Lecture Notes from April 13, 2023

taken by Cristian Meraz

Last time: From the GNS construction to an isometric isomorphism of C^*-algebras.

0 Warm-up

Let $\mathcal{A} = M_n(\mathbb{C})$ and $\varphi \in \mathcal{S}(\mathcal{A})$. Recall that states on a C^*-algebra $\mathcal{A}$ are positive, bounded, linear functionals $\varphi : \mathcal{A} \to \mathbb{C}$ with unit norm. (We proved this on 2023-03-28.)

0.1 Exercise. Part 1: Show that φ is given by

$$\varphi : \mathcal{A} \to \mathbb{C}, \qquad \varphi : \mathbf{A} \mapsto \varphi(\mathbf{A}) := \mathrm{tr}[\mathbf{AW}],$$

where $\mathbf{W} \in \mathcal{A}$ is such that $\mathbf{W} \geq 0$ and $\mathrm{tr}[\mathbf{W}] = 1$.

Part 2: Moreover, φ is pure if and only if $\mathbf{W}^2 = \mathbf{W}$.

Proof of Part 1. First note that since $\mathcal{A}$ is finite-dimensional, we can equip it with the Hilbert-Schmidt norm:

$$\|\mathbf{A}\|_{\mathsf{HS}} = (\mathrm{tr}[\mathbf{A}^*\mathbf{A}])^{\frac{1}{2}},$$

as all norms $\| \ \|$ we consider are equivalent. Then, as φ is a bounded linear functional on a Hilbert space, we have:

$$\varphi(\mathbf{A}) = \mathrm{tr}[\mathbf{AW}^*] \qquad \text{with some } \mathbf{W} \in \mathcal{A},$$

by the Riesz representation theorem for Hilbert spaces. If $\mathbf{I}$ denotes the $n \times n$ identity matrix then

$$\begin{aligned}\varphi(\mathbf{I}) &= 1 \\ &= \mathrm{tr}[\mathbf{IW}^*] = \mathrm{tr}[\mathbf{W}^*],\end{aligned}$$

since φ is a state. Next, we will show that $\mathbf{W} \geq 0$. This will imply that all eigenvalues are non-negative real numbers, and $\mathrm{tr}[\mathbf{W}] = 1$ follows.

To this end, note that if $\mathbf{X} = \mathbf{xx}^*$ is a rank-one matrix for some for some $\mathbf{x} \in \mathbb{C}^n$, then

$$\begin{aligned}\varphi(\mathbf{X}) &= \mathrm{tr}[\mathbf{XW}^*] \\ &= \mathrm{tr}[\mathbf{W}^*\mathbf{X}] = \langle \mathbf{W}^*\mathbf{x}, \mathbf{x} \rangle.\end{aligned}$$

I.e. this is the quadratic form of $\mathbf{W}^*$. Hence, if $\lambda \in \sigma(\mathbf{W}) \setminus \{0\}$ and $\mathbf{v} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ are an eigenvalue-eigenvector pair for $\mathbf{W}$, i.e. $\mathbf{W}\mathbf{v} = \lambda\mathbf{v}$, then

$$\varphi(\mathbf{v}\mathbf{v}^*) = \overline{\lambda}\|\mathbf{v}\|^2 \geq 0 \qquad \text{by positivity of } \varphi.$$

Thus $\overline{\lambda} \in \mathbb{R}^+$, and therefore $\mathbf{W} \geq 0$ and $\mathrm{tr}[\mathbf{W}] = 1$.

□

Proof of Part 2. Assuming Part 1, and that $\mathbf{W}$ also satisfies $\mathbf{W}^2 = \mathbf{W}$, we can conclude that $\mathbf{W}$ is an orthogonal rank-one projection. This means that there exists a vector $\mathbf{v} \in \mathbb{C}^n$ with $\|\mathbf{v}\| = 1$ such that $\mathbf{W} = \mathbf{v}\mathbf{v}^*$.

Moreover, this is equivalent to the conditions that $\mathrm{tr}[\mathbf{W}] = 1$, $\mathbf{W} \geq 0$, and $\mathbf{W}^2 = \mathbf{W}$. To see this, write

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{W} := \mathbf{v}\mathbf{v}^* = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} \overline{v}_1 & \cdots & \overline{v}_n \end{bmatrix}$$

so that

$$\mathrm{tr}[\mathbf{W}] = \sum_{i=1}^{n} |v_i|^2 = \mathbf{v}^*\mathbf{v} = \|\mathbf{v}\|^2 = 1, \qquad \mathbf{W}^2 = (\mathbf{v}\mathbf{v}^*)\mathbf{v}\mathbf{v}^* = (\mathbf{v}^*\mathbf{v})\mathbf{v}\mathbf{v}^* = \mathbf{W},$$

and

$$\sigma(\mathbf{W}) \subset [0,1) \qquad \Longrightarrow \qquad \mathbf{W} \geq 0.$$

Proof strategy: We show that if $\mathbf{W}$ cannot be written in this form for any $\mathbf{v}$, then this is equivalent to φ not being a pure state.

So suppose $\mathbf{W}$ is not of this form. Then

$$\mathbf{W} = \sum_{j=1}^{n} \lambda_j \mathbf{P}_j \qquad \text{by Spectral Theorem,}$$

and $\lambda_j \in [0,1)$. By $\mathrm{tr}[\mathbf{W}] = \sum_{j=1}^{n} \lambda_j = 1$, we know there exist λ_k, $\lambda_\ell \in (0,1)$ such that we can choose $\varepsilon > 0$ and $\lambda_k \pm \varepsilon$, $\lambda_\ell \mp \varepsilon \in (0,1)$. But then we can write

$$\begin{cases} \mathbf{W}' = \sum_{\substack{j=1 \\ j\neq k \\ j \neq \ell}} \lambda_j \mathbf{P}_j + (\lambda_j + \varepsilon)\mathbf{P}_j + (\lambda_j - \varepsilon)\mathbf{P}_j \\ \mathbf{W}'' = \sum_{\substack{j=1 \\ j\neq k \\ j \neq \ell}} \lambda_j \mathbf{P}_j + (\lambda_j - \varepsilon)\mathbf{P}_j + (\lambda_j + \varepsilon)\mathbf{P}_j \end{cases}$$

Thus $\mathbf{W}$ does not represent a pure state.

Conversely, if $\mathbf{W}$ does not represent a pure state, we can use the Spectral Theorem to obtain

$$\mathbf{W} = \sum_{j=1}^{n} \lambda_j \mathbf{P}_j$$

with each $\mathbf{P}_j$ an orthogonal rank-one projection, and $\mathbf{P}_j\mathbf{P}_k = \mathbf{0} = \mathbf{P}_k\mathbf{P}_j$ if $j \neq k$.

We can then note that

$$\mathbf{W}^2 = \sum_{j=1}^{n} \lambda_j^2 \mathbf{P}_j$$

and since

$$\begin{cases} \lambda_k^2 < \lambda_k & \text{if } k \in \{1, \dots, n\} \text{ with } \lambda_k < 1 \\ \lambda_j^2 \le \lambda_j & \text{for all } j, \end{cases}$$

we get

$$\begin{aligned} \operatorname{tr}[\mathbf{W}^2] = \sum_{j=1}^{n} \lambda_j^2 &< \sum_{j=1}^{n} \lambda_j = \operatorname{tr}[\mathbf{W}] \\ \implies \quad & \mathbf{W}^2 \neq \mathbf{W}. \end{aligned}$$

Therefore, $\mathbf{W}$ is not an orthogonal rank-one projection.

□

1 Representations of pure states are irreducible

We will now continue proving the theorem from the previous class.

1.1 Theorem. *Let $\mathcal{A}$ be a C^*-algebra with unit, and $\varphi \in \mathcal{S}(\mathcal{A})$. Then $(\pi_\varphi, \mathcal{H}_\varphi)$ (given by the GNS construction) is irreducible if and only if φ is a pure state.*

Proof. We had shown if φ is not irreducible, then φ is not pure.

Next, we assume $(\pi_\varphi, \mathcal{H}_\varphi)$ is irreducible and let

$$\varphi = \lambda \varphi_1 + (1 - \lambda)\varphi_2, \qquad \varphi_1 \neq \varphi_2.$$

By the GNS construction and relationship between

$$K_1^{(1)} := \varphi_1, \qquad K_1^{(2)} := \varphi_2,$$

we have that

$$\mathcal{H}_{\varphi_1} \subset \mathcal{H}_\varphi,$$

and $i : \mathcal{H}_{\varphi_1} \to \mathcal{H}_\varphi$ intertwines actions of $\mathcal{A}$ on $\mathcal{H}_\varphi$ and $\mathcal{H}_{\varphi_1}$.

We observed $ii^*\pi_\varphi(s) = \pi_\varphi(s)ii^*$. By irreducibility and *Schur's lemma* (proved on 2023-02-09),

$$ii^* \in (\pi_\varphi(\mathcal{A}))' = \mathbb{C}\mathbb{1}.$$

From this, we know that $ii^* = \lambda \mathbb{1}$, $\lambda \in \mathbb{C}$. Then,

$$\begin{aligned} ii^*\varphi &= i\varphi_1 = \varphi_1 \\ &= \lambda \varphi, \end{aligned}$$

and together with $1 = \varphi_1(1) = \lambda\varphi(1) = \lambda$, this shows that $\lambda = 1$, and hence $\varphi_1 = \varphi$. Similarly, one can show that $\varphi_2 = \varphi$. Thus, φ is an extreme point in $\mathcal{S}(\mathcal{A})$, and denotes a pure state. □

2 Preview for next class

We will prove in the next class that if $\mathcal{A}$ is an *abelian* C^*-algebra, then the states on $\mathcal{A}$ are precisely the *characters* on $\mathcal{A}$, i.e., the bounded linear homomorphisms on $\mathcal{A}$.