SELECTED KEY PUBLICATIONS BY BERNHARD G. BODMANN

The main technique in Bernhard Bodmann's research is the use of non-orthogonal stable expansions with frames. In contrast to the more rigid structure of orthonormal bases, the flexibility in the design of such frames permits including many types of optimality principles in their design.

1. Frames and phase retrieval

Although many Nobel prizes have been awarded for work on X-ray crystallography[1], our mathematical understanding of what makes this technology work for uncovering molecular structures still needs to grow. The following papers explore fundamental challenges and achievements in an idealized setting. The key to signal recovery is in redundant information encoded in a sequence of intensity measurements.

R. Balan, B. G. Bodmann, P. G. Casazza, D. Edidin (2009): Painless reconstruction from magnitudes of frame coefficients, J. Fourier Anal. Appl. 15, 488-501. This is arguably the first paper in which the equivalence of signal reconstruction from magnitude measurements and matrix completion is exploited. The problem of phase retrieval is important to several areas of research in signal and image processing, especially X-ray crystallography, speech recognition technology, as well as state tomography in quantum theory. The linear reconstruction algorithms for tight frames associated with projective 2-designs in finite-dimensional real or complex Hilbert spaces presented here have been combined with more sophisticated linear[2] and non-linear reconstruction techniques in subsequent works[3].

B. G. Bodmann and N. Hammen (2017): Algorithms and error bounds for noisy phase retrieval with low-redundancy frames, Appl. Comput. Harmon. Anal. 43, 482-503. After establishing injectivity for phase retrieval in a complex d-dimensional Hilbert space based on a frame of $4d-4$ vectors[4], the authors studied how enlarging the redundancy may yield explicit reconstruction algorithms and error bounds for stable recovery. This was achieved with frames consisting of $6d-3$ vectors and with an algorithm that was polynomial time in the dimension d. If the noise is sufficiently small compared to the squared norm of the vector to be recovered, the error bound is inverse proportional to the signal-to-noise ratio.

D. Domel-White and B. G. Bodmann (2022): Phase retrieval by random binary questions: Which complementary subspace is closer? Constructive Approx. 56, 1-33. In this paper, phase retrieval with approximate recovery is performed when the acquired data only consists of qualitative information, binary measurements. The work with Domel-White improves on results by Vershynin and Plan[5]. The

[1] Simona Galli, X-ray Crystallography: One Century of Nobel Prizes, J. Chem. Educ. 91 2009-2012 (2014).

[2] B. Alexeev, A. S. Bandeira, M. Fickus, and D. Mixon, Phase Retrieval with Polarization, SIAM J. Imaging Sci., 7(1), 3566 (2014).

[3] E. Candès, T. Strohmer, and V. Voroninski, Exact and Stable Phase Retrieval via Semidefinite Programming, Communications on Pure and Applied Mathematics 66 (8), 1241-1274 (2012).

[4] B. G. Bodmann and N. Hammen, Stable phase retrieval with low-redundancy frames, Adv. Comput. Math. 41, 317–331 (2015).

[5] Y. Plan and R. Vershynin, Robust 1-bit compressed sensing and sparse logistic regression: a convex programming approach, IEEE Trans. Inform. Theory 59, 482-494 (2013).

improvement is based on the fact that measurements are chosen in a more structured manner, which provides explicit expressions that can be estimated more accurately.

2. Frames and quantum state tomography

The construction of maximal sets of equiangular complex lines is related to the existence of statistically optimal measurement strategies to determine quantum states from the outcomes of experiments[6]. Even without the maximality requirement, the design of equiangular frames is already challenging.

B. G. Bodmann and H. J. Elwood (2010): Complex equiangular Parseval frames and Seidel matrices containing pth roots of unity, Proc. Amer. Math. Soc. 12, 4387-4404. Another geometric goal in frame design is equiangularity. For real Hilbert spaces, the seminal work of Seidel and collaborators[7] remains the standard source of constructions, while the few known examples in the complex case leave fundamental, unanswered questions such as whether maximal families of complex equiangular tight frames exist in any dimension. In a paper with Paulsen and Tomforde[8], Bodmann investigated whether Seidel's combinatorial approach can be used to find necessary conditions for the existence of equiangular tight frames of which the vectors have inner products that are, up to a common factor, cube roots of unity. Bodmann and his student Elwood then generalized these results to pth roots of unity if p is prime. The key to unlocking the structure of the more general case was to transform the system of conditions into a way that made it manifestly invariant under a large group of symmetry operations. In addition, this paper contains the construction of a previously unknown family of Butson-type complex Hadamard matrices which deserves to be studied further.

B. G. Bodmann and E. J. King (2020): Optimal arrangements of classical and quantum states with limited purity, J. London Math. Soc. 101, 393-431. This paper explores the question of how the transmission of classical information should be implemented when it is based on quantum states that degrade by interaction with an environment. In a sense, this can be seen as the quantum analogue of results on sphere packings explored by Conway, Hardin and Sloane [9]. The optimal packings are called spectrahedron arrangements, and solve semidefinite programming problems. The construction of such packings lead to specific reducible representations of the Heisenberg-Weyl group that deserve to be studied more.

3. Biomedical imaging

Part of these results are related to CT scanning equipment, the other to the processing of images obtained with CT scanners.

[6]A. J. Scott, Tight informationally complete quantum measurements, J. Phys. A: Math. Gen. 39, 13507-13530 (2006).

[7]see J. H. van Lint and J. J. Seidel, Equilateral point sets in elliptic geometry, Indag. Math. 28, 335-348 (1966), or P. W. H. Lemmens and J. J. Seidel, Equiangular lines, J. Algebra 24, 494-512 (1973)

[8]B. G. Bodmann, V. I. Paulsen, and M. Tomforde, Equiangular tight frames from complex Seidel matrices containing cube roots of unity, Linear Algebra Appl. 430, 396-417 (2009)

[9]J. H. Conway, R. H. Hardin, and N. J. A. Sloane, Packing lines, planes, etc.: Packings in Grassmannian space, Experimental Math. 5, 139-159 (1996).

M. Papadakis, B. G. Bodmann, S. K. Alexander, D. Vela, S. Baid, A. A. Gittens, D. J. Kouri, S. D. Gertz, S. Jain, J. R. Romero, X. Li, P. Cherukuri, D. D. Cody, G. W. Gladish, I. Aboshady, J. L. Conyers, and S. W. Casscells (2009): Texture-based Tissue Characterization for High-resolution CT Scans of Coronary Arteries, Communications in Numerical Methods in Engineering 25, 597-613. This paper is an early foray into the use of machine learning techniques for image analysis. In order to discriminate different types of tissues in CT scanning data, filters are trained and textural characteristics are extracted. The findings of the image analysis method were compared with ground truth based on histopathology. The method for texture classification was combined with the design of redundant frame wavelets that behave well under image rotations[10].

B. Goossens, D. Labate, and B. G. Bodmann (2020), Robust and stable region-of-interest tomographic reconstruction using a robust width prior, Inverse Problems and Imaging 14, 291-316. The main objective pursued in this paper is that of targeted image reconstruction. To obtain an accurate image for part of a structure being imaged does not require measuring X-ray absorption characteristics in all positions and with all orientations. This paper is a continuation of an earlier work that used an alternating projection method for image reconstruction[11].

4. Frames and compressed sensing

The terminology of frame theory is natural for studying the phenomenon of compressed sensing, where sparsity of a signal permits to solve an otherwise underdetermined system for the recovery of an unknown vector.

B. G. Bodmann and C. L. Liner (2012): Spikes, nodes and aliasing: Signal recovery from the roots of the spectrogram, SIAM J. Appl. Math. 72, 1449-1473. A main focus in the compressed sensing literature is to provide recovery guarantees for sparse signals that permit reducing the number of measured quantities to a small fraction of the dimension in which the sparse signal resides. In this work, a related signal recovery method is developed that is based on nodes in the short-time Fourier transform, which act as a type of fingerprint for sparse signals. In more geometric terms, orthogonality relations and sparsity allow to recover signals, thus being less susceptible to the modulation of a signal in the course of it being transmitted.

B. G. Bodmann, J. Cahill and P. G. Casazza (2012): Fusion Frames and the Restricted Isometry Property, Numerical Functional Analysis and Optimization 33, 770-790. Applications in compressed sensing motivated a large amount of literature on the construction of matrices with the restricted isometry property (RIP). This paper leverages the results on these constructions in order to create fusion frames with near optimal properties. Fusion frames are closed subspaces of a Hilbert space whose associated orthogonal projection operators give rise to an approximate resolution of the identity. The optimality properties of fusion frames had been characterized in a geometric fashion, for example as fusion frames with equal weights and equi-dimensional,

[10] M. Do and M. Vetterli, Rotation invariant texture characterizaton and retrieval using steerable wavelet domain hidden Markov models, IEEE Trans. Multimedia 4(4): 517-527 (2002).

[11] R. Azencott, B. G. Bodmann, T. Chowdhury, D. Labate, A. Sen, and D. Vera, ROI reconstruction from truncated cone-beam projections, Inverse Problems and Imaging 12, 29-57 (2017).

equi-isoclinic subspaces[12]. The authors show that RIP matrices, viewed as tight frames satisfying the restricted isometry property, give rise to nearly tight fusion frames which are nearly equi-isoclinic. The authors also show how to replace parts of the RIP frame with orthonormal systems while maintaining the restricted isometry property. The paper thus allows methods for the construction of RIP matrices to be used for the construction of fusion frames.

B. G. Bodmann (2013): Random fusion frames are nearly equiangular and tight, Linear Algebra Appl. 439, 1401-1414. This paper continues the work on the connection between compressed sensing and near-optimal fusion frames. When the usual randomized contructions are used for the construction of RIP matrices, then the fusion frames resulting from the orthonormalization procedure outlined in the preceding paper are equidistant and tight. The main challenge in this paper was to establish a "central-limit theorem" for the cosines of the principal angles between uniformly randomly selected subspaces. This was achieved by a decorrelation strategy where first random, near-orthonormal bases were chosen for the subspaces which were then orthonormalized to obtain the orthogonal projection operators corresponding to the subspaces. The desired properties of the principal angles were then derived from a perturbation argument. These properties show that such fusion frames are near-optimal for packet-based coding[13].

5. Frames as codes

Frames, incorporating linear dependencies between the vectors, are a natural structure that permits controlling the effects of data loss, for example when packets are dropped in transmissions across the internet.

B. G. Bodmann and V. I. Paulsen (2005): Frames, graphs and erasures, Linear Algebra Appl. 404, 118-146. In this work, Bodmann and Paulsen investigated optimal linear codes for real or complex Hilbert spaces for the purpose of correcting or suppressing the effect of erasures, that is partial data loss. While linear binary codes have a long history in information theory, codes over the real or complex numbers have only been examined since the 80s[14]. Such codes have a natural formulation in terms of frame theory, spanning sets of vectors which are used to expand the signal in terms of its frame coefficients. In the encoding process, a vector in a finite dimensional real or complex Hilbert space is mapped to the sequence of its inner products with the frame vectors. An erasure occurs when part of these frame coefficients is no longer accessible after the transmission. This paper focuses on the worst-case scenario. Optimality for one or two erasures then leads to geometric conditions such as the classes of equal-norm and equiangular tight frames[15]. Bodmann and Paulsen investigated optimality for a slightly larger number of erasures. In this case, there is no simple geometric characterization of the best frames, but it can be phrased in graph-theoretic terms. The analytic

[12] B. G. Bodmann, "Optimal linear transmission by loss-insensitive packet encoding," Appl. Comput. Harmon. Anal. **22**, 274-285 (2007)

[13] B. G. Bodmann and P. K. Singh. Random fusion frames for loss-insensitive packet encoding, in: D. V. D. Ville, V. K. Goyal, and M. Papadakis, editors, Proceedings of the SPIE: Wavelets and Sparsity XV, volume 8858, (2013)

[14] T. G. Marshall Jr., Coding of Real-Number Sequences for Error Correction: A digital signal processing problem, IEEE J. on Selected Areas in Communications SAC-2, 381-392 (1984)

[15] R. Holmes and V. I. Paulsen, Optimal frames for erasures, Linear Algebra Appl. 377, 31-51 (2004)

results in this paper are complemented by examples that were generated numerically with combinatorial search algorithms.

B. G. Bodmann and P. G. Casazza (2010): The road to equal-norm Parseval frames, J. Funct. Anal. 258, 397-420. Optimality principles for signal processing led Bodmann and other researchers to study geometric design principles in frame theory. The most fundamental type is that of equal-norm Parseval frames. Paulsen had raised the question of how close a nearly equal-norm, nearly Parseval frame is to an equal-norm Parseval frame. Holmes and Paulsen[16] constructed an algorithm which transforms a frame into an equal-norm Parseval frame in finitely many steps, but which lacked a distance estimate. To address this deficiency, Bodmann and Casazza used ordinary differential equations generating a flow on the space on frames combined with distance estimates. Benedetto and Fickus had already presented the use of the frame potential and physical principles in frame design[17]. The flow Bodmann and Casazza chose was different from a gradient descent for the frame potential in order to avoid a problem of having too many undesirable stationary points. The essence of the strategy is a dilation argument, an energy-velocity estimate, and a compactness argument which allows to patch the flow together.

Casazza and Fickus[18] subsequently developed an alternative proof for a distance estimate which does not require the dilation argument or a patching of locally defined flows.

B. G. Bodmann and J. Haas (2015): Frame potentials and the geometry of frames, Journal of Fourier Analysis and Applications 21(6) 1344-1383. After investigating equal-norm Parseval frames with Casazza, Bodmann and Haas broadened the class of frame potentials and studied the geometric structure of resulting optimizers. Next to the known classes of equal-norm and equiangular Parseval frames, this lead to equidistributed Parseval frames, which are more general than the equiangular type but have more structure than equal-norm ones. Based on results by Lojasiewicz, they show that the gradient descent for a real analytic frame potential on the manifold of Gram matrices belonging to Parseval frames always converges to a critical point.

6. Uncertainty principles: From quantum mechanics to optimality in filter design

Uncertainty principles often dictate a trade-off in applications. Finding uncertainty minimizers then gives examples of optimal designs.

B. G. Bodmann (2004): A lower bound for the Wehrl entropy of quantum spin with sharp high-spin asymptotics, Commun. Math. Phys. 250, 287-300. Uncertainty principles and optimization constitute an important theme in Bodmann's research related to harmonic analysis and its applications. For many Hilbert spaces equipped with a reproducing kernel, it seems plausible that kernel functions should be most concentrated among all vectors. In quantum physics, this idea is often phrased as "coherent states are closest to classical". In a previous paper, Lieb[19] had derived a

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[16] R. Holmes and V. I. Paulsen, *ibid.*

[17] J. J. Benedetto and M. Fickus, Finite Normalized Tight Frames, Adv. Comp. Math. 18, 357-385 (2004)

[18] P. G. Casazza and M. Fickus, Auto-tuning unit norm frames, Appl. Comput. Harmon. Anal. 32, 1-15 (2012).

[19] E. H. Lieb, Proof of an entropy conjecture of Wehrl, Commun. Math. Phys. 62, 35-41 (1978)

quantitative form of this statement with an entropy bound for the case of Bargmann space, as conjectured by Wehrl[20]. In his paper, Lieb conjectured that an analogous entropy estimate should be true for the SU(2) representation spaces of Bloch coherent states. To establish an estimate of this type, Bodmann proved a variant of an inequality for Dirichlet forms by Carlen[21] in the setting of highest weight SU(2) representations together with a sharp hypercontractivity estimate. To derive the inequality for the Dirichlet form needed for Lieb's conjecture, Bodmann adapted techniques for finding radial solutions to quasilinear elliptic problems by Serrin and Tang[22].

Both components, the inequalities for Dirichlet forms in holomorphic representation spaces and the radial solutions to quasilinear elliptic problems were employed by Bandyopadhyay[23] to prove analogous results for SU(1,1). This suggests that the techniques developed by Bodmann carry over to a wider class of highest-weight Lie group representation spaces.

B. G. Bodmann, M. Papadakis and Q. Sun (2006): An inhomogeneous uncertainty principle for digital low-pass filters, J. Fourier Anal. Appl. 12, 181-211. This paper is Bodmann's first work that investigates the role of uncertainty principles in signal processing. In previous works with Hoffman, Kouri and others, Papadakis[24] had examined approximations to ideal filters. These works demonstrate a well-known trade-off: due to the discontinuities of an ideal low-pass filter in the frequency domain, it cannot be approximated well without using an increasing filter-length in the time domain. In this paper, Bodmann and collaborators systematically developed a quantitative version of this statement in the form of an uncertainty inequality for filters which approximate an ideal filter. There are no minimizers for this fundamental inequality, but minimizing sequences. The variational techniques used for the characterization of minimizing sequences benefited from rearrangement inequalities that Bodmann had already used in the quantum setting. When a maximal filter length is imposed, then there are uncertainty minimizers that can be computed numerically from a one-parameter minimization problem. As the limit for the filter length increases, the uncertainty product approaches the global infimum. It turns out that the limiting value of the uncertainty products for digital Butterworth filters or Daubechies interpolatory filters is only slightly above the global infimum.

[20] A. Wehrl, On the relation between classical and quantum mechanical entropy. Rep. Mat. Phys. 16, 353-358 (1979)

[21] E. A. Carlen, Some integral identities and inequalities for entire functions and their application to the coherent state transform, J. Funct. Anal. 97, 231-249 (1991)

[22] J. Serrin and M. Tang, Uniqueness of ground states for quasilinear elliptic equations, Indiana Univ. Math. J. 49, 897-923 (2000)

[23] J. Bandyopadhyay, Optimal Concentration for SU(1, 1) Coherent State Transforms and An Analogue of the Lieb-Wehrl Conjecture for SU(1, 1), Commun. Math. Phys. 285 1065-1086 (2009)

[24] D. J. Kouri, M. Papadakis, I. Kakadiaris, and D. K. Hoffman, Properties of minimum-uncertainty wavelets and their relations to the harmonic oscillator and the coherent states, J. Phys. Chem. 107, 7318-7327 (2003)