*s)\}.$$

We know there is $\varphi \in \Phi(\mathcal{A})$ such that $\varphi(s^*s) = ||s^*s||$. Thus,

$$||\pi_\varphi(s)||^2 \geq ||\pi_\varphi(s)\varphi||^2 =< \pi_\varphi(s)\varphi, \pi_\varphi(s)\varphi >= \varphi(s^*s),$$

which gives

$$||\pi(s)||^2 \geq ||\pi_\varphi(s)||^2 \geq \varphi(s^*s)$$

and we can choose $\varphi$ such that

$$||\pi(s)||^2 \geq ||s^*s|| = ||s||^2$$

We conclude that $\pi$ is an isometry, thus giving us an isomorphism of $C^*-$ algebras. □

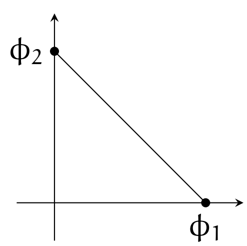
We study a relationship between representations and states.

**1.7 Definition.** A state $\varphi \in \Phi(\mathcal{A})$ on a $C^*-$algebra is called a *pure state* if it is an extreme point of $\Phi(\mathcal{A})$, so it cannot be obtained as a non-trivial convex combinations of distinct states.

*1.8 Example.* $\mathcal{A} = l^\infty(\{1,2\}), \Phi(\mathcal{A})$. By $\Phi(\mathcal{A}) \subset \mathcal{A}'$, we see any $\varphi \in \Phi(\mathcal{A})$ is given by

$$\varphi(a) = \varphi_1 a_1 + \varphi_2 a_2$$

and $\varphi_1\varphi_2 \geq 0,\ \varphi_1 + \varphi_2 = 1$. We note $\Phi(\mathcal{A})$ forms a simplex, figure below:




This allows us to identify the pure states as $\varphi_1 = (1,0)$ or $\varphi_2 = (0,1)$.

**1.9 Theorem.** *Let $\mathcal{A}$ be a $C^*-$ algebra with unit , $\varphi \in \Phi(\mathcal{A})$, $(\pi_\varphi, \mathcal{H}_\varphi)$ the GNS representation then $(\pi_\varphi, \mathcal{H}_\varphi)$ is irreducible if and only if $\varphi$ is a pure state.*

*Proof.* Let $(\pi_\varphi, \mathcal{H}_\varphi)$, then there is $\mathcal{H}_1 = \overline{\mathcal{H}_1} \subset \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{H}_1^\perp \neq \{0\}$ that are invariant under $\mathcal{A}$, giving us $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$.
Assume $\varphi$ is cyclic, then $\varphi \notin \mathcal{H}_1$ and $\varphi \notin \mathcal{H}_2$, so $\varphi = (\varphi_1, \varphi_2)$ with $\varphi_i \in \mathcal{H}_i \backslash \{0\}$ for $i \in \{1,2\}$.
For $s \in \mathcal{A}$, we get

$$\varphi_1(s) =< \pi_\varphi(s)(\varphi_1, 0), (\varphi_1, 0) >=< \pi_\varphi(s)\varphi_1, \varphi_1 > .$$

Let us define

$$\tilde{\varphi_1} = \frac{1}{||\varphi_1||^2}\varphi_1$$

then this is a state, and so is

$$\tilde{\varphi_2} = \frac{1}{||\varphi_2||^2}\varphi_2.$$

Thus, we obtain

$$\varphi = ||\varphi_1||^2\tilde{\varphi_1} + ||\varphi_2||^2\tilde{\varphi_2}$$

with

$$1 = ||\varphi||^2 = ||\varphi_1||^2 + ||\varphi_2||^2.$$

This shows $\varphi$ is not an extreme point.
Conversely, let $(\pi_\varphi, \mathcal{H}_\varphi)$ be an irreducible representation and $\lambda \in (0,1),\ \varphi_1, \varphi_2 \in \Phi(\mathcal{A})$ with $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$. For the kernels (and spaces) associated with $\varphi_1$ and $\varphi_2$, say $\mathcal{K}_1^{(1)}, \mathcal{K}_1^{(2)}$ we get

$$\mathcal{K}_1 = \lambda\mathcal{K}_1^{(1)} + (1-\lambda)\mathcal{K}_1^{(2)}.$$

By our lemma on reproducing kernels,

$$\mathcal{H}_{\varphi_1} = \mathcal{H}_{\mathcal{K}^{(1)}} \subset \mathcal{H}_{\mathcal{K}} = \mathcal{H}_{\varphi}$$

and the map

$$i : \mathcal{H}_{\varphi_1} \to \mathcal{H}_{\varphi}$$

is continuous. Moreover,

$$i^*(\pi_\varphi(s^*)\varphi) = i^*(\mathcal{K}_s) = \mathcal{K}_s^{(1)} = \pi_{\varphi_1}(s^*)\varphi_1$$

and hence $i^*\varphi = \varphi_1$. By

$$\pi_\varphi(s)(i(f))(x) = i(f)(xs) = f(xs) = (\pi_{\varphi_1}(s)f)(x) = i(\pi_{\varphi_1}(s)f)(x)$$

We observe $i$ intertwining $\pi_{\varphi_1}$ and $\pi_\varphi$. A similar intertwining relationship holds for $i^* : \mathcal{H}_\varphi \to \mathcal{H}_{\varphi_1}$. □