REVIEW MATERIAL AND PROBLEM SETS FOR 2022 HOUSTON WORKSHOP ON HYPERBOLIC DYNAMICAL SYSTEMS

Compiled by Vaughn Climenhaga – at whose feet all errors and inaccuracies should be laid – using exercises and suggestions provided by the speakers: Keith Burns, Mark Demers, Alena Erchenko, Todd Fisher, Yan Mary He, Kathryn Lindsey, Matt Nicol, Victoria Sadovskaya, Fan Yang, Agnieszka Zelerowicz, Hong-Kun Zhang, Pengfei Zhang.

HYPERBOLICITY

Structure of hyperbolic sets.

Basic definition. Let M be a compact Riemannian manifold and $f\colon M \to M$ a diffeomorphism. A closed f-invariant set $\Lambda \subset M$ is called *hyperbolic* if for every $x \in \Lambda$ there is a splitting $T_xM = E^u \oplus E^s$ such that the following are true.

- The splitting is invariant: $Df_x E^u_x = E^u_{f(x)}$ and $Df_x E^s_x = E^s_{f(x)}$ for all $x \in \Lambda$.
- E^s and E^u contract uniformly in forward and backward time, respectively: there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \Lambda$ and $n \in \mathbb{N}$ we have
$$\|Df^n_x(v^s)\| \leq C\lambda^n \|v^s\| \text{ for all } v^s \in E^s_x,$$
$$\|Df^{-n}_x(v^u)\| \leq C\lambda^n \|v^u\| \text{ for all } v^u \in E^u_x.$$

Λ is *locally maximal* if there is an open $U \supset \Lambda$ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$.

The following are some concepts you should know: if they are unfamiliar, try asking a colleague, a professor, or the internet.

- Anosov diffeomorphism (M is a hyperbolic set); hyperbolic toral automorphisms.
- Local and global stable and unstable manifolds of points in hyperbolic sets, including hyperbolic periodic points.
- Topological conjugacy, structural stability.
- Topological transitivity, topological mixing.

It will be helpful to work through some of the following exercises.

1. Give an example of a hyperbolic set that is *not* locally maximal.

2. Prove that a locally maximal hyperbolic set Λ has *local product structure*: there are $\delta, \epsilon > 0$ such that if $x, y \in \Lambda$ have $d(x, y) < \delta$, then the local stable manifold $W^s_\epsilon(x)$ intersects the local unstable manifold $W^u_\epsilon(y)$ in exactly one point z, and this point lies in Λ. (Here $W^s_\epsilon(x)$ denotes the ball of radius ϵ in the local stable manifold of x, and similarly for $W^u_\epsilon(y)$.) The orbit of z shadows y in the past and x in the future; z is often denoted $[x, y]$ and is called the *bracket* of x and y.)

3. Prove that every hyperbolic set is *expansive*: there is $\delta > 0$ such that if $x, y \in \Lambda$ have the property that $d(f^n x, f^n y) < \delta$ for all $n \in \mathbb{Z}$, then $x = y$.

4. The following steps outline a proof that topologically mixing locally maximal hyperbolic sets have the *specification property*: for every $\delta > 0$ there exists $\tau \in \mathbb{N}$ such that given any $x_1, \dots, x_k \in \Lambda$ and $n_1, \dots, n_k \in \mathbb{N}$, there is $y \in \Lambda$ whose orbit has the property that

$$d(f^{N_j+i}(y), f^i(x_j)) < \delta \text{ for all } 1 \le j \le k \text{ and } 0 \le i \le n_j,$$

where $N_j = \sum_{i=0}^{j-1}(n_i + \tau)$. (Writing (x_j, n_j) for the orbit segment $x_j \to f(x_j) \to f^2(x_j) \to \cdots \to f^{n_j}(x_j)$, this says that the orbit of y shadows each orbit segment (x_j, n_j) in turn, to within an accuracy of δ, with a gap of τ iterates between each one.)

(a) To prove this, start by showing that for every $\delta > 0$, there exists $\tau \in \mathbb{N}$ such that for every $x, y \in \Lambda$ and every $n \ge \tau$, we have $f^n(W^u_\delta(x)) \cap W^s_\delta(y) \ne \emptyset$.

(b) Use the previous part together with local product structure to prove the specification property in the case $k = 2$.

(c) Iterate this process and prove specification for arbitrary k.

(d) You may find it easier to start by doing all of this in the setting of uniformly expanding maps; see if you can formulate the analogous properties and argument there.

Hyperbolic periodic points.

5. *Transverse homoclinic intersection implies horseshoe implies positive topological entropy.*

(a) Let p be a hyperbolic fixed point for a diffeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, and suppose that the global stable and unstable manifolds of p intersect transversely at $q \ne p$. Draw a rectangle R near p whose sides are parallel to E^s_p and E^u_p. Investigate what happens to R under iteration by f and/or f^{-1}. Show that you can obtain a picture reminiscent of the Smale horseshoe (under some large iterate f^n).

(b) Use the previous part to show that f has positive topological entropy.

6. Consider the standard map $f_k(x, y) = (x + y + k \sin x, y + k \sin x)$ for some constant $0 < k < 4$.

(a) Find the fixed points of f_k.

(b) Determine if they are hyperbolic (one eigenvalue inside the unit circle and one outside) or elliptic (both eigenvalues on the unit circle).

7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving diffeomorphism, and $O = (0, 0)$ be a hyperbolic fixed point of f. For simplicity, we assume

$$f\colon \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \lambda x + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \lambda^{-1}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{bmatrix} + O(r^3),$$

where $\lambda > 1$, and $r = \sqrt{x^2 + y^2}$.

(a) Show that $2a_{20} + \lambda^2 b_{11} = 0$ and $a_{11} + 2\lambda^2 b_{02} = 0$.

(b) Find a coordinate transformation

$$h : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x + p_{20}x^2 + p_{11}xy + p_{02}y^2 \\ y + q_{20}x^2 + q_{11}xy + q_{02}y^2 \end{bmatrix} + O(r^3),$$

such that $g = h^{-1} \circ f \circ h$ satisfies

$$g\colon \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \lambda x \\ \lambda^{-1} y \end{bmatrix} + O(r^3).$$

(c) Does your argument in the previous part work if $O = (0, 0)$ is an elliptic fixed point of f?

Cocycles and Lyapunov exponents.

Basic definition. Let M be a smooth Riemannian manifold and $f\colon M \to M$ a smooth map. Given $x \in M$ and $v \in T_xM$, the *forward Lyapunov exponent* is

$$(\star) \qquad \chi(x, v) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \|Df_x^n(v)\|.$$

(In general when one talks about Lyapunov exponents, one is also concerned with whether or not the limit exists, with the filtration by Oseledets subspaces, and in the case when f is invertible, with the backward exponents, Lyapunov–Perron regularity, and the Oseledets splitting. If these things are unfamiliar to you, stick with $(\star)$ and see how far you can get.)

More generally, given $A\colon M \to GL(d, \mathbb{R})$, we can define the forward Lyapunov exponents of the *linear cocycle* $\mathcal{A}(x, n) = A(f^{n-1}x) \cdots A(fx)A(x)$ by

$$\chi(x, v) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \|\mathcal{A}(x, n)(v)\| \quad \text{for each } v \in \mathbb{R}^d.$$

8. Prove that when M is a torus, the derivative cocycle $(x, n) \mapsto Df_x^n$ can be interpreted as a linear cocycle in the sense above. What is the obstruction to this interpretation when M is not a torus? (There is a more general definition of "linear cocycle" suitable for this case as well, but we do not go into it here.)

9. Prove that when M is a torus, and f is Anosov, we can interpret $Df|_{E^u}$ and $Df|_{E^s}$ as linear cocycles in this same sense.

10. Prove that a linear cocycle as defined above satisfies $\mathcal{A}(x, m + n) = \mathcal{A}(f^m x, n)\mathcal{A}(x, m)$ for all $x \in M$ and $m, n \geq 0$. In the case when f is invertible, how should $\mathcal{A}(x, n)$ be defined for $n < 0$ to make this *cocycle property* hold for all $m, n \in \mathbb{Z}$?

11. Say that linear cocycles generated by $A, B\colon M \to GL(d, \mathbb{R})$ are *conjugate* if there is a function $C\colon M \to GL(d, \mathbb{R})$ such that $A(x) = C(fx)B(x)C(x)^{-1}$ for all $x \in M$. Find a relationship between the Lyapunov exponents of the two cocycles in this case.

12. Prove that if $f(x) = x$, then the set of values attained by $\chi(x, v)$ (as v varies over all possibilities) is exactly the set of eigenvalues of $A(x)$. More generally, prove that for each x there are only finitely many values that $\chi(x, v)$ can take.

13. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus, with volume inherited from $\mathbb{R}^d$, and suppose that $f\colon \mathbb{T}^d \to \mathbb{T}^d$ is a diffeomorphism that preserves volume. Show that if x is any periodic point of f, so that $f^n(x) = x$ for some $n \in \mathbb{N}$, then the matrix representing Df^n_x has determinant 1.

14. Let X be a compact metric space and $f\colon X \to X$ a continuous map. Say that $\phi\colon X \to \mathbb{R}$ is a *coboundary* if there exists a continuous $\psi\colon X \to \mathbb{R}$ such that $\phi(x) = \psi(x) - \psi(fx)$ for all $x \in X$.

(a) Prove that if ϕ is a coboundary and x is a periodic point with period p, so that $f^p(x) = x$, then $\sum_{k=0}^{p-1} \phi(f^k x) = 0$.

(b) More generally, prove that if ϕ is a coboundary and μ is an f-invariant Borel probability measure on X, then $\int \phi\, d\mu = 0$.

(c) Show that the converse of these statements is not always true.

(However, in the case when X is a locally maximal hyperbolic set and ϕ is Hölder continuous, the Livšic theorem says that the converse *is* true: vanishing periodic data implies that ϕ is a coboundary. See `https://vaughnclimenhaga.wordpress.com/2018/10/02/cohomologous-functions-and-the-livsic-theorem/` for a proof.)

MEASURES

Space of invariant measures.

Basic definition. Let X be a compact metric space and $f\colon X \to X$ continuous. Write $\mathcal{M}(X)$ for the space of all Borel probability measures on X, equipped with the weak* topology. The map f acts on $\mathcal{M}(X)$ via the pushforward $f_*\colon \mathcal{M}(X) \to \mathcal{M}(X)$ defined by $(f_*\mu)(E) = \mu(f^{-1}E)$. A measure μ is *f-invariant* if $f_*\mu = \mu$. Write $\mathcal{M}_f(X) \subset \mathcal{M}(X)$ for the space of f-invariant Borel probability measures on X. This space is a simplex, whose extreme points are the *ergodic* measures.

Concepts you should know:

- Full shift on a finite alphabet; subshift of finite type (SFT).
- Bernoulli measure on a full shift; Markov measure on an SFT.
- Properties in the mixing hierarchy: mixing, K (Kolmorogorov), Bernoulli.
- Birkhoff ergodic theorem.

15. Given $\mu \in \mathcal{M}(X)$ and $\phi \in C(X)$, show that $\int \phi \, d(f_*\mu) = \int \phi \circ f \, d\mu$. (One way of thinking of this is that if we interpret ϕ as a measurement of the system, and we choose the current state of the system at random according to the measure μ, then $\int \phi \, d\mu$ represents the mean value of the measurement if we make it immediately, while $\int \phi \, d(f_*\mu)$ represents the mean value of the measurement if we wait one time step. Thus $f_*\mu$ represents the distribution of states after one time step has elapsed.)

16. Prove that $\mathcal{M}_f(X)$ is nonempty (the *Krylov–Bogolyubov theorem*) using the following argument: start with an arbitrary measure $m \in \mathcal{M}(X)$ and consider the sequence $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j m$. Use a weak* compactness result to show that μ_n has a convergent subsequence μ_{n_k}, and then use continuity of f to show that the limit lies in $\mathcal{M}_f(X)$.

17. Let (X, σ) be an SFT that contains infinitely many points, and show that $\mathcal{M}_\sigma(X)$ is infinite-dimensional.

18. More difficult: strengthen the previous exercise by showing that if X is a topologically mixing SFT, then the set of ergodic measures is weak*-dense in $\mathcal{M}_\sigma(X)$ (so this is a simplex whose extreme points are dense). You may find it easier to start with the case when X is the full shift.

Thermodynamic formalism.

Basic definitions. Let X be a compact metric space and $f\colon X \to X$ continuous. Fix an invariant measure $\mu \in \mathcal{M}_f(X)$. The *measure-theoretic entropy* of μ is defined as follows.

- Start with a finite or countable *partition* α of X.

- The *static entropy* of α is $H_\mu(\alpha) = \sum_{A\in\alpha} -\mu(A)\log\mu(A)$. (Interpret $-\log(A)$ as the information you gain by learning that a randomly chosen point happened to land in $A \subset X$. Then $H(\alpha)$ is the expected information you gain by learning which element of α a randomly chosen point lands in.)
- Given $n \in \mathbb{N}$, let α_n be the *dynamically refined partition* into sets of the form $A_0 \cap f^{-1}(A_1) \cap f^{-2}(A_2) \cap \cdots \cap f^{-(n-1)}(A_{n-1})$, where each A_i is an element of α. Thus α_n records which element of α the first n iterates land in.
- The *entropy of f with respect to the partition α* is $h_\mu(f,\alpha) = \lim_{n\to\infty} \frac{1}{n} H(\alpha_n)$. (Interpret this as the linear growth rate of the expected information you gain by observing the first n iterates w.r.t. α.)
- The measure-theoretic (Kolmogorov–Sinai) entropy of μ is $h_\mu(f) = \sup_\alpha h_\mu(f,\alpha)$, where the supremum is taken over all finite partitions α. If α is *generating* in the sense that for μ-a.e. $x \neq y$, there exists n such that $f^n(x)$ and $f^n(y)$ lie in different elements of α, then $h_\mu(f) = h_\mu(f,\alpha)$.

The *topological entropy* of (X, f) is defined as follows.

- Given $x \in X$, $n \in \mathbb{N}$, and $\epsilon > 0$, the *Bowen ball of radius ϵ and order n centered at x* is
$$B_n(x,\epsilon) = \{y \in X : d(f^k x, f^k y) < \epsilon \text{ for all } 0 \le k < n\}.$$
- A set $E \subset X$ is (n,ϵ)-separated if for every $x \neq y \in E$, we have $y \notin B_n(x,\epsilon)$.
- The *topological entropy of (X, f) at scale ϵ* is
$$h_{\text{top}}(f,\epsilon) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sup\{\#E : E \subset X \text{ is } (n,\epsilon)\text{-separated}\}.$$
(Interpret this as the exponential growth rate of the number of "distinguishable" orbit segments of length n.)
- The *topological entropy* of (X, f) is $h_{\text{top}}(f) = \lim_{\epsilon\to 0} h_{\text{top}}(f,\epsilon)$. If (X, f) is expansive and ϵ is an expansivity constant, then $h_{\text{top}}(f) = h_{\text{top}}(f,\epsilon)$.

Thermodynamic formalism centers around the following ideas.

- *Variational principle for entropy:* $h_{\text{top}}(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}_f(X)\}$.
- A measure satisfying $h_\mu(f) = h_{\text{top}}(f)$ is called a *measure of maximal entropy.*
- The *topological pressure* of a *potential function* $\phi \in C(X)$ is
$$P(\phi) = \lim_{\epsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sup\Big\{\sum_{x\in E} e^{S_n\phi(x)} : E \subset X \text{ is } (n,\epsilon)\text{-separated}\Big\},$$
where $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(f^k x)$ is the nth Birkhoff sum. (Interpret this as an exponential growth rate, as with topological entropy, but now each orbit segment of length n is given a weight determined by $e^{S_n\phi}$.)
- *Variational principle for pressure:* $P(\phi) = \sup\{h_\mu(f) + \int \phi\, d\mu : \mu \in \mathcal{M}_f(X)\}$.
- A measure satisfying $h_\mu(f) + \int \phi\, d\mu = P(\phi)$ is called an *equilibrium state for ϕ.*

19. Prove that if X is a shift space and α is the partition into 1-cylinders, then α is generating, and the elements of α_n are n-cylinders. Use this to write an expression for $h_\mu(\sigma)$ in terms of cylinders.

20. Prove that if X is a shift space, then every Bowen ball is a cylinder. Use this to write an expression for $h_{\mathrm{top}}(\sigma)$ in terms of cylinders.

21. Let X be an SFT with transition matrix A. Find a relationship between $h_{\mathrm{top}}(\sigma)$ and the eigenvalues of A.

22. Without referring to the variational principle, show that if X is a full shift and μ is a Bernoulli measure, then $h_\mu(\sigma) \leq h_{\mathrm{top}}(\sigma)$, with equality if and only if μ gives equal weight to every symbol.

23. Prove that for any finite partition α and any measure μ, we have $H_\mu(\alpha) \leq \log \#\alpha$, with equality if and only if all elements of α have equal weight. (This should remind you of the previous exercise.) Use this to prove half of the variational principle in the setting of a shift space: $h_\mu(\sigma) \leq h_{\mathrm{top}}(\sigma)$.

24. *Bowen's formula.* Let (Σ^+, σ) be a topologically mixing SFT and let (Σ_r, ϕ_t) be a suspension flow over (Σ^+, σ) with a Hölder continuous roof function $r \colon \Sigma^+ \to (0, \infty)$. Let P denote the pressure over (Σ^+, σ) and P_ϕ the pressure over (Σ_r, ϕ_t). Show that for all Hölder continuous functions $G : \Sigma_r \to \mathbb{R}$ we have

$$P(\Delta_G - P_\phi(G)r) = 0$$

where $\Delta_G : \Sigma^+ \to \mathbb{R}$ is given by $\Delta_G(\underline{x}) = \int_0^{r(\underline{x})} G(\underline{x}, t)\, dt$. In particular, when $G \equiv 0$, we get $P(-h_{\mathrm{top}}(\phi)r) = 0$, which is the famous Bowen's formula. (Note that here the letter ϕ is used to refer to a flow, rather than to a potential – there are various notational conventions in use by different authors.)

REPELLERS

Repelling sets.

Basic definition. Let X be a compact metric space and $f\colon X \to X$ a continuous map. Say that f is *expanding* if there are $\delta > 0$ and $\lambda > 1$ such that

$$d(fx, fy) \geq \lambda d(x, y) \text{ whenever } d(x, y) < \delta. \tag{$\star\star$}$$

Iterating this we see that

$$d(x, y) \leq \lambda^{-n} d(f^n x, f^n y) \text{ whenever } y \in B_n(x, \delta),$$

where $B_n(x, \delta)$ is the Bowen ball defined in the "Thermodynamic formalism" section. A more general definition of *expanding map* would be to require that there are $C, \delta > 0$ and $\lambda > 1$ such that

$$d(x, y) \leq C\lambda^{-n} d(f^n x, f^n y) \text{ whenever } y \in B_n(x, \delta).$$

(Compare this to the "backwards contraction" requirement on E^u in the definition of a hyperbolic set.) In fact it is possible to show that if this condition holds, then there is an equivalent *adapted* metric in which the stronger "one-step" condition ($\star\star$) holds.

A compact f-invariant set Λ is a *locally maximal repeller* if $f|_\Lambda$ is expanding and there is an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{n=0}^{\infty} f^{-n}(U)$. (Compare this to the definition of locally maximal hyperbolic set.)

Other concepts you should know.

- *α-dimensional Hausdorff measure* m_α has the property that if f expands distances by a factor of r, then $m_\alpha(fZ) = r^\alpha m_\alpha(Z)$.
- *Hausdorff dimension of a set*: $\dim_H(Z)$ is the unique number with the property that $m_\alpha(Z) = 0$ for all $\alpha > \dim_H(Z)$ and $m_\alpha(Z) = \infty$ for all $\alpha < \dim_H(Z)$.
- *Hausdorff dimension of a measure*: $\dim_H(\mu) = \inf\{\dim_H(Z) : \mu(Z^c) = 0\}$.

25. Find a map f defined on the real line (or a subset of it) that has the middle-third Cantor set as a locally maximal repeller.

26. An *iterated function system* (IFS) on the real line is a finite set of maps $f_i\colon \mathbb{R} \to \mathbb{R}$ ($1 \leq i \leq k$), all of which are contractions: there is $\lambda \in (0, 1)$ such that $|f_i(x) - f_i(y)| \leq \lambda|x - y|$ for all $x, y \in \mathbb{R}$. It can be proved that there is a unique non-empty compact set $\Lambda \subset \mathbb{R}$ such that $\Lambda = \bigcup_{i=1}^{n} f_i(\Lambda)$.

(a) Obtain the standard middle-third Cantor set C as the attractor of an iterated function system (IFS).

(b) What is the relationship between the IFS from part (a) and the expanding map from the previous problem?

27. Compute the Hausdorff dimension of the standard middle-third Cantor set.

28. Let R_1, R_2 be two disjoint closed subintervals of $[0,1]$ and $f\colon R_1 \cup R_2 \to [0,1]$ be a map such that $f|_{R_i}\colon R_i \to [0,1]$ is linear and surjective for $i = 1, 2$.

(a) Describe the largest set $\Lambda \subset [0,1]$ such that $\Lambda = f^{-1}(\Lambda)$; this is the maximal repeller of the map f.

(b) Find the Hausdorff dimension of Λ.

(c) Describe the measure of maximal entropy of f. Find its dimension, the entropy of f with respect to this measure, and the Lyapunov exponent of f with respect to this measure. (The *Lyapunov exponent with respect to an ergodic measure m* is the value χ such that $\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| = \chi$ for m-a.e. x.; the fact that such a value exists is a consequence of the Birkhoff ergodic theorem, and is a useful exercise in itself if you have not seen it before.)

(d) Let μ be a Bernoulli invariant measure on Λ such that $\mu(R_1) = p$ and $\mu(R_2) = 1-p$ where $0 < p < 1$. Find its dimension, the entropy of f with respect to μ, and the Lyapunov exponent of f with respect to μ.

(e) Find the invariant measure of full dimension of f. Find its dimension, the entropy of f with respect to this measure, and the Lyapunov exponent of f with respect to this measure.

29. *Bounded distortion lemma:* Let R_1, R_2 be two disjoint closed subintervals of $[0,1]$ and $f\colon R_1 \cup R_2 \to [0,1]$ be a C^∞ expanding map such that $f' \geq L > 1$ for a constant L and $f|_{R_i}$ is a homeomorphism onto $[0,1]$. Define

$$R_{i_1, i_2 \dots, i_n} = R_{i_1} \cap f^{-1}(R_{i_2}) \cap \dots \cap f^{-n+1}(R_{i_n}).$$

Find $C > 0$ such that for every $n \in \mathbb{N}$ and every n-tuple $(i_1, i_2, \dots, i_n)$ of 1s and 2s,

$$\frac{(f^n)'(x)}{(f^n)'(y)} \leq C \text{ for every } x, y \in R_{i_1,\dots,i_n}.$$

30. In the setting of the previous exercise, can you say anything about the Hausdorff dimension of the maximal repeller?

Julia sets in complex dynamics.

31. Consider the map $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$. Show that the unit circle $S^1 \subset \mathbb{C}$ is a locally maximal repeller. This is the *Julia set* of f; the next few exercises lead towards a general definition of Julia set.

32. Given $c \in \mathbb{C}$ with $c \approx 0$, show that the map $f_c(z) = z^2 + c$ has a locally maximal repeller that is close to the unit circle S^1.

33. Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomorphic. Consider f as a map of the real plane, and show that the forward Lyapunov exponent $\chi(z, v)$ (defined earlier in these exercises) depends only on z and not on v; thus we can reasonably speak about "the Lyapunov exponent $\chi(z)$" without specifying a direction.

34. Consider the holomorphic map $f(z) = z^2$ and prove that:
- $\chi(z) = -\infty$ for all $|z| < 1$;
- $\chi(z) = \log 2$ for all $|z| = 1$;
- $\chi(z) = +\infty$ for all $|z| > 1$.

35. One might reasonably hope to identify the locally maximal repellers in Exercises 31 and 32 as the sets of points with positive Lyapunov exponent. The previous exercise demonstrates that this does not work, at least not without modification. To get closer to making it work, let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the *Riemann sphere*, equipped with a metric as follows. Consider the stereographic projection $\pi \colon \mathbb{C} \to S^2$ (the unit sphere in $\mathbb{R}^3$) defined by observing that given any $x + iy \in \mathbb{C}$, the line in $\mathbb{R}^3$ connecting $(0, 0, 1)$ and $(x, y, 0)$ intersects S^2 in exactly two points: one of these is $(0, 0, 1)$, and the other is what we take as $\pi(x+iy)$. Then extend π to the Riemann sphere by putting $\pi(\infty) = (0, 0, 1)$. Now define a metric on $\widehat{\mathbb{C}}$ by declaring π to be an isometry. Show that if we consider $\widehat{\mathbb{C}}$ as a Riemannian manifold with this *spherical metric* and consider $f(z) = z^2$ as a smooth map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, then we have $\chi(z) = -\infty$ for all $z \in \widehat{\mathbb{C}} \setminus S^1$, while $\chi(z) = \log 2$ for all $z \in S^1$.

36. Now we might be tempted to conjecture that we can identify the sets of interest by considering all points with positive Lyapunov exponent *in the spherical metric*. To see that this does not quite work, find $c \in \mathbb{C}$ such that the map $f_c(z) = z^2 + c$ has a fixed point with Lyapunov exponent 0. This fixed point cannot be in a set on which f is expanding (hence it cannot be in a repeller under our definition); can we reasonably exclude it from the "maximal compact invariant set of interest"?

37. Let $U \subset \mathbb{C}$. A family $\mathcal{F}$ of holomorphic functions $U \to \mathbb{C}$ is said to be *normal* if it is equicontinuous on every compact subset of U, where we use the spherical metric on $\mathbb{C} \subset \widehat{\mathbb{C}}$. Prove that if $f \colon \mathbb{C} \to \mathbb{C}$ is holomorphic and the family of iterates $\{f^n|_U : n \in \mathbb{N}\}$ is normal, then the Lyapunov exponent of f at z is ≤ 0 for every $z \in U$.

38. The *Fatou set* of a polynomial f is the set of all points z that have a neighborhood U on which the family $f^n|_U$ is normal. Its complement in $\mathbb{C}$ is the *Julia set* J_f. Prove that the Julia set is compact and invariant, and that in Exercises 31 and 32 it agrees with the locally maximal repeller you found there.

39. The *Mandelbrot set* is the set of parameters $c \in \mathbb{C}$ for which the trajectory $\{f_c^n(0) : n \in \mathbb{N}\}$ is bounded. It is known that if f_c has an attracting periodic orbit, then the trajectory $f_c^n(0)$ converges to this orbit, and thus is bounded.

(a) Find and sketch the set of parameters $c \in \mathbb{C}$ for which the map $f_c(z) = z^2 + c$ has an attracting fixed point. This is the *main cardioid* of the Mandelbrot set.

(b) Find and sketch the set of parameters c for which f_c has an attracting period-2 orbit.

OPERATORS

Perron–Frobenius theory.

Basic theorem. Let A be a $d \times d$ matrix such that $A_{ij} \geq 0$ for all i, j, and such that there is $N \in \mathbb{N}$ with $(A^N)_{ij} > 0$ for all i, j. The *Perron–Frobenius theorem* says that:

- A has a simple eigenvalue λ such that every other eigenvalue ρ has $|\rho| < \lambda$;
- A has positive left and right eigenvectors u and v for λ ("positive" meaning that all entries are positive), and no other eigenvalues have any positive eigenvectors.

From this one can deduce that if the eigenvectors u (row vector) and v (column vector) are normalized so that $uv = 1$, then $P = vu$ is a projection matrix ($P^2 = P$) onto the eigenspace of λ along the direct sum of the generalized eigenspaces corresponding to other eigenvalues. Since all of these eigenvalues have absolute value $< \lambda$, this means that $R = A - \lambda P$ has spectral radius $< \lambda$, and thus $\lambda^{-n} R^n \to 0$ (exponentially quickly). Consequently we have

$$A = \lambda P + R, \qquad P^2 = P, \qquad PR = RP = 0,$$

and deduce that

$$\lambda^{-n} A^n = P + \lambda^{-n} R^n \to P \text{ exponentially quickly.}$$

The following exercises walk you through some generalizations of this that are relevant to dynamical systems.

40. The *Ruelle–Perron–Frobenius* theorem appears in various guises. As a starting point, let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle and let $T \colon S^1 \to S^1$ be a C^2 expanding map with $\inf_x |T'(x)| \geq \lambda > 1$. If we want to prove that T has an invariant measure *that is absolutely continuous with respect to Lebesgue*, we might follow the approach from Exercise 16: start with Lebesgue measure m and consider the averaged pushforwards $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} T_*^n m$. One may reasonably expect that each μ_n is absolutely continuous w.r.t. Lebesgue, but whether this absolute continuity survives passage to the limit $\mu_{n_k} \to \mu \in \mathcal{M}_T(S^1)$ is a subtler question. To begin addressing this, prove that given a Borel measure $\nu \ll m$ whose density is $h \in L^1(m)$ (this means that $\int \psi \, d\nu = \int \psi h \, dm$ for all $\psi \in C(S^1)$), the pushforward $T_*\nu$ satisfies $T_*\nu \ll m$ and has density $\mathcal{L}h$, where

$$(\mathcal{L}h)(x) = \sum_{y \in T^{-1}x} \frac{h(y)}{|T'(y)|}.$$

Thus a measure $\nu \ll m$ is T-invariant if and only if its density $h = d\nu/dm$ satisfies $\mathcal{L}h = h$, and to control convergence of the averaged pushforwards μ_n described above, we need to control the iterates $\mathcal{L}^n$.

41. More generally, given a continuous potential $\phi \colon S^1 \to \mathbb{R}$, we can define the Ruelle–Perron–Frobenius operator $\mathcal{L}_\phi \colon C(S^1) \to C(S^1)$ by

$$(\mathcal{L}_\phi h)(x) = \sum_{y \in T^{-1}x} e^{\phi(y)} h(y).$$

(When $\phi = -\log|T'|$ we recover the operator in the previous exercise.) Prove that the operator $\mathcal{L}_\phi$ acts on $C(S^1)$, so its dual $\mathcal{L}_\phi^*$ acts on $C(S^1)^*$ (the space of finite signed Borel measures on S^1); to understand how this dual operator acts, show that

$$\mathcal{L}_\phi^* \delta_x = \sum_{y \in T^{-1}x} e^{\phi(y)} \delta_y.$$

Thus $\mathcal{L}_\phi^*$ "pushes mass backwards to preimages and multiplies it by e^ϕ".

42. Prove that when $\phi = -\log|T'|$ (the *geometric potential*), Lebesgue measure m satisfies $\mathcal{L}^* m = m$. This suggests the following generalization of Exercise 40: if ϕ is *any* potential and m is *any* measure with $\mathcal{L}_\phi^* m = m$, then $T_*(h\, dm) = (\mathcal{L}h)\, dm$ for all h.

43. The *Ruelle–Perron–Frobenius theorem* (in one of its incarnations) says that if $T\colon S^1 \to S^1$ is uniformly expanding and $\phi\colon S^1 \to \mathbb{R}$ is Hölder continuous, then there exists $\lambda > 0$, a measure $m \in \mathcal{M}(S^1)$, and a function $h \in C(S^1)$ such that $\int h\, dm = 1$ and:

- $\mathcal{L}_\phi^* m = \lambda m$ and $\mathcal{L}_\phi h = \lambda h$;
- for every Hölder continuous $\psi\colon S^1 \to \mathbb{R}$ with $\int \psi\, dm = 1$, we have $\lambda^{-n}\mathcal{L}^n \psi \to h$.

(In fact this convergence happens exponentially quickly.)

(a) Prove that if we write $\bar\phi = \phi - \log\lambda$, then $\bar\phi$ satisfies the same result but with RPF eigenvalue equal to 1: that is, $\mathcal{L}_{\bar\phi}^* m = m$ and $\mathcal{L}_{\bar\phi} h = h$. In particular, by the result of the previous exercise, $h\, dm$ defines an invariant probability measure.

(b) Prove that if we go one step further and write $\bar\phi = \phi - \log\lambda + \log h - \log h \circ T$, then $\mathcal{L}_{\bar\phi}\mathbf{1} = \mathbf{1}$ (so the normalized operator has a constant eigenfunction), and the invariant measure μ defined by $\mu(E) = \int_E h\, dm$ is the eigenmeasure: $\mathcal{L}_{\bar\phi}^* \mu = \mu$.

(c) Prove that $\lambda = e^{P(\phi)}$.

44. The exponential rate of convergence in the RPF theorem can be used to prove *exponential decay of correlations*, and relies on the *Lasota–Yorke inequality*, which establishes *quasi-compactness* of $\mathcal{L}_\phi$ acting on the space of Hölder continuous functions (with an appropriate exponent): apart from the RPF eigenvalue λ, the remainder of the spectrum of $\mathcal{L}$ lies inside a disc of radius $< \lambda$, so there is a *spectral gap*. Here are some of the key ingredients in the case when $\phi = -\log|T'|$ so that $\mathcal{L}$ is as in Exercise 40.

(a) Show the bounded distortion estimates: $\exists C > 0$ s.t.

$$\frac{|D^2T^n(x)|}{|DT^n(x)|^2} \le C, \qquad |DT^n(x)|^{-1} \le C|J_{n,i}|,$$

where $x \in J_{n,i}$ and $J_{n,i}$ is an interval of monotonicity of T^n with $T^n(J_{n,i}) = S^1$.

(b) Show that the transfer operator $\mathcal{L}\colon C^1 \to C^1$ defined in Exercise 40 satisfies the inequalities

$$\|\mathcal{L}^n f\|_{C^0} \le C\|f\|_C^0, \qquad \|\mathcal{L}^n f(x)\|_{C^1} \le C\lambda^{-n}\|f\|_{C^1} + C\|f\|_{C^0}.$$

(c) Show that the unit ball of $C^1(S^1)$ is compactly embedded in $C^0(S^1)$. More generally, the unit ball of $C^\alpha(S^1)$ is compactly embedded in $C^\beta(S^1)$ for any $0 \le \beta < \alpha \le 1$.

45. *Lasota–Yorke inequalities for a contracting map.* Let $I = [0,1]$, $T\colon I \to I$ be a C^1 contracting map, i.e., $\sup |DT| \le \lambda < 1$. For $\alpha \in [0,1]$, let $(C^\alpha)^*$ denote the dual of $C^\alpha(I)$ and for $\mu \in (C^\alpha)^*$, define

$$\|\mu\|_\alpha = \sup_{\substack{\varphi \in C^\alpha(I) \\ |\varphi|_{C^\alpha} \le 1}} \mu(\varphi).$$

(a) Verify that $\|\mu\|_\alpha$ is a norm. Show that the unit ball of $\mathcal{B}_\alpha$ is compactly embedded in $\mathcal{B}_\beta$ for any $0 \le \alpha < \beta \le 1$, where $\mathcal{B}_\alpha$ is $(C^\alpha)^*$ equipped with the $\|\cdot\|_\alpha$ norm.

(b) For $\mu \in (C^\alpha)^*$, define the transfer operator for T by

$$\mathcal{L}\mu(\varphi) = \mu(\varphi \circ T), \qquad \text{for all } \varphi \in C^\alpha(I).$$

Thus $\mathcal{L}\mu \in (C^\alpha)^*$. Prove that $\mathcal{L}$ satisfies

$$\|\mathcal{L}^n \mu\|_\alpha \le \lambda^{\alpha n}\|\mu\|_\alpha + \|\mu\|_1, \qquad \|\mathcal{L}^n \mu\|_1 \le \|\mu\|_1.$$

(c) In general, elements of $\mathcal{B}_\alpha$ are distributions. Yet suppose that μ is an element of the peripheral spectrum of $\mathcal{L}$ acting on $\mathcal{B}_\alpha$, i.e., $\mathcal{L}\mu = e^{i\theta}\mu$. Show that for all $\varphi \in C^\alpha$,

$$|\mu(\varphi)| \le |\varphi|_{C^0}\|\mu\|_\alpha,$$

so that in fact μ is a measure. (Hint: Use $\mathcal{L}^n\mu(\varphi) = \mu(\varphi \circ T^n)$ for each n.)

46. *Ruelle–Perron–Frobenius theorem as generalization of Perron–Frobenius theorem.* Let X be the one-sided full shift on d symbols, and consider a potential function $\phi\colon X \to \mathbb{R}$ that is constant on 2-cylinders: thus there is a $d \times d$ matrix A with positive entries such that $e^{\phi(x)} = A_{ij}$ whenever x begins with the symbols ij, and the corresponding RPF operator $\mathcal{L}\colon C(X) \to C(X)$ is given by

$$(\mathcal{L}h)(x) = \sum_{i=1}^{d} A_{ix_0} h(ix_0).$$

(a) Define $\iota\colon \mathbb{R}^d \to C(X)$ by $\iota(u)(x) = u_{x_0}$, and show that the subspace $\iota(\mathbb{R}^d)$ is $\mathcal{L}$-invariant, with $\mathcal{L}h = \iota(uA)$ for $h = \iota(u)$. (This is the space of functions that are constant on 1-cylinders.) In particular, $\mathcal{L}h = \lambda h$ if and only if $uA = \lambda u$ (treating u as a row vector).

(b) Prove that the dual operator $\mathcal{L}^*$ has the property that

$$(\mathcal{L}^* m)[w_1 \dots w_n] = A_{w_1 w_2} m[w_2 \dots w_n].$$

Use this to prove that m is an eigenmeasure for $\mathcal{L}$ if and only if

$$m[w_1 \dots w_n] = A_{w_1 w_2} \cdots A_{w_{n-1} w_n} v_{w_n} \lambda^{-(n-1)},$$

where $v \in \mathbb{R}^d$ is a right eigenvector for A: $Av = \lambda v$.

(c) Prove that the measure m from the previous part is a Markov measure and find the associated stochastic matrix Π. Prove that this measure need not be shift-invariant, and find the equivalent invariant Markov measure by showing that $(u_i v_i)_i$ gives a stationary vector for Π, where u, v are left and right Perron–Frobenius eigenvectors for A.

GEOMETRY

Geodesic flows, Jacobi fields, Riccati equation.

Basic definitions. Let M be a smooth surface with a Riemannian metric. Each unit tangent vector $v \in T^1M$ determines a unique geodesic γ_v such that $\dot\gamma_v(0) = v$. The *geodesic flow* on T^1M is defined by

$$f_t(v) = \dot\gamma_v(t).$$

Many dynamical properties of the geodesic flow are related to curvature. Let $\kappa(x)$ denote the Gaussian curvature of the surface M at $x \in M$. Given a smooth one-parameter family of geodesics γ_s, the vectors $\frac{\partial \gamma_s(t)}{\partial s}|_{s=0}$ are said to form a *Jacobi field* along the geodesic γ_0. If these vectors are orthogonal to $\dot\gamma_0(t)$ for each t, then the Jacobi field is said to be orthogonal.

Given a geodesic $\gamma\colon \mathbb{R} \to M$, fix a continuous unit vector field along γ that is orthogonal to γ; that is, $X(t) \in T^1_{\gamma(t)}M$ and $X(t) \perp \dot\gamma(t)$. Then any smooth orthogonal vector field along γ can be written as $J(t)X(t)$, where $J\colon \mathbb{R} \to \mathbb{R}$. It can be shown that this is an orthogonal *Jacobi* field if and only if J satisfies the Jacobi equation

$$J''(t) + \kappa(\gamma(t))J(t) = 0.$$

47. Let $\mathcal{J}(\gamma)$ be the set of orthogonal Jacobi fields along a geodesic γ. Prove that $\mathcal{J}(\gamma)$ is a vector space, and determine its dimension.

48. Let M be a surface with constant Gaussian curvature, and determine $\mathcal{J}(\gamma)$ explicitly.

49. Two points $x, y \in M$ are said to be *conjugate* along a geodesic γ connecting them if there is a non-zero orthogonal Jacobi field along γ that vanishes at both x and y.

(a) Prove that if M has constant *positive* Gaussian curvature, then it has conjugate points.

(b) Prove that if M has nonpositive curvature (not necessarily constant), then $t \mapsto \|J(t)\|$ is convex, and thus M has no conjugate points.

50. An orthogonal Jacobi field is said to be *stable* if $\|J(t)\|$ is bounded for $t \geq 0$, and *unstable* if $\|J(t)\|$ is bounded for $t \leq 0$. Write $\mathcal{J}^s(\gamma)$ and $\mathcal{J}^u(\gamma)$ for the sets of stable and unstable orthogonal Jacobi fields.

(a) Prove that $\mathcal{J}^s(\gamma)$ and $\mathcal{J}^u(\gamma)$ are both subspaces.

(b) Prove that if M has constant *negative* curvature, then $\mathcal{J}(\gamma) = \mathcal{J}^s(\gamma) \oplus \mathcal{J}^u(\gamma)$, and stable and unstable Jacobi fields contract exponentially quickly in the future and in the past, respectively. (Thus the geodesic flow is hyperbolic, and these give the stable and unstable directions.)

51. Show that if J solves the Jacobi equation $J'' + \kappa J = 0$, then $U = J'/J$ solves the Riccati equation $U' + U^2 + \kappa = 0$. Solve this equation when κ is constant.

Hyperbolic geometry.

52. *Coding of limit set.* Let S be a hyperbolic one holed torus. In other words, $S = \Gamma\backslash\mathbb{D}$ where Γ is a Schottky group. More precisely, $\Gamma = \langle g_1, g_2\rangle$ where g_i are Möbius transformations given as below:

(a) g_1 and g_2 are positive isometries on $\mathbb{D}$ not fixing o;

(b) Let $D(g)$ denote the semi-disk given the perpendicular bisector of o and $g(o)$ containing o. Assume $\overline{D(g_i^{\pm})}$ are all disjoint (see Fig 1);

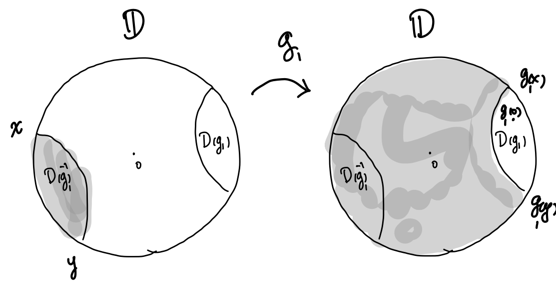


Figure 1. A positive isometry g_1 and $D(g_1)$

(c) $D(g_i^{\pm})$ on $\mathbb{D}$ are ordered as in Fig 2.

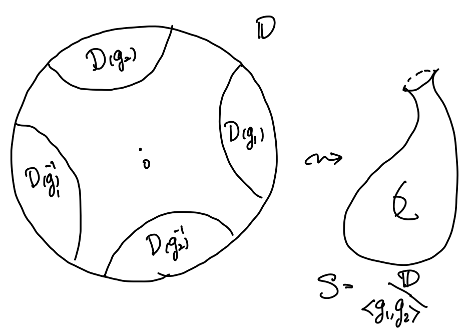


Figure 2. Schottky group for one holed torus

Let $\Lambda_\Gamma \subset \partial\mathbb{D}$ be the limit set of Γ. Show there there exists a homeomorphism $\omega : \Sigma_A^+ \to \Lambda_\Gamma$ where Σ_A^+ is the SFT given by $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.

BILLIARDS

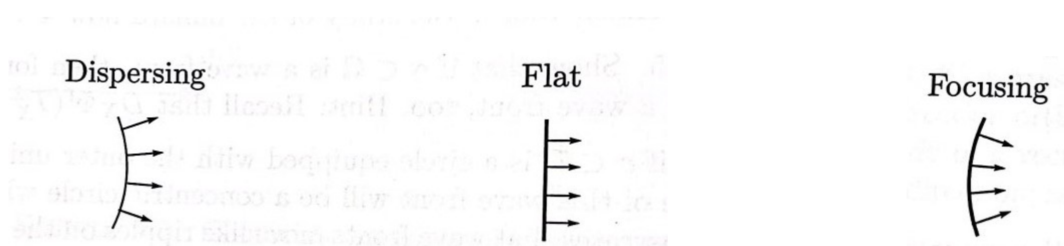
Basic definitions. Let Q be a connected flat surface with boundary; we will consider both the cases $Q \subset \mathbb{R}^2$ and $Q \subset \mathbb{T}^2$. Assume that the boundary ∂Q is a finite union of smooth compact closed curves, and that each curve is either:

- *flat* (a straight line segment);
- *dispersing* (curves away from the interior of Q); or
- *focusing* (curves towards the interior of Q).

For example, if Q is a square then all its boundary components are flat. If Q is a disc then it has a focusing boundary. If Q is the torus $\mathbb{T}^2$ with several discs removed, then Q has dispersing boundary.

We call Q a *billiard table.* It defines a billiard system in which a particle located in Q moves in a straight line until it encounters ∂Q, at which point it reflects, with outgoing angle (to the tangent line of ∂Q) equal to incoming angle. Each collision is completely described by a pair (q, φ), where $q \in \partial Q$ and φ is the outgoing angle; usually φ is measured with respect to the inward-pointing normal, so that $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, but sometimes φ is measure with respect to the tangent line, so that $\varphi \in (0, \pi)$. The *billiard map* F takes a pair (q, φ) to the pair (q_1, φ_1) that represents the next collision with the boundary. The set $M = \partial Q \times (-\frac{\pi}{2}, \frac{\pi}{2})$ is called the *phase space of the billiard map.*

It is sometimes useful to consider a family of trajectories given by a *wavefront*: a smooth curve in Q equipped with a continuous family of unit normal vectors. The figure shows three important kinds of wavefronts (pictures copied from Chernov–Markarian).



Useful references:

- "Geometry and Billiards", Serge Tabachnikov, AMS-STML volume 30, `http://www.personal.psu.edu/sot2/books/billiardsgeometry.pdf`
- "Mechanisms of chaos in billiards: dispersing, defocusing and nothing else", Leonid A Bunimovich, Nonlinearity, 2018.
- `https://blogs.ams.org/visualinsight/2016/11/15/bunimovich-stadium/`
- "Hyperbolic billiards", Maciej P. Wojtkowski, `https://www.math.arizona.edu/~maciejw/hb11.pdf`
- "Design of hyperbolic billiards", Maciej P. Wojtkowski, `https://www.math.arizona.edu/~maciejw/dhb1.pdf`
- "Chaotic billiards", book by Chernov and Markarian, 2015.

53. Let Q be a square in $\mathbb{R}^2$.

(a) Show that flat wavefronts remain flat after hitting the (flat) boundary of Q.

(b) Show that nearby trajectories diverge at most linearly fast, so the billiard is not hyperbolic.

54. Let Q be a disc in $\mathbb{R}^2$.

(a) Show that a flat wavefront becomes focusing after hitting the (focusing) boundary of Q.

(b) Show that for every billiard trajectory, there is a smaller circle C concentric with Q such that the billiard trajectory has a tangency with C after every collision. C is called a *caustic*.

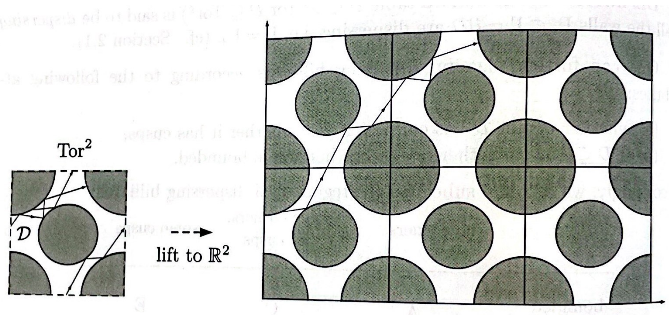
(c) Show that the billiard map is topologically conjugate to the *twist map* $(x, y) \mapsto (x+y, y)$ on $S^1 \times (0, 1)$, and that nearby trajectories diverge at most linearly fast, so the billiard is not hyperbolic.

55. Show that a flat wavefront that hits a dispersing boundary becomes dispersing. If a general argument is too hard, consider the case when the dispersing boundary is a circle (but now we are hitting it from outside, instead of from inside as in the previous exercise). One can also show that a dispersing wavefront remains dispersing after hitting a dispersing boundary.

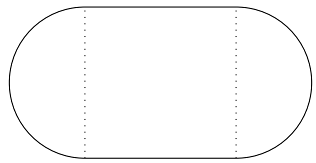
56. Consider a wavefront that is dispersing immediately after a collision; sketch the curve in phase space M corresponding to this wavefront. What can you say about the slope of this curve? Do the same thing for a focusing wavefront.

57. Show that if a focusing wavefront evolves for long enough without hitting a boundary, then it becomes dispersing. (This is the *defocusing* mechanism.)

58. If Q is obtained by removing a finite number of strictly convex regions (*scatterers*) from $\mathbb{T}^2$ then we refer to the resulting billiard system as a *Sinai billiard*. All the boundary components are dispersing so one can show that the system is hyperbolic. One can also consider the corresponding billiard in the plane by repeating the scatterers periodically, as shown in the picture below of a *periodic Lorentz gas* (again taken from Chernov–Markarian); then each billiard trajectory on the torus lifts to a billiard trajectory in the plane. Find a periodic orbit on the torus that lifts to a nonperiodic orbit.



59. Bunimovich showed that if Q is the *stadium* shown in the picture, which consists of a rectangle with two half-discs attached to opposite sides, then the corresponding billiard is (non-uniformly) hyperbolic. Thus it is possible for billiards in convex domains to display hyperbolic behavior.



(a) Why is this surprising? What is the difficulty for proving hyperbolicity in convex billiards?

(b) What type of tools are available to establish such a result?

60. Let $Q \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, $\gamma(s) \in \partial Q$ be the arclength parameterization of the boundary ∂Q (oriented counterclockwise). Let $M = \partial Q \times (0, \pi)$ be the phase space, where $x = (s, \theta)$ represents an orbit starting at $\gamma(s)$ with initial direction $R_\theta(\dot{\gamma}(s))$. Let $F \colon M \to M$ be the billiard map on M.

(a) Let $h(s, s_1) = |\gamma(s_1) - \gamma(s)|$ be the Euclidean distance between any two points on ∂Q. Show that $dh = -\cos\theta ds + \cos\theta_1 ds_1$.

(b) Show that F preserves the 2-form $\omega = \sin\theta ds \wedge d\theta$.