Class Notes
Manpreet Singh
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Last Time

- Pure States on $M_n(\mathbb{C})$.
- Pure States and irreducibility for the GNS construction.

Warm up

- For $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that for $A \in M_2(\mathbb{C})$, $A = A^* \ \exists\ c_i \in \mathbb{R}$ such that $A = c_i I + c_2 X + c_3 Y + c_4 Z$.

  Solution: We note that $tr[X] = tr[Y] = tr[Z] = 0$. Also $XY = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = iZ$, $YZ = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = iX$, $ZX = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = iY$ and thus $tr[XY] = tr[YZ] = tr[ZX] = 0$. Therefore, these are Hilbert-Schmidt orthogonal. Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A = A^*$, then

$$a_{11} = \bar{a}_{11} = \alpha_{11} \in \mathbb{R}$$

$$a_{12} = \bar{a}_{21} = \alpha_{12} + i\beta_{12}$$

$$a_{22} = \bar{a}_{22} = \alpha_{22} \in \mathbb{R}$$

  We see $dim_{\mathbb{R}}\{A \in M_n(\mathbb{C}) : A = A^*\} = 4$. So the linearly independent set $\{I, X, Y, Z\}$ forms a basis.

- Let $W \in M_2(\mathbb{C})$ represent a pure state ϕ on $M_2(\mathbb{C})$. Characterize it in terms of

$$W = c_0 I + c_1 X + c_2 Y + c_3 Z$$

  Solution: From $tr[W] = \phi(I) = 1$, we get $2c_0 = 1$ i.e. $c_0 = \frac{1}{2}$.
  For example $W = \frac{1}{2} I + c_3 Z = \begin{bmatrix} \frac{1}{2} + c_3 & 0 \\ 0 & \frac{1}{2} - c_3 \end{bmatrix}$. In this case, we get from the fact that W is an orthogonal projection, $c_3 = \frac{1}{2}$ or $\frac{-1}{2}$.

We recall a necessary and sufficient condition for $W = W^*$, $tr[W] = 1$ is $W^2 = W$. We compute

$$W^2 = (\frac{1}{2}I + c_1X + c_2Y + c_3Z)^2$$

$$= \frac{1}{2}W + c_1XW + c_2YW + c_3ZW$$

$$= \frac{1}{2}W + c_1(\frac{1}{2}X + c_1I + ic_2Z - ic_3Y) + c_2(\frac{1}{2}Y - ic_1Z + c_2I + ic_3X) + c_3(\frac{1}{2}Z + ic_1Y - ic_2X + c_3I)$$

$$= \frac{1}{2}W + (c_1^2 + c_2^2 + c_3^2)I + \frac{1}{2}c_1X + \frac{1}{2}c_2Y + \frac{1}{2}c_3Z$$

$$= \frac{1}{2}W + (c_1^2 + c_2^2 + c_3^2 - \frac{1}{2}) + \frac{1}{2}W$$

$$= W + (c_1^2 + c_2^2 + c_3^2 - \frac{1}{2})I$$

For W to be a pure state we need $\|c\|^2 = \frac{1}{2}$. With a view to the linearity of the representation of the state ϕ, the set of all states is obtained from the convex hull, $\|c\|^2 \leq 1/2$, and the sphere parametrizes the pure states as extreme points.

We return to Spectral theory.

0.1 Corollary. *Let $\mathcal{A}$ be commutative C^*- algebra with unit. The set of pure states is identical to the set of characters on $\mathcal{A}$.*

Proof. By the preceding theorem in the last lecture, a state ϕ is extremal if and only if $(\pi_\phi, \mathcal{H}_\phi)$ is irreducible. In that case, $dimH_\phi = 1$, since $\mathcal{A}$ is abelian. Thus, ϕ is extremal if and only if $H_\phi = \mathbb{C}\phi$. This is equivalent to $\pi_\phi(S)\phi = \chi(S)\phi$ for each $S \in \mathcal{A}$ or by construction $\phi(xS) = \chi(S)\phi(x)$. By fixing x and varying S, this shows $\chi \in H_\phi$. So $\chi = \alpha\phi$ for some $\alpha \in \mathbb{C}$ and by $\chi(1) = 1 = \phi(1)$, we get $\alpha = 1$, that is $\chi = \phi$. □

Next we return to the question of how coarse the topology introduced by characters on $\mathcal{A}$ is in comparison with the weak topology induced by all of $\mathcal{A}'$

0.2 Corollary. *Let $\mathcal{A}$ be a C^*-algebra and $0 \neq S \in \mathcal{A}$, then there is an irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ with $\pi(S) \neq 0$.*

Proof. **We denote the set of states by $\rho(\mathcal{A})$.** We recall that a compact convex set is the closed convex hull of its extreme points COMPACTNESS IN WHICH SENSE? ALSO, GIVE A **[REFERENCE]**. Applying this to $\rho(\mathcal{A})$, we see that it is the closed convex hull of pure states. Let $S \in \mathcal{A} \setminus \{0\}$ then $S^*S \neq 0$ and there is $\phi \in \rho(\mathcal{A})$ such that $\phi(S^*S) > 0$. By $\rho(\mathcal{A})(S^*S) \in \mathbb{R}^+$ there exists a pure state such that $\phi(S^*S) > 0$ WHY?. The resulting irreducible GNS representation satisfies

$$\phi(S^*S) = \|\pi_\phi(S)\phi\|^2 > 0$$

So, $\pi_\phi(S) \neq 0$. □

0.3 Corollary. *Let $\mathcal{A}$ be a commutative C^*-algebra, then $\hat{\mathcal{A}} \subset \mathcal{A}'$ separates points in $\mathcal{A}$.*

Proof. From the preceding corollary, we know irreducible representation separate points because for $x, y \in \mathcal{A}$, $S = x - y$, there is a pure state ϕ such that

$$\phi(S^*S) = \|\pi_\phi(S)\phi\|^2 > 0$$

For the abelian case, we get $\chi \in \hat{\mathcal{A}}$ such that

$$\|\chi(S)\phi\| = |\chi(S)|\|\chi\| > 0$$

So $\chi(S) \neq 0$. □