ELECTROSTATIC APPROXIMATION OF VECTOR FIELDS.

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Abstract. This paper provides expressions for the boundary potential that provides the best electrostatic potential approximation of a given L^2vector field on a nice bounded region in $\mathbb{R}^N$ The permittivity of the region is assumed to be known and the potential is required to be zero on the conducting part of the boundary. The boundary potential is found by solving the minimization conditions and using a special basis of the trace space for the space of allowable potentials. The trace space is identified by its representation with respect to a basis of $\Sigma-$Steklov eigenfunctions.

1. Introduction

Quite often in physical applications one wishes to produce electrostatic fields in a region of known permittivity $\epsilon(x)$ that approximate a prescribed (given) field $\mathbf{F}$. Such fields are determined by their boundary values, so a natural question is what imposed potentials on the boundary provide good approximations, in an energy norm, to $\mathbf{F}$? Very often the boundary includes surface patches that are conductors as well as patches where nonzero potentials may be imposed. This may be regarded as a problem of stationary control or approximation.

The difficulty with such problems has been how to work with the control space of allowable boundary conditions as it will be a trace space of allowable H^1 functions. The standard Lions-Magenes description of trace spaces is not amenable to nice constructions of solutions for problems of this type. In this paper this problem is treated using methods based on the spectral characterization of trace spaces as described in Auchmuty [3] and [4] which provides constructive methods using explicit bases. The allowable trace space is characterized as being isomorphic to a class of weak solutions of a linear elliptic equation and a basis of $\Sigma-$Steklov eigenfunctions of this space is identified.

Here it is shown that the best approximation has an explicit expression in terms of the data and such a basis. Moreover boundary data that is close in an explicit boundary norm to this optimal solution will provide good approximations to the field $\mathbf{F}$ in the energy norm on Ω. The analysis described here is described for quite general N-dimensional regions since the results are essentially independent of the dimension $N \geq 2$ and the methods may be applicable to other approximation questions.

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2. Definitions and Requirements.

To analyze this problem, standard definitions, terminology and assumptions will be used as in Attouch, Buttazzo and Michaille [1]. All functions in this paper will take values in $\overline{\mathbb{R}} := [-\infty, \infty]$, derivatives should be taken in a weak sense and $N \geq 2$ throughout.

A *region* is a non-empty, connected, open subset of $\mathbb{R}^N$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. Let $L^p(\Omega), H^1(\Omega)$ be the usual real Lebesgue and Sobolev spaces of functions on Ω. The norm on $L^p(\Omega)$ is denoted $\|.\|_p$ and the inner product on $L^2(\Omega)$ by $\langle ., . \rangle$. The basic requirement on Ω is

(B1): *Ω is a bounded region in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area.*

The region Ω is said to satisfy *Rellich's theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ where $p_S(N) := 2N/(N-2)$ when $n \geq 3$, or $p_S(2) = \infty$ when $N = 2$.

The *trace map* is the linear extension of the map restricting Lipschitz continuous functions on $\overline{\Omega}$ to $\partial\Omega$. When (B1) holds, this map has an extension to $W^{1,1}(\Omega)$ and then the trace of u on $\partial\Omega$ will be Lebesgue integrable with respect to σ, see [5], Section 4.2 for details. The region Ω is said to satisfy the *compact trace theorem* provided the trace mapping $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is compact. We will use the inner product

$$[u, v]_\partial := \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \int_{\partial\Omega} \gamma(u)\, \gamma(v)\, d\sigma \tag{2.1}$$

on $H^1(\Omega)$ and the associated norm is denoted $\|u\|_\partial$. This is an equivalent inner product to the usual inner product when Ω obeys (B1) - see [2] for a proof. Here $\nabla u := (D_1 u, \ldots, D_n u)$ is the gradient of the function u

Our interest is in a problem that arises in electrostatics where part of the boundary Σ is a conductor and a potential can be imposed on the complementary part of the boundary $\tilde{\Sigma} := \partial\Omega \setminus \overline{\Sigma}$. Mathematically our requirements are

(B2): Σ is an nonempty open subset of $\partial\Omega$, Σ and $\tilde{\Sigma}$ have strictly positive surface measure and $\sigma(\partial\Sigma) = 0$.

A function $u \in H^1(\Omega)$ is said to be in $H^1_{\Sigma 0}(\Omega)$ provided $\gamma(u) = 0$, $\sigma a.e.$ on Σ. This is equivalent to requiring that

$$\int_{\partial\Omega} \gamma(u)\, \gamma(v)\; d\sigma = 0 \quad \text{for all} \quad v \in X_\Sigma \tag{2.2}$$

Let X be the space $H^1(\Omega) \cap C(\overline{\Omega})$ and X_Σ be the subspace of functions in X with $supp\, v \cap \partial\Omega \subset \Sigma$. The space $H^1_{\Sigma 0}(\Omega)$ is a closed subspace of $H^1(\Omega)$ that contains $H^1_0(\Omega)$.

3. The Boundary Control Problem

The problem to be studied here is given a vector field $\mathbf{F}$ on Ω to find the potential φ that provides the best L^2-approximation when the region Ω has known permittivity tensor $\epsilon(.)$ and part of the boundary Σ is a conductor held at zero potential. That is we want to find the function $\tilde{\varphi}$ that minimizes

$$\| \epsilon\nabla\varphi - \mathbf{F} \|_{\mathbf{2}}^{\mathbf{2}} \ := \ \int_{\mathbf{\Omega}} |\epsilon\nabla\varphi \ - \ \mathbf{F}|^{\mathbf{2}} \ \mathbf{dx} \qquad \text{over all} \quad \varphi \in \mathbf{H^1_{\Sigma 0}(\Omega)}. \tag{3.1}$$

with $|.|$ is the Euclidean norm on $\mathbb{R}^N$. Since $\mathbf{F}$ is known this reduces to minimizing the functional

$$\mathcal{E}(\varphi) \ := \ \int_{\Omega} \left[\, (A(x)\nabla\varphi)\cdot\nabla\varphi \ - \ 2\,\mathbf{G}\cdot\nabla\varphi \right] \mathbf{dx} \qquad \text{over all} \quad \varphi \in \mathbf{H^1_{\Sigma 0}(\Omega)}. \tag{3.2}$$

where $A(x) \ := \ \epsilon(x)^T \, \epsilon(x)$ is real symmetric, $\mathbf{G(x)} \ := \ \epsilon\mathbf{(x)^T F(x)}$ on Ω and the superscript T denotes the vector transpose. The following will be assumed.

(A1): $A(x) \ := \ (a_{jk}(x))$ is a real symmetric matrix whose components are bounded Lebesgue-measurable functions on Ω and there exist constants c_0, c_1 such that

$$c_0\,|\xi|^2 \ \leq \ \xi^T A(x)\xi \ \leq \ c_1\,|\xi|^2 \qquad \text{for all} \quad \xi \in \mathbb{R}^N, \ x \in \Omega. \tag{3.3}$$

Existence uniqueness and extremality conditions for this problem may be obtained using standard methods. The problem is a convex quadratic minimization problem on a Hilbert space so the existence may be stated as follows.

Theorem 3.1. *Assume that (A1), (B1), (B2) hold and* $\mathbf{F} \in \mathbf{L^2(\Omega : \mathbb{R}^N)}$ *is given, then there is a unique minimizer* $\tilde{\varphi}$ *of* $\mathcal{E}$ *on* $H^1_{\Sigma 0}(\Omega)$.

The functional $\mathcal{E}$ also is G-differentiable so the minimizers satisfy the following.

Theorem 3.2. *Assume that (A1), (B1), (B2) hold and* $\mathbf{F} \in \mathbf{L^2(\Omega : \mathbb{R}^N)}$ *is given, then the minimizer* $\tilde{\varphi}$ *of* $\mathcal{E}$ *on* $H^1_{\Sigma 0}(\Omega)$ *satisfies the equation*

$$\int_{\Omega} (A(x)\nabla\varphi \ - \ \mathbf{G})\cdot\nabla\psi \ \mathbf{dx} \ = \ \mathbf{0} \qquad \textit{for all} \quad \psi \in \mathbf{H^1_{\Sigma 0}(\Omega)}. \tag{3.4}$$

To obtain further results about this problem a decomposition of the space $H^1_{\Sigma 0}(\Omega)$ will be used. Consider the bilinear form $a : H^1_{\Sigma 0}(\Omega) \to \mathbb{R}$ defined by

$$a(u,v) \ := \ [u,v]_a \ := \ \int_{\Omega} \ (A\nabla u)\cdot\nabla v \ \ d^N x \, + \int_{\tilde{\Sigma}} \ \gamma(u)\,\gamma(v) \ \ d\sigma \tag{3.5}$$

This bilinear form defines the a-inner product on $H^1_{\Sigma 0}(\Omega)$ and is equivalent to the $\partial-$norm (2.1) when A satisfies (A1).

Observe that a function $w \in H^1_{\Sigma 0}(\Omega)$ is a-orthogonal to $H^1_0(\Omega)$ if and only if

$$\int_{\Omega} (A\nabla w)\cdot\nabla v \, dx \ = \ 0 \quad \text{for all} \quad v \in H^1_0(\Omega). \tag{3.6}$$

That is $\gamma(w)$ is zero on Σ and $\mathcal{L}_A w(x) := \text{div}\,(A\nabla w)(x) = 0$ in a weak sense on Ω. The space of all such functions will be denoted $N(\mathcal{L}_A, \Sigma)$ so we have the orthogonal decomposition

$$H^1_{\Sigma 0}(\Omega) \;=\; H^1_0(\Omega) \oplus_a N(\mathcal{L}_A, \Sigma) \tag{3.7}$$

where $\oplus_a$ indicates that the a-inner product is used.

In light of this result, the minimizer $\tilde{\varphi}$ has a decomposition of the form $\tilde{\varphi} = \phi_0 + \phi_b$ where ϕ_0 is the minimizer of $\mathcal{E}$ on $H^1_0(\Omega)$ and $\phi_b \in N(\mathcal{L}_A, \Sigma)$ is the solution of

$$\int_\Omega (A\nabla\varphi)\cdot\nabla\psi\,dx \;=\; \int_\Omega (\mathbf{G} \,-\, \mathbf{A}\nabla\varphi_{\mathbf{0}})\cdot\nabla\psi\;\mathbf{dx} \quad \text{for all} \quad \psi \in \mathbf{N}(\mathcal{L}_{\mathbf{A}}, \mathbf{\Sigma}). \tag{3.8}$$

The fact that ϕ_0 is a solution of the extremality condition on $H^1_0(\Omega)$ implies that $\text{div}\,(\,A\nabla\varphi_0 \,-\, \mathbf{G}\,) = \mathbf{0}$ on Ω in a weak sense so this last equation may be written

$$\int_\Omega (A\nabla\varphi)\cdot\nabla\psi\,dx \;=\; \int_{\tilde{\Sigma}} \psi\,(\mathbf{G} \,-\, \mathbf{A}\nabla\varphi_{\mathbf{0}})\cdot\nu\;\mathbf{d}\sigma \quad \text{for all} \quad \psi \in \mathbf{N}(\mathcal{L}_{\mathbf{A}}, \mathbf{\Sigma}). \tag{3.9}$$

This is the weak form of the equation $\mathcal{L}_A\varphi = 0$ on Ω subject to the boundary conditions

$$\varphi(z) \;=\; 0 \qquad \text{on} \quad \Sigma \quad \text{and} \quad A\nabla\varphi\cdot\nu \;=\; (\mathbf{G} \,-\, \mathbf{A}\nabla\varphi_{\mathbf{0}})\cdot\nu \qquad \text{on} \quad \tilde{\mathbf{\Sigma}}. \tag{3.10}$$

The solution of the problem for ϕ_0 is a standard Dirichlet boundary value problem and it is worth noting that the value of $\mathcal{E}(\phi_0) = 0$ if and only if $\text{div}\;\mathbf{G} = \mathbf{0}$ on Ω in a weak sense. In this case the general problem reduces to that of solving (3.8) or (3.9) alone.

Our interest is in the problem of finding an expression for the boundary trace of $\tilde{\varphi}$ or ϕ_b on $\tilde{\Sigma}$. That is what boundary data gives the best approximating potential for the given field $\mathbf{F}$ on Ω?

4. Bases and Representations of $N(\mathcal{L}_A, \Sigma)$

To find the boundary data that provides the best L^2-approximation to the field $\mathbf{F}$, an orthogonal basis of the space consisting of certain Steklov-type eigenfunctions of $\mathcal{L}_A$ is constructed and used. This will yield a spectral representation of ϕ_b as described in theorem 5.xx below.

An a-orthonormal basis of $N(\mathcal{L}_A, \Sigma)$ may be found using the algorithm described in Auchmuty [4]. In the notation of that paper take $V = N(\mathcal{L}_A, \Sigma)$, a as above and

$$m(u,v) \;:=\; \int_{\tilde{\Sigma}} \gamma(u)\,\gamma(v)\;d\sigma. \tag{4.1}$$

A function $\chi \in N(\mathcal{L}_A, \Sigma)$ is said to be a $\Sigma-$Steklov eigenfunction of $\mathcal{L}_A$ on Ω provided it is a solution of

$$\int_\Omega (A\nabla\chi)\cdot\nabla\psi\,dx \;=\; \lambda\int_{\tilde{\Sigma}} \gamma(\chi)\,\gamma(\psi)\;d\sigma \quad \text{for all} \quad \psi \in H^1_{\Sigma 0}(\Omega). \tag{4.2}$$

This problem has the form of equation 2.1 of [4] and the bilinear forms a, m satisfy conditions (A1)-(A4) of that paper. Moreover condition (A5) there holds with $H = L^2(\tilde{\Sigma}, d\sigma)$.

Define $\mathcal{A}(\chi) = a(\chi,\chi)$, $\mathcal{M}(\chi) := m(\chi,\chi)$ and C_1 to be the closed unit ball in $H^1_{\Sigma 0}(\Omega)$ with respect to the a-norm. Consider the variational problem of maximizing $\mathcal{M}$ on C_1. This problem has maximizers $\pm\chi_1$ that have a-norm 1 and are solutions of (4.2) associated with an eigenvalue $\lambda_1 > 0$. Moreover one has the coercivity inequality

$$\mathcal{A}(\chi) \;\geq\; (\lambda_1 + 1) \int_{\tilde{\Sigma}} \gamma(\chi)^2 \, d\sigma \quad \text{for all} \quad \chi \in H^1_{\Sigma 0}(\Omega). \tag{4.3}$$

Using the construction of section 4 of [4], a countably infinite a-orthonormal basis $\mathcal{B} := \{\chi_j : j \geq 1\}$ of $N(\mathcal{L}_A, \Sigma)$ may be constructed using a sequence of constrained maximization problems for $\mathcal{M}$.

These eigenfunctions also are m-orthogonal so that $m(\chi_j, \chi_k) = 0$ when $j \neq k$. Define $\tilde{\chi_j} := \chi_j/\sqrt{\lambda_j + 1}$ then $\tilde{\mathcal{B}} := \{\, \gamma(\tilde{\chi_j}) : j \geq 1\}$ will be an m-orthonormal basis of $L^2(\tilde{\Sigma}, d\sigma)$ from theorem 4.6. Since these functions constitute bases of the various Hilbert spaces, there is a spectral representation of functions $\varphi \in N(\mathcal{L}_A, \Sigma)$ in terms of their boundary values. Namely when $\varphi \in N(\mathcal{L}_A, \Sigma)$ then,

$$\varphi(x) \;=\; \sum_{j=1}^{\infty} c_j \; \tilde{\chi_j}(x) \qquad \text{on} \quad \Omega \qquad \text{with } c_j := m(\varphi, \tilde{\chi_j}) \tag{4.4}$$

and this series converges strongly in $H^1_{\Sigma 0}(\Omega)$. Thus these constructions yield the following result.

Theorem 4.1. *Assume that (A1), (B1), (B2) hold and $\mathcal{B}, \tilde{\mathcal{B}}$ are defined as above. If $\varphi \in N(\mathcal{L}_A, \Sigma)$ then* (4.4) *holds and the series converges in a-norm. Moreover $a(\varphi,\varphi) = \sum_{j=1}^{\infty} (1+\lambda_j)\, c_j^2$.*

Let $H^{1/2}(\tilde{\Sigma})$ be the subspace of $L^2(\tilde{\Sigma}, d\sigma)$ of all functions with $\sum_{j=1}^{\infty} (1+\lambda_j)\, c_j^2 < \infty$. It will be a Hilbert space with respect to the inner product

$$\langle \varphi, \psi \rangle_{1/2, \tilde{\Sigma}} \;:=\; \sum_{j=1}^{\infty} (1+\lambda_j)\, m(\phi, \chi_j)\, m(\psi, \chi_j) \tag{4.5}$$

In particular this yields an isomorphism between functions in $N(\mathcal{L}_A, \Sigma)$ and functions in $H^{1/2}(\tilde{\Sigma})$. Thus $H^{1/2}(\tilde{\Sigma})$ may be regarded as the boundary trace subspace (on $\tilde{\Sigma}$) of functions in $N(\mathcal{L}_A, \Sigma)$ and this is an isometry with $a(\varphi,\varphi) \;=\; \langle \varphi, \varphi \rangle^2_{1/2, \tilde{\Sigma}}$.

5. The best approximating potential

We are now in a position to specify the boundary data for the potential that minimizes (3.1). The preceding analysis enables the derivation of an explicit representation of the solution ϕ_b of (3.8) or (3.9). Equation (3.8) implies that ϕ_b satisfies

$$a(\phi_b, \tilde{\chi_j}) \;=\; g_j \;:=\; \int_{\Omega} (\mathbf{G} \,-\, \mathbf{A}\nabla\varphi_{\mathbf{0}}) \cdot \nabla\tilde{\chi}_{\mathbf{j}} \; \mathbf{dx} \quad \text{for} \quad \mathbf{j} \geq \mathbf{1}.$$

Note that the g_j depend onlds only on the data, the eigenfunction $\tilde{\chi_j}$ and the solution ϕ_0 of the zero-Dirichlet variational problem. Then the eigenfunction equation (4.2) yields that the solution is

$$\phi_b(x) \ = \ \sum_{j=1}^{\infty} \frac{g_j}{(1+\lambda_j)} \ \tilde{\chi_j}(x). \tag{5.1}$$

Thus the boundary trace on $\tilde{\Sigma}$ of the best approximation is given by the boundary trace of this right hand side. That is imposing Dirichlet boundary data $\gamma(\phi_b)$ on $\tilde{\Sigma}$ given by (5.1) yields the minimizing potential ϕ_b of $\mathcal{E}$.

Theorem 5.1. *Assume that (A1), (B1), (B2) hold and $\tilde{\mathcal{B}}, g_j$ are defined as above. Then the potential $\tilde{\varphi}$ that minimizes the norm in* (3.1) *or $\mathcal{E}$ on $H^1_{\Sigma 0}(\Omega)$ is given by $\tilde{\varphi} = \phi_0 + \phi_b$ where ϕ_0 minimizes $\mathcal{E}$ on $H^1_0(\Omega)$ and ϕ_b is given by* (5.1). *When* div $\mathbf{G} = \mathbf{0}$ *on Ω, then $\mathbf{F} = \epsilon \nabla \tilde{\varphi}$ on Ω.*

Proof. This is a restatement of the preceding results. Note that when div $\mathbf{G} = \mathbf{0}$ on Ω, then $\phi_0 = 0$ so the last sentence holds. □

Moreover when the boundary potential φ is a good approximation of this ϕ_b in the norm of $H^{1/2}(\tilde{\Sigma})$ then the fact that the a-norm and the norm on $H^{1/2}(\tilde{\Sigma})$ are isometric implies that such potentials φ will provide a good approximation of $\mathbf{F}$ on Ω in any equivalent norm on $H^1_{\Sigma 0}(\Omega)$.

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