FINITE ENERGY SOLUTIONS OF SELF-ADJOINT ELLIPTIC MIXED BOUNDARY VALUE PROBLEMS.

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Abstract. This paper describes existence, uniqueness and special eigenfunction representations of H^1-solutions of second order, self-adjoint, elliptic equations with both interior and boundary source terms. The equations are posed on bounded regions with Dirichlet conditions on part of the boundary and Neumann conditions on the complement. The system is decomposed into separate problems defined on orthogonal subspaces of $H^1(\Omega)$. One problem involves the equation with the interior source term and the Neumann data. The other problem just involves the homogeneous equation with Dirichlet data. Spectral representations of the solution operators for each of these problems are found. The solutions are described using bases that are, respectively, eigenfunctions of the differential operator with mixed null boundary conditions, and certain mixed Steklov eigenfunctions. These series converge strongly in $H^1(\Omega)$. Necessary and sufficient conditions for the Dirichlet part of the boundary data to have a finite energy extension are described. The solutions for a problem that models a cylindrical capacitor is found explicitly.

1. Introduction

The *mixed Dirichlet-Neumann boundary value problems* to be investigated here involve second order uniformly elliptic equations with Dirichlet data imposed on part of the boundary and Neumann (given flux, or conormal) data on the complementary part of the boundary. Dirichlet-Neumann (DN) mixed problems sometimes are called Zaremba boundary value problems in recognition of [25].

First conditions for the existence of finite energy (H^1-)solutions of self-adjoint, second-order equations of the form

$$\mathcal{L}\, u(x) \;:=\; -\operatorname{div}\left(A(x)\,\nabla u(x)\right) \,+\, a_0(x)u(x) \;=\; \rho(x) \quad \text{on } \Omega, \tag{1.1}$$

subject to the mixed Dirichlet and Neumann boundary conditions

$$u(x) \;=\; \eta_1(x) \quad \text{on } \Sigma, \text{ and} \qquad (A(x)\,\nabla u(x))\cdot\nu(x) \;=\; \eta_2(x) \quad \text{on } \tilde{\Sigma}. \tag{1.2}$$

are described. Then uniqueness is shown and explicit spectral representations of these solutions are obtained. Here Ω is a region in $\mathbb{R}^n, \partial\Omega$ is its boundary, Σ is a nonempty open subset of $\partial\Omega$ and $\tilde{\Sigma} := \partial\Omega \setminus \overline{\Sigma}$. Further assumptions on the coefficients and other data will be specified later. The case $A(x) \equiv I_n,\ \mathcal{L} = c - \Delta$ with c a constant is the standard model for these systems. In both electromagnetic field theory and fluid mechanics, the restriction

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to finding solutions in $H^1(\Omega)$ is a standard physical restriction as well as being a natural setting for variational methods.

The analysis to be described here is based on a decomposition of this inhomogeneous system into two distinct problems. One is an elliptic boundary value problem with homogeneous Dirichlet data $\eta_1 \equiv 0$. This will be investigated using variational methods and the solutions are found using expansions involving the mixed D-N eigenfunctions. These solutions are described in sections 4 and 6, while the descriptions of the associated basis of mixed D-N eigenfunctions is described in section 5. The strong convergence of the expansion to the exact solution is proved.

The second problem involves non-zero Dirichlet data η_1 on Σ with the source ρ and the boundary flux η_2 taken to be zero. This is treated as an extension problem for the boundary data η_1. The solution is found as an infinite series involving certain mixed Steklov eigenfunctions. Necessary and sufficient conditions for the function η_1 to be a trace of an $H^1(\Omega)$ function on Σ are found. This provides an intrinsic description of the spaces of acceptable Dirichlet boundary data. The associated eigenfunction expansion may be regarded as an spectral representation of certain integral operators associated with the Dirichlet data. This generalizes similar results described in [4] for the standard $(\Sigma = \partial\Omega)$ Dirichlet boundary value problem for elliptic problems on general regions in $\mathbb{R}^n$.

The eigenproblems are described in sections 5 and 8 respectively. The eigenfunctions are constructed using a sequence of constrained variational problems. The variational principles used here are different to those usually used for elliptic eigenproblems and enable straightforward proofs that appropriately normalized eigenfunctions constitute orthonormal bases of the respective Hilbert subpaces of $H^1(\Omega)$. One eigenproblem generates mixed Dirichlet-Neumann eigenfunctions associated with Ω, Σ. The other generates certain *mixed Steklov* eigenfunctions that provide a basis of solutions of the homogeneous equation $\mathcal{L}\,u = 0$. For the case $A(x) \equiv A$ is constant, these eigenfunction expansions may be interpreted as spectral representations of some classical integral formulae. The methodology to be used here is independent of the dimension n ≥ 2 and may be regarded as an alternative approach to some uses of boundary integral equations.

The analysis here is illustrated by the explicit calculation, in section 10, of the series representation of the solution of a mixed boundary value problem for Poisson's equation in a circular cylinder. This equation is a standard model for a cylindrical capacitor. This series converges in $H^1(\Omega)$ under very simple, and easily verifiable, conditions on the source and the boundary data.

Mixed Dirichlet-Neumann elliptic problems arise in a wide range of physical and engineering applications. They are common in the modeling of electromagnetic fields where different parts of the boundary have different physical properties. They also arise in some fluid mechanical situations including sloshing problems, situations where there are immersed surfaces and in engineering applications - including situations involving controllers placed on a small subset of the boundary. Some specific mixed div-curl systems from electromagnetic field theory are analyzed in [3].

Many texts on second order elliptic partial differential equations and boundary integral methods describe existence results for finite energy (H^1-)solutions of this problem; see

Hsiao and Wendland [15] or McLean [18] for detailed treatments of more general problems and Steinbach [21] section 4.1.4 for a specific discussion. There has been considerable work on the numerical computation of solutions. See Steinbach [21] for an introduction to the numerical analysis of these problems.

There also is a considerable applied literature describing approximate and series solutions of problems involving physical models with mixed boundary conditions. Much of the work is formal and requires equations with constant coefficients and domains with some symmetry. Classical treatments of the analysis of mixed boundary value problems, including many applications, may be found in the monographs of Sneddon [20] and Duffy [11]. These texts both emphasize special Fourier series solutions of the problems.

2. Definitions and Notation.

To analyze this problem, standard definitions, terminology and assumptions will be used as in Evans and Gariepy [12], save that $\sigma,\ d\sigma$ will represent Hausdorff $(n-1)-$dimensional measure and integration with respect to this measure respectively. All functions in this paper will take values in $\overline{\mathbb{R}} := [-\infty, \infty]$, derivatives should be taken in a weak sense and $n \geq 2$ throughout.

A *region* is a non-empty, connected, open subset of $\mathbb{R}^n$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. The basic requirement on Ω is

(B1): *Ω is a bounded region in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area.*

When this holds there is an outward unit normal ν defined at σ *a.e.* point of $\partial\Omega$. The real Lebesgue space $L^q(\partial\Omega, d\sigma)$ may be defined in the usual way and its norm will be denoted $\|.\|_{q,\partial\Omega}$.

Let $L^p(\Omega), H^1(\Omega)$ be the usual real Lebesgue and Sobolev spaces of functions on Ω. The norm on $L^p(\Omega)$ is denoted $\|.\|_p$ and the inner product on $L^2(\Omega)$ by $\langle ., . \rangle_2$. $H^1(\Omega)$ is a real Hilbert space under the standard H^1- inner product

$$[u, v]_{1,2} := \int_\Omega \ [u(x).v(x) \ + \ \nabla u(x) \cdot \nabla v(x)] \ \ d^n x. \tag{2.1}$$

Here ∇u is the gradient of the function u and the associated norm is denoted $\|u\|_{1,2}$.

When Ω satisfies (B1), then the *Gauss-Green* theorem holds in the form

$$\int_\Omega \ u(x)\, D_j v(x) \ \ d^n x \ = \ \int_{\partial\Omega} u\, v\, \nu_j \ d\sigma \ - \ \int_\Omega \ v(x)\, D_j u(x) \ \ d^n x \quad \text{for } 1 \leq j \leq n. \tag{2.2}$$

and all $u,\ v$ in $H^1(\Omega)$ and related versions of the divergence theorem.

The region Ω is said to satisfy *Rellich's theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ where $p_S(n) \ := \ 2n/(n-2)$ when $n \geq 3$, or $p_S(2) = \infty$ when $n = 2$.

There are a number of different criteria on Ω and $\partial\Omega$ that imply this result. When (B1) holds it is theorem 1 in section 4.6 of [12]; see also Amick [1].

The *trace map* is the linear extension of the map restricting Lipschitz continuous functions on $\overline{\Omega}$ to $\partial\Omega$. When (B1) holds, this map has an extension to $W^{1,1}(\Omega)$ and then the trace of u on $\partial\Omega$ will be Lebesgue integrable with respect to σ, see [12], Section 4.2 for details. The region Ω is said to satisfy the *trace theorem* provided the trace mapping $\gamma : H^1(\Omega) \to L^q(\partial\Omega, d\sigma)$ is continuous when either $n = 2$ and $1 \le q < \infty$ or else $n \ge 3$ and $1 \le q \le 2(n-1)/(n-2)$. Conditions on the region Ω under which this theorem holds are stated and proved in Necas [19], Chapter 2, theorem 4.7 and Adams and Fournier [2], Theorem 5.36. The conditions required by Necas' result hold when (B1) is satisfied. Moreover, in theorem 6.2 of chapter 2, Necas shows that the trace map is compact when $1 \le q < 2(n-1)/(n-2)$. In surface integrals we will often use u in place of γu for simplicity as was done above in (2.2).

A real number is *positive* if it is greater than, or equal to, zero; *strictly positive* if it is greater than zero. Similarly a function u is said to be *(strictly) positive* on a set E, if $u(x) \ge (>)\ 0$ on E. Similarly a real sequence $\{x_m : m \ge 1\}$ is said to be *increasing* if $x_{m+1} \ge x_m$ for all m; it is *strictly increasing* provided strict inequality holds here for all m.

When $\Sigma, \tilde{\Sigma}$ as above, let $\partial\Sigma$ to be the common boundary of these sets. $\partial\Sigma$ is called the *transition set* or *interface* and may be empty. We have $\partial\Sigma = \overline{\Sigma} \cap \overline{\tilde{\Sigma}}$ and usually require the following

(B2): Σ is an nonempty open subset of $\partial\Omega$, Σ and $\tilde{\Sigma}$ have strictly positive surface area, and $\sigma(\partial\Sigma) = 0$.

Our requirements on the data for the boundary value problem (1.1)−(1.2) will generally include the following

(A1): $A(x) := (a_{jk}(x))$ is a real symmetric matrix whose components are bounded Lebesgue-measurable functions on Ω and there exist constants c_0, c_1 such that

$$c_0\,|\xi|^2 \;\le\; \langle A(x)\xi, \xi\rangle \;\le\; c_1\,|\xi|^2 \qquad \text{for all} \quad \xi \in \mathbb{R}^n,\ x \in \Omega. \tag{2.3}$$

Here $\langle .,. \rangle$ is the usual Euclidean inner product and $|.|$ is the Euclidean norm on $\mathbb{R}^n$. The boundary functions η_1, η_2 in (1.2) will be extended to $\partial\Omega$ via being identically zero on $\tilde{\Sigma}, \Sigma$ respectively. Since $\sigma(\partial\Sigma) = 0$, the values of the data on the interface need not be specified for this analysis.

(A2): When $n \ge 3$, a_0 is in $L^p(\Omega)$ for some $p \ge n/2$, with $a_0 \ge 0$ a.e on Ω. When $n = 2$, $p > 1$ here.

(A3): When $n \ge 3$, ρ is in $L^p(\Omega)$ for some $p \ge 2n/(n+2)$ and $\eta_2 \in L^q(\partial\Omega, d\sigma)$ for some $q \ge 2(1 - n^{-1})$. (For n = 2, we require $p > 1$ and $q > 1$.) η_1 is in $\gamma(H^1(\Omega))$.

(A3) implies that the linear functionals associated with integration against ρ, η_2 are in the dual space of $H^1(\Omega)$; see the next section for more comments on this. They are imposed so that the analysis may be done within the context of the calculus of variations.

In this paper we shall use various standard results from the calculus of variations and convex analysis. Background material on such methods may be found in Blanchard and Bruning [7] or Zeidler [27], both of which have discussions of the variational principles for the Dirichlet eigenvalues and eigenfunctions of second order elliptic operators. Here a similar theory for the spectrum of these mixed problems will be described for use in our analysis.

All the variational principles, and functionals to be discussed here will be defined on (closed convex subsets of) $H^1(\Omega)$. When $\mathcal{F} : H^1(\Omega) \to (-\infty, \infty]$ is a functional, then $\mathcal{F}$ is said to be *G-differentiable* at a point $u \in H^1(\Omega)$ if there is a continuous linear functional G on $H^1(\Omega)$ such that

$$\lim_{t \to 0} t^{-1} [\mathcal{F}(u + tv) - \mathcal{F}(u)] \; = \; G(v) \qquad \text{for all } v \in H^1(\Omega).$$

In this case we write $\mathcal{F}'(u)$ for G and this functional is said to be the G-derivative of $\mathcal{F}$ at u.

3. Bilinear Forms and Equivalent Inner Products

The results to be described here are based on special choices for inner products and associated orthogonal decompositions of $H^1(\Omega)$ and certain of its subspaces. These special choices simplify much of the analysis and allow the decomposition of the problem into two quite different subproblems - each of which is analyzed using special bases.

Consider the bilinear form $\mathcal{A} : H^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{A}(u, v) \; := \; [u, v]_{A\Sigma} \; := \; \int_\Omega \; (A(x)\nabla u) \cdot \nabla v \;\; d^n x \, + \, \int_\Sigma \; u \; v \;\; d\tilde{\sigma} \tag{3.1}$$

Here $\tilde{\sigma}(E) := \sigma(E)/\sigma(\Sigma)$ is a normalized surface area measure on Σ.

The following result shows that this bilinear form defines an inner product on $H^1(\Omega)$ that is equivalent to the standard H^1-norm (2.1). The corresponding norms is denoted by $\|u\|_{A\Sigma}$ and is called the $A\Sigma$-norm on $H^1(\Omega)$. When $A(x) \equiv I_n$ this inner product and norm will be denoted $[u, v]_\Sigma$, and $\|u\|_\Sigma$, respectively and called the Σ-inner product, or norm, on $H^1(\Omega)$.

Theorem 3.1. *Assume (A1), (A2), (B1) and (B2) hold, then (3.1) defines an inner product on $H^1(\Omega)$ and there are constants C_1, C_2 such that*

$$C_1 \; \|u\|_{1,2} \; \leq \; \|u\|_{A\Sigma} \; \leq \; C_2 \; \|u\|_{1,2} \quad \textit{for any} \quad u \in H^1(\Omega). \tag{3.2}$$

The norms $\|u\|_{A\Sigma}, \|u\|_\Sigma$ are equivalent to the standard norm on $H^1(\Omega)$.

Proof. When (A1) holds, $\mathcal{A}(u, v)$ is finite for each u,v and is symmetric. $\mathcal{A}(u, u) = 0$ implies $u = 0$ in $H^1(\Omega)$ since $c_0 > 0$. To prove the first inequality in (3.2), we first show that there is a $\lambda_1 > 0$ such that

$$\mathcal{A}_0(u, u) \; := \; \int_\Omega \; (A\nabla u) \cdot \nabla u \; d^n x + \int_\Sigma \; u^2 \; d\tilde{\sigma} \; \geq \; \lambda_1 \; \int_\Omega \; u^2 \; d^n x \qquad \text{for all } u \in H^1(\Omega). \tag{3.3}$$

Let $B := \{u \in H^1(\Omega) : \|u\|_{A\Sigma} \leq 1\}$ be the unit $A\Sigma-$ball in $H^1(\Omega)$ and define the functional $\mathcal{Q} : H^1(\Omega) \to [0, \infty)$ by

$$\mathcal{Q}(u) \ := \ \|u\|_2^2 \ := \ \int_\Omega \ u^2 \ d^n x \tag{3.4}$$

Consider the problem of maximizing the functional $\mathcal{Q}$ on B. (B1) and Rellich's theorem imply that $\mathcal{Q}$ is weakly continuous on $H^1(\Omega)$, so $\mathcal{Q}$ attains a finite maximum γ on B. Then by homogeniety

$$\mathcal{Q}(u) \ \leq \ \gamma \, [u, u]_{A\Sigma} \qquad \text{for all} \quad u \in H^1(\Omega). \tag{3.5}$$

Thus (3.3) holds with $\lambda_1 = 1/\gamma$. The definition (2.1) and assumption (A1) yield

$$c_0 \|u\|_{1.2}^2 \ \leq \ \int_\Omega \ (A\nabla u) \cdot \nabla u \ d^n x \ + \ c_0 \mathcal{Q}(u)$$

Combine these last two inequalities to obtain the left inequality in (3.2).

To prove the second inequality in (3.2), observe that $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is a continuous linear map, so there is a constant C_b such that

$$\int_{\partial\Omega} \ u^2 \ d\sigma \ \leq \ C_b \ \|u\|_{1,2}{}^2 \tag{3.6}$$

Substitute this in (3.1), then assumption (A1) yields

$$\|u\|_{A\Sigma}^2 \ \leq \ (c_1 \ + \ \sigma(\Sigma)^{-1} \, C_b) \, \|u\|_{1,2}{}^2$$

so the right inequality holds. These inequalities combine to show that these norms are equivalent. □

Associated with the lowest order term in $\mathcal{L}$ are the bilinear and quadratic forms $\mathcal{A}_0, \mathcal{Q}_0$ defined by

$$\mathcal{A}_0(u, v) \ := \ \int_\Omega \ a_0 \, u \ v \ \ d^n x \qquad \text{and} \quad \mathcal{Q}_0(u) \ := \ \int_\Omega \ a_0 \, u^2 \ \ d^n x \tag{3.7}$$

The following result will be used repeatedly.

Proposition 3.2. *Assume (B1) holds and a_0 satisfies (A2), then the bilinear form $\mathcal{A}_0$ is continuous and the quadratic form $\mathcal{Q}_0$ is convex and continuous on $H^1(\Omega)$. When $p > n/2$ in (A2), then $\mathcal{A}_0$ is weakly continuous in each variable separately and $\mathcal{Q}_0$ is weakly continuous on $H^1(\Omega)$.*

Proof. Our assumptions on $\Omega, \partial\Omega$ are such that the Sobolev imbedding theorems hold for this region. Thus $u \in H^1(\Omega)$ and $n \geq 3$ implies that $u \in L^s(\Omega)$ for $1 \leq s \leq 2n/(n-2)$. Let $s_m := 2n/(n-2)$ and use Holder's inequality to see that

$$|\mathcal{A}_0(u, v)| \ \leq \ \|a_0\|_p \ \|u\|_{s_m} \ \|v\|_{s_m} \qquad \text{with } p = n/2.$$

Thus (A2) implies $\mathcal{A}_0$ is bounded, so it is continuous. The Sobolev imbedding is compact when $s < s_m$ so this bilinear form is weakly continuous in each variable when $p > n/2$.

The associated quadratic form $\mathcal{Q}_0$ is convex since it is positive on $H^1(\Omega)$. It is continuous when $p \geq n/2$. When $p > n/2$, the using Holder as above with some $s < s_m$, yields that $\mathcal{Q}_0$ is weakly continuous. □

A consequence of this result is that when (A2) holds then the bilinear form $\mathcal{A}_1 : H^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{A}_1(u,v) \ := \ \mathcal{A}(u,v) + \mathcal{A}_0(u,v) \ = \ \int_\Omega \ [(A\nabla u)\cdot\nabla v \ + \ a_0\,u\,v] \ \ d^n x \ + \ \int_\Sigma \ u\,v \ \ d\tilde{\sigma} \quad (3.8)$$

also defines an equivalent inner product on $H^1(\Omega)$. This is called the $\mathcal{L}\,\Sigma-$inner product on $H^1(\Omega)$ and denoted $[.,.]_{\mathcal{L}\,\Sigma}$.

Define the linear functional $F : H^1(\Omega) \to \mathbb{R}$ by

$$F(u) \ := \ \int_\Omega \ \rho\,u\;d^n x \ + \ \int_{\tilde{\Sigma}} \ \eta_2\,u\;d\sigma. \quad (3.9)$$

This linear functional represents some of the "source terms" for our mixed boundary value problem. Straightforward analysis, using the Sobolev theorems and Holder's inequality leads to the following result.

Proposition 3.3. *Assume (B1), (B2) and (A3) hold, then the linear functional F defined by* (3.9) *is continuous.*

The special decompositions of $H^1(\Omega)$ to be used here are determined by the set Σ where the Dirichlet boundary condition holds. When E is a nonempty open subset of $\partial\Omega$, the characteristic function of E is the function that is 1 on E and zero otherwise. It is a Borel measurable function. Define $P_E : L^2(\partial\Omega, d\sigma) \to L^2(\partial\Omega, d\sigma)$ by $P_E\,u(x) \ := \ \chi_E(x)\,u(x)$, then P_E is a self-adjoint projection on $L^2(\partial\Omega, d\sigma)$. P_E has infinite dimensional range whenever $\sigma(E) > 0$ as the space of continuous functions on $\partial\Omega$ with support in E is a subset of this range.

Define $H^1_{\Sigma 0}(\Omega)$ to be the subspace of $H^1(\Omega)$ of all functions that satisfy $P_\Sigma \gamma u = 0$. This is equivalent to requiring that

$$\gamma u(x) \ = \ 0 \qquad \sigma \ a.e. \text{ on } \Sigma. \quad (3.10)$$

$H^1_{\Sigma 0}(\Omega)$ is a closed subspace of $H^1(\Omega)$ when (B1) and (B2) hold as γ and P_Σ are continuous linear operators. The $A\Sigma-$ and $\mathcal{L}\,\Sigma-$inner products on $H^1_{\Sigma 0}(\Omega)$ reduce respectively to

$$[u,v]_{A\Sigma} \ = \ \int_\Omega (A\nabla u)\cdot\nabla v \ \ d^n x \quad \text{and} \qquad [u,v]_{\mathcal{L}\,\Sigma} \ = \ \int_\Omega [(A\nabla u)\cdot\nabla v \ + \ a_0\,uv] \ \ d^n x \quad (3.11)$$

as the boundary integrals vanish when $u \in H^1_{\Sigma 0}(\Omega)$.

The weak form of our system is to first find solutions $\hat{u} \in H^1_{\Sigma 0}(\Omega)$ of

$$\mathcal{A}_1(u,v) \ = \ F(v) \qquad \text{for all} \quad v \in H^1_{\Sigma 0}(\Omega). \quad (3.12)$$

The space $H^1(\Omega)$ has an $\mathcal{L}\,\Sigma-$orthogonal decomposition given by (7.3). Suppose $\hat{w} \in H^1(\Omega)$ is a solution of the system

$$\mathcal{A}_1(w,v) \ = \ 0 \quad \text{for all} \quad v \in H^1_{\Sigma 0}(\Omega) \text{ and } \quad \gamma w \ = \ \eta_1 \text{ on } \partial\Omega. \quad (3.13)$$

Then $\hat{w}$ will be in the complementary subspace and, by linearity, $\hat{u}+\hat{w}$ will be a finite energy solution of $(1.1)-(1.2)$. In the following these problems will be investigated separately since their analysis is quite different.

4. Variational Principles for the Component in $H^1_{\Sigma 0}(\Omega)$

To describe the H^1-solvability of the mixed boundary problem (1.1) – (1.2), we shall first consider the case with zero Dirichlet data on Σ. That is, take $\eta_1 \equiv 0$ and seek functions $\hat{u} \in H^1_{\Sigma 0}(\Omega)$ satisfying

$$\int_\Omega [(A \cdot \nabla u) \cdot \nabla w \;+\; (a_0\, u \;-\; \rho) w]\; d^n x \;-\; \int_{\tilde{\Sigma}} \eta\; w\; d\sigma \;=\; 0 \qquad \text{for all } w \in H^1_{\Sigma 0}(\Omega). \quad (4.1)$$

Such a function may be regarded as a weak solution of the mixed boundary value problem

$$-\operatorname{div}(A\,\nabla u) \;+\; a_0\, u \;\;=\;\; \rho \qquad \text{on } \Omega, \;\text{ subject to} \qquad (4.2)$$

$$u \;=\; 0 \qquad \text{on } \Sigma, \;\text{ and} \qquad (A\,\nabla u)\cdot \nu \;=\; \eta \qquad \text{on } \tilde{\Sigma}. \qquad (4.3)$$

When $\mathcal{A}_1$, F as in section 3, this equation has the form

$$\mathcal{A}_1(u, w) \;=\; F(w) \qquad \text{for all } w \in H^1_{\Sigma 0}(\Omega).$$

Existence results for this problem in Sobolev spaces are straightforward; see Steinbach [21] theorem 4.11 for a recent exposition. Here we are interested in certain representation and approximation results that hold under further conditions on the data. When (ρ, η) obey (A3) the following result guarantees the existence of a unique solution and provides an H^1-estimate on the solution.

Theorem 4.1. *Assume (A1) - (A3), (B1) and (B2) hold. Then there is a unique solution $\hat{u}$ of (4.1) in $H^1_{\Sigma 0}(\Omega)$ and there exist constants k_1, k_2 such that*

$$\|\hat{u}\|_{1,2} \;\le\; k_1\; \|\rho\|_p \;+\; k_2\; \|\eta\|_{q,\tilde{\Sigma}} \qquad (4.4)$$

where p, q as in condition (A3).

To prove this theorem, we shall show that there is a convex variational principle for the solutions of (4.1) and that the variational problem has a minimizer on $H^1_{\Sigma 0}(\Omega)$.

Define the functionals $\mathcal{D}_0$ and $\mathcal{D}$ on $H^1_{\Sigma 0}(\Omega)$ by

$$\mathcal{D}_0(u) \;:=\; \int_\Omega \;[(A\nabla u \cdot \nabla u) \;+\; a_0\, u^2]\;\; d^n x, \qquad \text{and} \qquad (4.5)$$

$$\mathcal{D}(u) \;:=\; \mathcal{D}_0(u) \;-\; 2 \int_\Omega \rho\, u\; d^n x \;-\; 2 \int_{\tilde{\Sigma}} \eta u\; d\sigma. \qquad (4.6)$$

Consider the variational problem of minimizing $\mathcal{D}$ on $H^1_{\Sigma 0}(\Omega)$. Note that $\mathcal{D}$ has the form $\mathcal{D}(u) \;=\; \mathcal{D}_0(u) \;-\; 2\, F(u)$ In particular, when (A2) holds then (3.11) shows that $\mathcal{D}_0(u)$ will be the square of the $\mathcal{L}\,\Sigma$ norm on $H^1_{\Sigma 0}(\Omega)$.

Theorem 4.2. *Assume (A1) - (A3), (B1) and (B2) hold. Then there is a unique minimizer $\hat{u}$ of $\mathcal{D}$ on $H^1_{\Sigma 0}(\Omega)$. This minimizer satisfies (4.1) and there are constants k_1, k_2 such that (4.4) holds.*

Proof. We first prove the existence of a unique minimizer of $\mathcal{D}$ on $H^1_{\Sigma 0}(\Omega)$. Assumptions (A1) and (A2) imply that $\mathcal{D}_0$ is continuous, strictly convex and coercive on $H^1_{\Sigma 0}(\Omega)$ from properties of the norm, the Sobolev imbedding theorem and inequality (3.2). When Rellich's theorem and the trace theorem hold, then (A3) guarantees that the integrals

$$\int_{\Omega} \rho u \, d^n x \qquad \text{and} \qquad \int_{\tilde{\Sigma}} \eta \, u \, d\sigma$$

define continuous linear functionals on $H^1_{\Sigma 0}(\Omega)$ when p, q as in (A3). Hence $\mathcal{D}$ will be continuous and has a unique minimizer $\hat{u}$ on $H^1_{\Sigma 0}(\Omega)$.

The functional $\mathcal{D}$ is G-differentiable with derivative given by

$$\langle \mathcal{D}'(u), w \rangle \;=\; 2 \int_{\Omega} [(A\nabla u) \cdot \nabla w + (a_0 u - \rho) w] \, d^n x \;-\; 2 \int_{\tilde{\Sigma}} \eta \, w \, d\sigma. \tag{4.7}$$

A minimizer of $\mathcal{D}$ will satisfy $\langle \mathcal{D}'(u), w \rangle = 0$ for all $w \in H^1_{\Sigma 0}(\Omega)$, so $\hat{u}$ satisfies (4.1).

Put $u = w = \hat{u}$ in (4.1), then

$$\int_{\Omega} [(A\nabla \hat{u}) \cdot \nabla \hat{u} + a_0 \hat{u}^2] \, d^n x \;=\; \int_{\Omega} \rho \hat{u} \, d^n x + \int_{\tilde{\Sigma}} \eta \hat{u} \, d\sigma. \tag{4.8}$$

Use (A2), the ellipticity inequality (2.3) together with Holder's inequality for the right hand side, then

$$c_0 \|\hat{u}\|^2_{\mathcal{L}\Sigma} \;\leq\; \|\rho\|_p \;\|\hat{u}\|_{p'} + \|\eta\|_{q,\tilde{\Sigma}} \;\|\hat{u}\|_{q',\tilde{\Sigma}}$$

where p', q' are the conjugate indices to p, q. Apply inequality (3.2) to the left hand side and use Rellich's theorem and the trace theorem to the terms on the right then

$$c_0 \, C_1^2 \, \|\hat{u}\|^2_{1,2} \;\leq\; [\|\rho\|_p + \|\eta\|_{q,\tilde{\Sigma},d\sigma} \,] \; \|\hat{u}\|_{1,2} \tag{4.9}$$

which yields (4.4) as desired. □

Proof. of Theorem 4.1. Theorem 4.2 shows that there is a solution $\hat{u}$ of (4.1) - and it is the unique minimizer of $\mathcal{D}$ on $H^1_{\Sigma 0}(\Omega)$. If there were another solution $\tilde{u} \in H^1_{\Sigma 0}(\Omega)$ of (4.1), then $\tilde{u}$ would be a critical point of $\mathcal{D}$ on $H^1_{\Sigma 0}(\Omega)$ from (4.7). Since $\mathcal{D}$ is convex, the only critical points are minimizers, so $\tilde{u}$ would also be a minimizer of $\mathcal{D}$. Since $\mathcal{D}$ is strictly convex, we must have $\tilde{u} = \hat{u}$ so the solution of (4.1) is unique. □

The inequality (4.4) provides an estimate for the continuous dependence of the solutions of this problem on the data ρ and η. This result shows that this problem for finite-energy solutions in $H^1_{\Sigma 0}(\Omega)$ is well-posed provided the data satisfies (A3), the boundary satisfies (B1) and (B2) and the equation satisfies (A1) and (A2). Specifically, we have the following corollary.

Corollary 4.3. *Assume (A1) - (A3), (B1) and (B2) hold. Then there are continuous linear transformations $\mathcal{G}_0 : L^p(\Omega) \to H^1_{\Sigma 0}(\Omega)$ and $\mathcal{G}_1 : L^q(\partial\Omega, d\sigma) \to H^1_{\Sigma 0}(\Omega)$ such that the unique solution $\hat{u}$ of (4.1) in $H^1_{\Sigma 0}(\Omega)$ is given by*

$$\hat{u}(x) \;=\; (\mathcal{G}_0 \rho)(x) + (\mathcal{G}_1 \eta)(x). \tag{4.10}$$

$\mathcal{G}_0$ is a compact linear mapping when $p > 2n/(n+2)$. $\mathcal{G}_1$ is a compact linear mapping when $q > 2(1 - n^{-1})$.

Proof. Put $\eta \equiv 0$ on $\tilde{\Sigma}$, and consider the solution of the problem (4.1). Denote the solution by $\mathcal{G}_0 \rho$, then $\mathcal{G}_0$ is linear, its range is a subspace of $H^1_{\Sigma 0}(\Omega)$ and the estimate (4.4) shows that $\mathcal{G}_0$ is continuous. When $p > 2n/(n+2)$ then the imbedding of $L^p(\Omega)$ into $H^{-1}(\Omega)$ is compact and the operator $\mathcal{G}_0$ is continuous from $H^{-1}(\Omega)$ to $H^1_{\Sigma 0}(\Omega)$ from theorem 4.11 of Steinbach [21], hence $\mathcal{G}_0$ is compact.

Simlarly put $\rho \equiv 0$ on Ω and solve (4.1). Denote the solution by $\mathcal{G}_1 \eta$, then $\mathcal{G}_1$ is linear, maps into $H^1_{\Sigma 0}(\Omega)$ and theorem 4.1 shows that $\mathcal{G}_1$ is continuous. For $q > 2(1 - n^{-1})$, the imbedding of $L^q(\partial\Omega, d\sigma)$ into $H^{-1/2}(\partial\Omega)$ is compact from duality and the fact that γ is compact. So $\mathcal{G}_1$ is a compact linear mapping. □

5. Mixed Dirichlet-Neumann Eigenfunctions.

In this section, some basic results about eigenvalues and eigenfunctions of the operator $\mathcal{L}$ subject to mixed zero boundary conditions will be derived. The results may be obtained by a number of standard approaches as described, for example, in the monographs of Weinberger [23] or Bandle [6]. Here, however, the results will be obtained in a manner that is similar to the way that the theory of Steklov eigenfunctions will be described. This avoids the use of Rayleigh quotients and enables a straightforward proof of the completeness of the eigenfunctions.

A real number λ is said to be a Dirichlet-Neumann, or DN, eigenvalue of $\mathcal{L}$ provided there is a non-zero function $v \in H^1_{\Sigma 0}(\Omega)$ such that

$$\int_\Omega [(A\nabla v) \cdot \nabla w + (a_0 - \lambda)\, v\, w]\, d^n x = 0 \qquad \text{for all } w \in H^1_{\Sigma 0}(\Omega). \tag{5.1}$$

Any such v will be said to be a DN eigenfunction of $\mathcal{L}$ corresponding to the eigenvalue λ. (5.1) is a weak form of the system

$$\mathcal{L}\, v = -\operatorname{div}(A\nabla v) + a_0 v = \lambda v \qquad \text{on } \Omega, \text{ subject to} \tag{5.2}$$

$$v = 0 \quad \text{on } \Sigma, \text{ and } \quad (A\nabla v)\cdot \nu = 0 \quad \text{on } \tilde{\Sigma}. \tag{5.3}$$

From (3.8), (5.1) may be written as

$$\mathcal{A}_1(v, w) = [v, w]_{\mathcal{L}\Sigma} = \lambda \int_\Omega v\, w\, d^n x \qquad \text{for all } w \in H^1_{\Sigma 0}(\Omega). \tag{5.4}$$

The successive eigenvalues, and corresponding eigenfunctions of this problem, can be constructed using variational principles. Let C_1 be the subset of functions in $H^1_{\Sigma 0}(\Omega)$ satisfying $\|u\|_{\mathcal{L}\Sigma} \leq 1$ and consider the problem of maximizing the usual L^2-norm $\mathcal{Q}(u) := \|u\|_2^2$ defined by (3.4) on C_1.

Theorem 5.1. *Assume (A1), (A2), (B1) and (B2) hold. Then there are maximizers $\pm v_1$ of $\mathcal{Q}$ on C_1. The maximizers satisfy (5.1) and $\|v_1\|_{\mathcal{L}\Sigma} = 1$. The corresponding eigenvalue λ_1 is positive and is the least eigenvalue of (5.1).*

Proof. $\mathcal{Q}$ is weakly continuous on $H^1(\Omega)$, from Rellich's theorem when (B1) holds. C_1 is weakly compact in $H^1(\Omega)$ as it is a closed, bounded, convex subset. Thus $\mathcal{Q}$ attains its supremum on C_1 and this supremum is finite. Let $\pm v_1$ be maximizers of $\mathcal{Q}$ on C_1.

Define the Lagrangian functional $\mathcal{M} : H^1_{\Sigma 0}(\Omega) \times [0,\infty) \to \mathbb{R}$ by

$$\mathcal{M}(v,\mu) \ := \ -\mathcal{Q}(v) \ + \ \mu\,(\|v\|_{\mathcal{L}\Sigma}{}^2 \ - \ 1). \tag{5.5}$$

This functional has the property that

$$\sup_{v\in C_1} \mathcal{Q}(v) \ = \ - \inf_{v\in H^1_{\Sigma 0}(\Omega)} \ \sup_{\mu\in[0,\infty)} \ \mathcal{M}(v,\mu) \tag{5.6}$$

Hence the maximizers of $\mathcal{Q}$ on C_1 occur at critical points of $\mathcal{M}$ from the inequality multiplier theorem for smooth convex constrained problems. $\mathcal{M}$ is quadratic in v and a critical point $\hat{v}$ satisfies

$$\langle D_v\,\mathcal{M}(v,\mu), w\rangle \ = \ \int_\Omega \{\,\mu[(A\nabla v)\cdot\nabla w \ + \ a_0\,vw] \ - \ vw\} \ \ d^nx \ = \ 0 \tag{5.7}$$

for all $w \in H^1_{\Sigma 0}(\Omega)$ and some $\mu \geq 0$. If $\mu = 0$, this implies that $\hat{v} = 0$ which cannot be a maximizer of $\mathcal{Q}$ on C_1. Thus $\mu > 0$ and $\hat{v}$ satisfies (5.4) with $\lambda \ = \ \mu^{-1} > \ 0$. Put $v = w = v_1$ in this equation then

$$1 \ \geq \ \|v_1\|^2_{\mathcal{L}\Sigma} \ = \ \lambda\mathcal{Q}(v_1) \tag{5.8}$$

At the maximizer v_1 of $\mathcal{Q}$ on C_1, we must have $\|v_1\|^2_{\mathcal{L}\Sigma} = 1$ by scaling. Then this equation yields

$$\mathcal{Q}(v_1) \ = \ 1/\lambda.$$

If v_1 is a maximizer of $\mathcal{Q}$, then λ must be the least eigenvalue λ_1 of this problem. □

Suppose that the first $M-1$ DN eigenvalues of $\mathcal{L}$ are $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{M-1}$. Let $\{v_j : 1 \leq j \leq M-1\}$ be an associated family of DN eigenfunctions that are L^2-orthonormal on Ω and have $\|v_j\|_{\mathcal{L}\Sigma} \ = \ 1$ for each j. To find the next eigenvalue and an eigenfunction corresponding to this eigenvalue, consider the problem of maximizing the L^2-norm $\mathcal{Q}$ on

$$C_M \ := \ \{u \in C_1 \ : \ \int_\Omega u\,v_j\,d^nx \ = \ 0 \quad \text{for } 1 \leq j \leq M-1\}$$

The following theorem describes the solutions of this problem.

Theorem 5.2. *Assume (A1), (A2), (B1) and (B2) hold. For each $M \geq 2$, there are maximizers $\pm v_M$ of $\mathcal{Q}$ on C_M that satisfy (5.1) with and $\|v_M\|_{\mathcal{L}\Sigma} = 1$. The corresponding eigenvalue λ_M is the least eigenvalue of this problem greater than or equal to λ_{M-1} with*

$$\|v_M\|_2 = \lambda_M{}^{-1/2} \qquad \textit{and} \qquad [v_M, v_j]_{\mathcal{L}\Sigma} \ = \ 0 \qquad \textit{for } 1 \leq j \leq M-1. \tag{5.9}$$

Proof. The existence follows just as for theorem 5.1, since each extra constraint involves a continuous linear functional so each C_M is closed and convex. For this problem the Lagrangian will be $\mathcal{M}_M : H^1_{\Sigma 0}(\Omega) \times [0,\infty) \times \mathbb{R}^{M-1} \to \mathbb{R}$ defined by

$$\mathcal{M}_M(v,\mu,\xi) \ := \ \mathcal{M}(v,\mu) \ - \ \sum_{j=1}^{M-1} \ \xi_j\,\langle v, v_j\rangle_2. \tag{5.10}$$

Here $\mathcal{M}$ is the Lagrangian of (5.5). Just as in the previous proof, this has the property that

$$\sup_{v \in C_M} \mathcal{Q}(v) \;=\; - \inf_{v \in H^1_{\Sigma 0}(\Omega)} \; \sup_{(\mu,\xi) \in [0,\infty) \times \mathbb{R}^{M-1}} \mathcal{M}_M(v,\mu,\xi) \tag{5.11}$$

Hence the maximizers of $\mathcal{Q}$ on C_M occur at critical points of $\mathcal{M}_M$ from the well-known multiplier theorem. The critical points of $\mathcal{M}_M$ satisfy

$$\int_\Omega \{ \, \mu[(A\nabla v) \cdot \nabla w \,+\, a_0\, vw] \,-\, vw \} \; d^n x \,-\, \sum_{j=1}^{M-1} \xi_j \, \langle w, v_j \rangle_2 \;=\; 0 \tag{5.12}$$

for all $w \in H^1_{\Sigma 0}(\Omega)$, some $\mu \geq 0$ and $\xi \in \mathbb{R}^{M-1}$. If $\mu = 0$, this implies that $\hat{v}$ is a linear combination of the v_j, so $\hat{v} = 0$ as it is in C_M. 0 cannot be a maximizer of $\mathcal{Q}$ on C_M so $\mu > 0$. Divide by μ then $\hat{v}$ satisfies

$$\mathcal{A}_1(v,w) \;=\; \lambda \left[\langle v, w \rangle_2 \,+\, \sum_{j=1}^{M-1} \xi_j \, \langle v_j, w \rangle_2 \right]$$

for all $w \in H^1_{\Sigma 0}(\Omega)$.

When $M = 2$, take $v = v_2,\ w = v_1$ here then $\mathcal{A}(v_1, v_2) \;=\; \xi_1/\lambda_1$ from the definition of v_2 and the value of $\mathcal{Q}(v_1)$. However (5.4) implies that $\mathcal{A}(v_1, v_2) \;=\; 0$ since $\langle v_1, v_2 \rangle_2 = 0$. Thus $\xi_1 = 0$ and v_2 is a solution of (5.1) corresponding to an eigenvalue λ_2. Since $\|v_2\|_{\mathcal{L}\,\Sigma} = 1$, this equation implies that $\|v_2\|_2$ satisfies the first part of (5.9).

Induction and minor modifications of these arguments then proves this result for arbitrary integers M. □

The preceding theorem generates a countably infinite L^2-orthogonal family $\{v_j : j \geq 1\}$ of DN eigenfunctions of $\mathcal{L}$. This sequence is $\mathcal{L}\,\Sigma-$orthonormal. For each $j \geq 1$, define

$$w_j(x) \;:=\; {\lambda_j}^{1/2} \, v_j(x) \qquad \text{for } x \in \Omega. \tag{5.13}$$

Then $\mathcal{E} \;:=\; \{w_j : j \geq 1\}$ will be an L^2- orthonormal subset of $H^1_{\Sigma 0}(\Omega)$ and $\mathcal{D}_0(w_j) = \lambda_j$ for each $j \geq 1$.

Theorem 5.3. *Assume (A1), (B1) and (B2) hold and the sequence $\mathcal{E}$ of mixed DN eigenvalues and eigenfunctions is obtained iteratively as above. Then*
(i) each eigenvalue λ_j has finite multiplicity and $\lambda_j \to \infty$ as $j \to \infty$,
(ii) $\mathcal{E}$ is a maximal L^2-orthonormal set in $H^1_{\Sigma 0}(\Omega)$, and
(iii) $\langle w_j, w_k \rangle_{\mathcal{L}\,\Sigma} \;=\; \lambda_j \, \delta_{jk}$ for all $j, k \geq 1$.

Proof. The infinite sequence of eigenfunctions $\{v_j : j \geq 1\}$ defined by theorems 5.1 and 5.2 obey $\|v_j\|_{\mathcal{L}\,\Sigma} = 1$ and this is an equivalent norm to the usual norm on $H^1_{\Sigma 0}(\Omega)$. Thus v_j must converge strongly to zero in $L^2(\Omega)$ from Rellich's theorem. Since $\|v_j\|_2 = {\lambda_j}^{-1/2}$ from theorem 5.2, (i) follows.

Suppose $\mathcal{E}$ is not a maximal L^2-orthonormal set in $H^1_{\Sigma 0}(\Omega)$. Then there is a $z \in H^1_{\Sigma 0}(\Omega)$ such that

$$\|z\|_{\mathcal{L}\,\Sigma} \;=\; 1 \quad \& \quad \langle z, w_j \rangle_2 \;=\; 0 \qquad \text{for all} \quad j \geq 1$$

$\|.\|_{\mathcal{L}\,\Sigma}$ is a norm so $\|z\|_2 = c > 0$. Suppose M is so large that $\lambda_j < c^2$ for $j > M$. Then v_{M+1} is not the next maximizer of $\mathcal{Q}$ on C_{M+1}. This contradicts the construction, so there is no such z and the sequence $\mathcal{E}$ is maximal.

The orthogonality in (iii) holds from the last part of (5.9). When $j = k$, this holds since $\|v_j\|_{\mathcal{L}\,\Sigma} \ = \ 1$. □

6. Spectral Representations of Solution Operators

Given that $\mathcal{E}$ is an L^2-orthonormal basis of $H^1_{\Sigma 0}(\Omega)$, one may ask about the possible representations, and approximations, of the solutions of (4.1) using eigenfunction expansions. Assume that the solution of (4.1) has a representation of the form

$$\hat{u}(x) \ := \ \sum_{j=1}^{\infty} \ c_j \, w_j(x) \qquad \text{with} \quad c_j := \langle \hat{u}, w_j \rangle_2. \tag{6.1}$$

Put $w = w_k$ in (4.1), then $\hat{u}$ satisfies

$$[\hat{u}, w_k]_{\mathcal{L}\,\Sigma} \ = \ \rho_k \ + \ \eta_k \qquad \text{for each } k \geq 1, \text{ with} \tag{6.2}$$

$$\rho_k \ := \ \int_{\Omega} \rho \, w_k \; d^n x \qquad \text{and} \quad \eta_k \ := \ \int_{\partial\Omega} \eta \, w_k \; d\sigma. \tag{6.3}$$

Substitute (6.1) in (6.2), then (5.1) and orthogonality yields

$$c_k \ \lambda_k \ = \ \rho_k \ + \ \eta_k \qquad \text{for all } k \geq 1 \tag{6.4}$$

so the solution of (4.1) is

$$\hat{u}(x) \ = \ \sum_{k=1}^{\infty} \ {\lambda_k}^{-1} \, (\rho_k \ + \ \eta_k) \ \ w_k(x). \tag{6.5}$$

Theorem 6.1. *Assume (A1)-(A3), (B1) and (B2) hold with w_j defined as in section 5. Then the unique solution $\hat{u}$ of (4.1) in $H^1_{\Sigma 0}(\Omega)$ is given by (6.5). This series converges in $H^1_{\Sigma 0}(\Omega)$ and*

$$\|\hat{u}\|_{A\Sigma}^{\ 2} \ \leq \ \|\hat{u}\|_{\mathcal{L}\,\Sigma}^{\ 2} \ = \ \sum_{k=1}^{\infty} \ {\lambda_k}^{-1} \ (\rho_k \ + \ \eta_k)^2 \tag{6.6}$$

Proof. From theorem 4.1, there is a unique solution of (4.1) and, from theorem 5.3, it has a representation of the form (6.1) as the set $\mathcal{E}$ is a maximal L^2-orthonormal set. The coefficients are given by (6.4) so from the orthogonality (5.9) we have

$$\|\hat{u}\|_{\mathcal{L}\,\Sigma}^{\ 2} \ = \ \sum_{k=1}^{\infty} \ c_k^2 \ \lambda_k,$$

which implies (6.6). □

It is worth noting that the minimizer $\hat{u}$ of $\mathcal{D}$ on $H^1_{\Sigma 0}(\Omega)$ obeys $\mathcal{D}_0(\hat{u}) = F(\hat{u})$ from (4.8). Thus the sum in (6.6) is finite and may be estimated in terms of the data (ρ, η) by using (4.4).

In particular, this shows that the solution of this boundary value problem is approximated by finite rank integral operators. Let V_m be the m-dimensional subspace of $H^1_{\Sigma 0}(\Omega)$ with the L^2-orthonormal basis $\{w_1, \ldots, w_m\}$ as in (5.13). Let P_m be the usual L^2- projection of $H^1_{\Sigma 0}(\Omega)$ onto V_m defined by

$$P_m v \;:=\; \sum_{j=1}^{m} c_j\, w_j \qquad \text{with} \quad c_j := \langle v, w_m \rangle_2. \tag{6.7}$$

Suppose $u_m := P_m \hat{u}$ is the m-th partial sum of the solution (6.5), then (6.4) may be written as

$$u_m(x) \;=\; \int_\Omega G_m(x,y)\,\rho(y)\,d^n y \;+\; \int_{\tilde{\Sigma}} G_m(x,y)\,\eta(y)\,d\sigma(y) \quad \text{where} \tag{6.8}$$

$$G_m(x,y) \;=\; \sum_{k=1}^{m} {\lambda_k}^{-1}\; w_k(x)\, w_k(y) \;=\; \sum_{k=1}^{m} v_k(x)\, v_k(y) \tag{6.9}$$

is a smooth kernel defined on $\overline{\Omega} \times \overline{\Omega}$ since each v_k, w_k is in $H^1_{\Sigma 0}(\Omega)$.

This sequence $\{u_m : m \geq 1\}$ converges strongly to the solution $\hat{u}$ in $H^1_{\Sigma 0}(\Omega)$ as $m \to \infty$ since $\mathcal{E}$ is a basis of this space

In Corollary 4.3 this solution was described in terms of continuous linear operators $\mathcal{G}_0, \mathcal{G}_1$. (6.8) shows that these operators can be represented as limits of the finite rank integral operators defined by this sequence of integral kernels G_m. Moreover the same sequence of kernel functions yields both of the operators - with $\mathcal{G}_0$ involving an integral over the region, while $\mathcal{G}_1$ only involves a boundary integral.

In the classical theory, the Green's function for this problem is often defined by

$$G(x,y) \;:=\; \lim_{m \to \infty} G_m(x,y) \qquad \text{for } (x,y) \in \overline{\Omega} \times \overline{\Omega}.$$

The sense in which this limit should be taken is often not clear when the G_m are regarded as functions. The preceding analysis shows that the finite rank integral operators defined in (6.8) converge in the strong operator topology to the solution operators $\mathcal{G}_0$, $\mathcal{G}_1$ of (4.10).

7. The Mixed Problem for the Homogeneous equation

Given a solution $\hat{u}$ of (4.1) in $H^1_{\Sigma 0}(\Omega)$, the solutions $\tilde{u} \in H^1(\Omega)$ of the problem (1.1) − (1.2) will be determined provided we can also find a function $\hat{v} \in H^1(\Omega)$ that satisfies

$$\mathcal{L}\, v(x) \;=\; -\operatorname{div}\left(A(x)\, \nabla v(x)\right) \,+\, a_0(x)\, v(x) \;=\; 0 \qquad \text{on } \Omega \text{ subject to} \tag{7.1}$$

$$v(y) \;=\; \eta(y) \qquad \text{on } \Sigma \;\; \text{and} \qquad (A(y)\nabla v(y)) \cdot \nu(y) \;=\; 0 \qquad \text{on } \tilde{\Sigma}. \tag{7.2}$$

in some sense. Even for the Laplacian and continuous functions μ on Σ, this system may not have finite energy solutions in $H^1(\Omega)$. Examples of this date back to results of Fichera from the early 1950's; see the references in Wendland, Stephan and Hsiao [24] for a discussion of this with n = 2.

To study this situation, the subspace of all finite energy solutions of (7.1) that satisfy the zero flux condition on $\tilde{\Sigma}$ is characterized using an orthogonal decomposition of $H^1(\Omega)$. Let $\ker \mathcal{L}\,(\Sigma)$ to be the orthogonal complement of $H^1_{\Sigma 0}(\Omega)$ with respect to the $\mathcal{L}\,\Sigma-$inner product. It is a closed subspace and we may write

$$H^1(\Omega) \;=\; H^1_{\Sigma 0}(\Omega) \;\oplus_{\mathcal{L}\,\Sigma}\; \ker \mathcal{L}\,(\Sigma) \tag{7.3}$$

where $\oplus_{\mathcal{L}\,\Sigma}$ indicates orthogonality with respect to the $\mathcal{L}\,\Sigma-$inner product.

We say that a function $v \in H^1(\Omega)$ is $\mathcal{L}$ *−homogeneous on* Ω provided it satisfies

$$\mathcal{A}_1(v,\varphi) \;=\; [v,\varphi]_{\mathcal{L}\,\Sigma} \;=\; 0 \qquad \text{for all } \varphi \in C^1_c(\Omega). \tag{7.4}$$

Here $C^1_c(\Omega)$ is the set of all C^1-functions on Ω with compact support in Ω. The following theorem shows that $\ker \mathcal{L}\,(\Sigma)$ is the requisite class of finite energy solutions.

Theorem 7.1. *Assume (B1) and (B2) hold, then a function $v \in H^1(\Omega)$ is in $ker \mathcal{L}\,(\Sigma)$ if and only if v is $\mathcal{L}$ −homogeneous on Ω and satisfies*

$$(A(y)\nabla v(y)) \cdot \nu(y) \;=\; 0 \quad \sigma\, a.e. \quad on\ \tilde{\Sigma} \tag{7.5}$$

Proof. The definition of the inner product (3.8) shows that v is in $\ker \mathcal{L}\,(\Sigma)$ if and only if

$$\int_\Omega \;[(A\nabla v)\cdot \nabla w \;+ a_0 v w]\; d^n x \,+\, \int_\Sigma \; v\, w \;\; d\tilde{\sigma} \;=\; 0 \qquad \text{for all } w \in H^1_{\Sigma 0}(\Omega). \tag{7.6}$$

Since $C^1_c(\Omega)$ is a subspace of $H^1_{\Sigma 0}(\Omega)$, this implies v satisfies (7.4) - so it is $\mathcal{L}$ −homogeneous. Choose w to also be continuous on $\overline{\Omega}$ and apply the Gauss-Green theorem to (7.6), then

$$\int_{\tilde{\Sigma}} \; w\;(A\nabla v)\cdot \nu) \;\; d\sigma \;=\; 0.$$

When (B2) holds, there are sufficiently many such w to yield (7.5). □

We will need the following result later.

Proposition 7.2. *Assume (B1) and (B2) hold, then the space $ker \mathcal{L}\,(\Sigma)$ is infinite dimensional and*

$$\langle u, v\rangle_\Sigma \;:=\; \int_\Sigma \; u\, v\, d\tilde{\sigma} \tag{7.7}$$

is an inner product on $ker \mathcal{L}\,(\Sigma)$.

Proof. Take E be an open subset of Σ with $\overline{E} \subset \Sigma$ and $\sigma(E) > 0$. Define P_E to be the projection as at the end of section 3, then $P_E\,\gamma H^1(\Omega)$ will be a subspace of $P_E L^2(\partial\Omega, d\sigma)$ as the trace theorem holds. Assume it is a finite dimensional subspace. Then there is a $v \in P_E L^2(\partial\Omega, d\sigma)$ such that $\|v\|_{2,\partial\Omega} = 1$ and $\|v - \gamma u\|_{2,\partial\Omega} = 1$ for all $u \in H^1(\Omega)$ since the range of P_E is infinite dimensional. This contradicts the fact that the space $\gamma H^1(\Omega) = H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega, d\sigma)$ from theorem 5.1 of [5]. So the first part holds. Suppose $u \in \ker \mathcal{L}\,(\Sigma)$ and $\langle u, u\rangle_\Sigma = 0$, then $\gamma u = 0$ $\sigma\, a.e.$ *on* Σ. So u is also in $H^1_{\Sigma 0}(\Omega)$, from (3.10). Thus $u = 0$ in $H^1(\Omega)$ as $\ker \mathcal{L}\,(\Sigma)$ and $H^1_{\Sigma 0}(\Omega)$ are $\mathcal{A}_1-$orthogonal. □

It may be worth noting that $\ker \mathcal{L}(\Sigma)$ will not be a Hilbert space with respect to this inner product (7.7). We say that a function $\hat{v} \in \ker \mathcal{L}(\Sigma)$ is a *finite energy solution* of (7.1)-(7.2) provided $P_\Sigma \hat{v} = \mu \quad \sigma$ *a.e.* on $\partial\Omega$. This is equivalent to saying that this equality holds as functions in $L^2(\Sigma,\ d\sigma)$.

Theorem 7.3. *Assume (A1), (A2), (B1) hold and there is a finite energy solution $\hat{v} \in$ ker$\mathcal{L}(\Sigma)$ of (7.1) − (7.2), then $\hat{v}$ is the unique solution of this problem in $H^1(\Omega)$.*

Proof. Suppose that v_1, v_2 are two solutions of the problem. Then $w := v_2 - v_1$ is an $\mathcal{L}$ −homogeneous function with $w = 0\ \sigma$ *a.e.* on Σ. This implies $w \in H^1_{\Sigma 0}(\Omega)$. If w is in both $H^1_{\Sigma 0}(\Omega)$ and its $\mathcal{A}_1$−orthogonal complement, it must be zero. Thus $v_2 = v_1$ in $H^1(\Omega)$. □

To obtain solvability conditions for this extension problem more information about the boundary traces of functions in $\ker \mathcal{L}(\Sigma)$ is required. To obtain this, an orthogonal basis of this Hilbert space will be described as a class of eigenfunctions in the next section. Then the functions μ for which the problem has an finite energy solution will be characterized in terms of their expansions with respect to this basis.

8. Mixed Steklov Eigenproblems

Here methods similar to those used in section 5 will be used to construct an orthogonal basis for the Hilbert space $\ker \mathcal{L}(\Sigma)$. The resulting functions obey Steklov-type eigenvalue conditions on Σ so they will be called mixed Steklov eigenproblems.

A real number δ is said to be a *mixed Steklov eigenvalue for* $\mathcal{L}, \Sigma$ provided there is a non-zero function $s \in H^1(\Omega)$ such that

$$\int_\Omega [(A\nabla s) \cdot \nabla u + a_0\, su]\ d^n x \ = \ \delta \int_\Sigma s\, u\, d\tilde{\sigma} \qquad \text{for all } u \in H^1(\Omega). \tag{8.1}$$

Any such s will be said to be called a *mixed Steklov eigenfunction for* $\mathcal{L}, \Sigma$ corresponding to the mixed Steklov eigenvalue δ. Since this right hand side is zero for all $u \in H^1_{\Sigma 0}(\Omega)$, each mixed Steklov eigenfunction s is in $\ker \mathcal{L}(\Sigma)$. Clearly (8.1) is the weak form of the system

$$\mathcal{L}\, s(x) \quad \equiv \quad 0 \qquad \text{on } \Omega \ \text{ subject to} \tag{8.2}$$

$$(A\nabla s)\cdot \nu \quad = \quad \delta\, \sigma(\Sigma)^{-1} s \qquad \text{on } \Sigma \ \text{ and } \qquad (A\nabla s)\cdot\nu \ = \ 0 \qquad \text{on } \tilde{\Sigma}. \tag{8.3}$$

Eigenfunctions of this type for the Laplacian with n = 2 or 3 have been studied as modes in the theory of sloshing of a fluid and some analyses of these problems is described in [13] and [17]. When $\Sigma = \partial\Omega$, solutions of (8.1) are called Steklov eigenfunctions and were studied by the author in [4] and [5]. Here similar methods will be adapted to this problem with Σ is a proper open subset of $\partial\Omega$ obeying condition (B2).

Put $u = s$ in (8.1), then

$$\int_\Omega [(A\nabla s) \cdot \nabla s \ + \ a_0 s^2]\ d^n x \ = \ \delta \int_\Sigma s^2\, d\tilde{\sigma}$$

so when a_0 is non-zero, or s is non-constant, on Ω, the corresponding mixed Steklov eigenvalues are strictly positive. When $a_0 \equiv 0$ on Ω, then $\delta_1 = 0$ is an eigenvalue of (8.1) with constant functions on Ω as associated eigenfunctions. In the following analysis, we will generally assume give the details for the case where a_0 is not identically zero on Ω. Any differences when $a_0 \equiv 0$ will be noted.

The successive Steklov eigenfunctions, will be characterized by variational principles that involve maximizing the boundary functional

$$\mathcal{Q}_\Sigma(u) \;:=\; \int_\Sigma u^2 \, d\tilde{\sigma} \tag{8.4}$$

on various closed convex subsets of $H^1(\Omega)$. To find the least mixed Steklov eigenvalue, consider the problem of minimizing $\mathcal{Q}$ on the closed convex set $B_1 := \{u \in H^1(\Omega) : \|u\|_{\mathcal{L}\,\Sigma} \le 1\}$.

Theorem 8.1. *Assume (A1), (A2), (B1) and (B2) hold then there are maximizers $\pm s_1$ of $\mathcal{Q}_\Sigma$ on B_1. The maximizers satisfy (8.1) and $\|u\|_{\mathcal{L}\,\Sigma} = 1$. The corresponding eigenvalue δ_1 is positive and is the least eigenvalue of this system.*

Proof. When (B1) holds the trace map γ is a compact map of $H^1(\Omega)$ into $L^2(\partial\Omega, d\sigma)$, so $\mathcal{Q}_\Sigma$ will be a weakly continuous functional on $H^1(\Omega)$. The set B_1 is closed, convex and bounded so it is weakly compact in $H^1(\Omega)$. Thus $\mathcal{Q}_\Sigma$ attains a finite strictly positive supremum on B_1 provided $\sigma(\Sigma) > 0$.

The functionals here all are quadratic, G-differentiable and convex, so it is straightforward to show that the extremality condition for this problem is (8.1) for some real δ. If $\|u\|_{\mathcal{L}\,\Sigma} < 1$, then there is a $c > 1$ such that $cs_1 \in B_1$ and $\mathcal{Q}_\Sigma(cs_1) = c^2 \mathcal{Q}_\Sigma(s_1) > \mathcal{Q}_\Sigma(s_1)$ which contradicts the assumption that s_1 is a maximizer. Hence we must have $\|u\|_{\mathcal{L}\,\Sigma} = 1$. When a_0 is not identically zero, then $\delta_1 > 0$ as discussed above. When $a_0 \equiv 0$, then the least eigenvalue will be zero.

Suppose s is a mixed Steklov eigenfunction with $\|s\|_{\mathcal{L}\,\Sigma} = 1$ and mixed Steklov eigenvalue δ. Put $u = s$ in (8.1), then

$$1 \;=\; (1 + \delta)\; \mathcal{Q}_\Sigma(s)$$

Suppose $\delta < \delta_1$ is a mixed Steklov eigenvalue and s is the associated eigenfunction, then this implies that $\mathcal{Q}_\Sigma(s) > \mathcal{Q}_\Sigma(s_1)$. This contradicts the assumption that s_1 maximizes $\mathcal{Q}_\Sigma$ on B_1. Hence δ_1 is the least mixed Steklov eigenvalue. □

Lemma 8.2. *Suppose (A1), (A2), (B1) and (B2) hold and s_j, s_k are mixed Steklov eigenfunctions corresponding to distinct mixed Steklov eigenvalues δ_j, δ_k. Then*

$$\int_\Sigma s_j\, s_k \, d\tilde{\sigma} \;=\; 0 \qquad \text{and} \qquad [s_j, s_k]_{\mathcal{L}\,\Sigma} \;=\; 0. \tag{8.5}$$

Proof. When s_j is a mixed Steklov eigenfunction corresponding to an eigenvalue δ_j then, from (8.1) and (3.8)

$$[u, s_j]_{\mathcal{L}\,\Sigma} \;=\; (1 \,+\, \delta_j) \int_\Sigma u\, s_j \, d\tilde{\sigma} \qquad \text{for all } u \in H^1(\Omega) \tag{8.6}$$

Put $u = s_k$ here, then

$$(1 \,+\, \delta_j) \int_\Sigma s_k\, s_j \, d\tilde{\sigma} \;=\; (1 \,+\, \delta_k) \int_\Sigma s_k\, s_j \, d\tilde{\sigma}$$

When $\delta_j \neq \delta_k$, this implies that

$$\int_\Sigma s_j\, s_k\, d\tilde{\sigma} \ = \ 0$$

so the first equality in (8.5) holds. Substitute this in (8.6) to obtain the second equality. □

Functions $u, v \ \in \ker \mathcal{L}\,(\Sigma)$ are said to be $\Sigma-$orthogonal provided $\langle u, v\rangle_\Sigma = 0$ with the inner product (7.7). When s_j is an mixed Steklov eigenfunction then (8.6) yields

$$\|s_j\|^2_{\mathcal{L}\,\Sigma} \ = \ (1 \ + \ \delta_j) \ \|s_j\|^2_\Sigma. \tag{8.7}$$

Given the first M-1 mixed eigenvalues of this problem and a corresponding family of mixed Steklov eigenfunction for $\mathcal{L}\,, \Sigma$, there is a variational principle for determining the next smallest mixed eigenvalue. Let the first $M-1$ mixed Steklov eigenvalues be $0 \leq \delta_1 \leq \delta_2 \leq \ldots \leq \delta_{M-1}$ and $\mathcal{S}_{M-1} \ := \ \{s_j : 1 \leq j \leq M-1\}$ be an associated set of mixed Steklov eigenfunctions. For $M \geq 2$, assume that the functions in $\mathcal{S}_{M-1}$ are normalized so that

$$\|s_j\|_{\mathcal{L}\,\Sigma} \ = \ 1 \qquad \text{and} \quad \langle s_j, s_k\rangle_\Sigma \ = \ 0 \qquad \text{for } 1 \leq j < k \leq M-1. \tag{8.8}$$

Consider the problem of maximizing the functional $\mathcal{Q}_\Sigma$ defined by (8.4) on

$$B_M := \{u \in B_1 : \langle u, s_j\rangle_\Sigma = 0 \ for \ 1 \leq j \leq M-1\}.$$

For each integer M, B_M is non-empty from proposition 7.2.

Theorem 8.3. *Assume (A1), (A2), (B1) and (B2) hold. Then there are maximizers $\pm s_M$ of $\mathcal{Q}_\Sigma$ on B_M that satisfy (8.1) and $\|s_M\|_{\mathcal{L}\,\Sigma} = 1$. The corresponding eigenvalue δ_M is the least mixed Steklov eigenvalue greater than or equal to δ_{M-1}.*

Proof. The set B_M is a bounded, closed convex set in $H^1(\Omega)$, so it is weakly compact. The functional $\mathcal{Q}_\Sigma$ is weakly continuous on $H^1(\Omega)$ so it attains its supremum on B_M. This supremum will be strictly positive from proposition 7.2. Let s_M be a maximizer then $-s_M$ is also in B_M and takes the same value so it is also a maximizer. The proof that $\|s_M\|_{\mathcal{L}\,\Sigma} = 1$ is the same as in Theorem 8.1.

Consider the Lagrangian functional $\mathcal{M}_{\,S} : H^1(\Omega) \times [0, \infty) \times \mathbb{R}^{M-1} \to \mathbb{R}$ defined by

$$\mathcal{M}_{\,S}(v, \mu, \xi) \ := \ -\mathcal{Q}_\Sigma(v) \ + \ \mu \ \left(\|v\|^2_{\mathcal{L}\,\Sigma} - 1\right) \ - \ \sum_{j=1}^{M-1} \ \xi_j \ \langle v, s_j\rangle_\Sigma. \tag{8.9}$$

This functional has the property that

$$\sup_{v\in B_M} \mathcal{Q}_\Sigma(v) \ = \ - \inf_{v\in H^1(\Omega)} \ \sup_{\mu,\xi} \ \mathcal{M}_{\,S}(v, \mu, \xi) \tag{8.10}$$

The maximizers of $\mathcal{Q}_\Sigma$ on B_M are critical points of $\mathcal{M}_{\,S}$ from the multiplier theorem for problems with convex constraints. The critical points of $\mathcal{M}_{\,S}$ satisfy

$$\mu \int_\Omega \ [(A\nabla v)\cdot\nabla w \ + \ a_0\, v w] \ dx \ + \ (\mu - 1) \int_\Sigma \ v\, w \ d\tilde{\sigma} \ = \ 0.5 \ \sum_{j=1}^{M-1} \ \xi_j \ \langle w, s_j\rangle_\Sigma. \tag{8.11}$$

for all $w \in H^1(\Omega)$, some $\mu \geq 0$ and $\xi \in \mathbb{R}^{M-1}$. If $\mu = 0$, this implies that $\hat{v}$ is a linear combination of the s_j, so $\hat{v} = 0$ as it is in B_M. Thus $\mu > 0$ as 0 cannot be a maximizer of $\mathcal{Q}_\Sigma$ on B_M. Divide by μ then $\hat{v}$ satisfies

$$\mathcal{A}_1(v, w) - \delta \int_\Sigma v\, w\, d\tilde{\sigma} = (2\mu)^{-1} \sum_{j=1}^{M-1} \xi_j \langle s_j, w \rangle_\Sigma$$

for all $w \in H^1(\Omega)$ and with $\delta := \mu^{-1} - 1$.

When $M = 2$, take $v = s_2,\ w = s_1$ here, then $\mathcal{A}(s_1, s_2) = 0$ so $\xi_1 = 0$ and s_2 is a solution of (8.1) corresponding to an eigenvalue δ_2. Just as for the DN eigenproblem case, we can now show that δ_2 is the least eigenvalue greater than δ_1. An induction argument generalizes this proof to an arbitrary integer M. □

This result shows that if (8.8) holds for $\mathcal{S}_{M-1}$ then it continues to hold for $\mathcal{S}_M$ with this choice of s_M. Iterate this process to obtain an increasing sequence $\{\delta_j : j \geq 1\}$ of eigenvalues and a corresponding $\mathcal{L}\,\Sigma-$orthonormal sequence of eigenfunctions $\mathcal{S} := \{s_j : j \geq 1\}$ of (8.1). The following theorems provide some standard properties of these eigenvalues and eigenfunctions.

Theorem 8.4. *Assume (A1), (A2), (B1) and (B2) hold and (δ_j, s_j) are successive mixed Steklov eigenvalues and eigenfunctions constructed iteratively so that (8.8) holds for all M. Then each eigenvalue δ_j has finite multiplicity, $\delta_j \to \infty$ as $j \to \infty$ and $\mathcal{S}$ is a maximal $\mathcal{L}\,\Sigma-$orthonormal set in ker$\,\mathcal{L}\,(\Sigma)$.*

Proof. Put $u = s_j$ in (8.7) then, for all $j \geq 1$,

$$(1 + \delta_j) \int_\Sigma {s_j}^2\, d\tilde{\sigma} = 1. \tag{8.12}$$

The sequence $\mathcal{S}$ of mixed Steklov eigenfunctions is an infinite $\mathcal{L}\,\Sigma$ orthonormal set in $H^1(\Omega)$, so it converges weakly to zero. Then γs_j converges strongly to zero in $L^2(\partial\Omega, d\sigma)$ as γ is compact. This together with (8.12) implies that δ_j cannot be bounded so δ_j must increase to ∞ as j increases.

Suppose the sequence $\mathcal{S}$ is not maximal. Then there is a $w \in \ker \mathcal{L}\,(\Sigma)$ with

$$\|w\|_{\mathcal{L}\,\Sigma} = 1 \qquad \text{and} \qquad [w, s_j]_{\mathcal{L}\,\Sigma} = 0 \qquad \text{for all } j \geq 1. \tag{8.13}$$

From proposition 7.2, $\mathcal{Q}_\Sigma(w)$ is strictly positive as w is non-zero and $\mathcal{Q}_\Sigma(s_j) \to 0$ as $j \to \infty$ from (8.12). Let J be the first value of j for which $\mathcal{Q}_\Sigma(s_J) < \mathcal{Q}_\Sigma(w)$. Then s_J can not be the maximizer of $\mathcal{Q}_\Sigma$ on B_{J-1}. This contradicts the definition of s_J so there is no such w and $\mathcal{S}$ is maximal as claimed. □

9. Finite Energy Solutions of the Extension problem.

It is well-known [16], [26] that the criteria for the existence of H^1-solutions of a Dirichlet problem for Poisson's equation on a bounded region satisfying (B1) is that the boundary trace η be in a space usually denoted $H^{1/2}(\partial\Omega)$. This is a proper dense subspace of $L^2(\partial\Omega, d\sigma)$. This criterion may be generalized to our mixed boundary value problem. That is, a necessary and sufficient criterion on the boundary data η, Σ is found, for the existence of a finite-energy solution of this extension problem. The criterion involves a spectral condition similar to the intrinsic definition of $H^{1/2}(\partial\Omega)$ described in Auchmuty [5].

Henceforth $L^2(\Sigma,\ d\tilde{\sigma})$ is the usual Lebesgue space with inner product defined by (7.7). An obvious necessary condition for there to be a finite energy solution of $(7.1)-(7.2)$ is that $\eta \in L^2(\Sigma,\ d\tilde{\sigma})$ since the trace map $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is linear and continuous.

Let $\mathcal{S}$ be the maximal $\mathcal{L}\,\Sigma-$ orthonormal sequence of mixed Steklov eigenfunctions defined in the previous section. For $j \geq 1$, define $g_j : \Sigma \to [-\infty, \infty]$ by

$$g_j(y) \ := \ (1 + \delta_j)^{1/2}\ (\gamma s_j)(y) \qquad \text{for } y \in \Sigma. \tag{9.1}$$

From (8.1), and (8.8) the sequence $\tilde{\mathcal{S}} \ := \ \{g_j : j \geq 1\}$ is an orthonormal subset of $L^2(\Sigma,\ d\tilde{\sigma})$.

Given $\eta \in L^2(\Sigma,\ d\tilde{\sigma})$, the *j-th* $\Sigma-$*Steklov coefficient* of η with respect to $\tilde{\mathcal{S}}$ is

$$c_j \ := \ \langle \eta, g_j \rangle_\Sigma \ := \ \int_\Sigma \ \eta\, g_j \ \ d\tilde{\sigma} \tag{9.2}$$

The results about the extension problem may be summarized as follows.

Theorem 9.1. *Assume (A1)-(A3), (B1) and (B2) hold and* (δ_j, s_j) *are successive mixed Steklov eigenvalues and eigenfunctions as above. Then there is a solution* $\hat{v} \in H^1(\Omega)$ *of* $(7.1)-(7.2)$ *provided the* $\Sigma-$*Steklov coefficients* $\{c_j : j \geq 1\}$ *of* η *satisfy*

$$\sum_{j=1}^{\infty} (1 + \delta_j)\ {c_j}^2 \ < \ \infty \tag{9.3}$$

When this holds, the solution of $(7.1)-(7.2)$ *is*

$$\hat{v}(x) \ = \ \sum_{j=1}^{\infty} (1+\delta_j)^{1/2}\ c_j\ s_j(x) \qquad \textit{and} \qquad \|\hat{v}\|^2_{\mathcal{L}\,\Sigma} \ = \ \sum_{j=1}^{\infty} (1 + \delta_j)\ {c_j}^2. \tag{9.4}$$

Proof. If $(7.1)-(7.2)$ has a solution in $H^1(\Omega)$, then it will be in $\ker \mathcal{L}\,(\Sigma)$, so from theorem 8.4 and the Riesz-Fischer theorem, it has an $\mathcal{L}\,\Sigma-$orthonormal representation of the form

$$\hat{v}(x) \ = \ \sum_{j=1}^{\infty} a_j\ s_j(x) \qquad \text{with} \qquad \|\hat{v}\|^2_{\mathcal{L}\,\Sigma} \ = \ \sum_{j=1}^{\infty} {a_j}^2 \tag{9.5}$$

as $\mathcal{S}$ is a basis of $\ker \mathcal{L}\,(\Sigma)$. Apply the trace operator to this, then $\hat{v}$ is the finite energy solution of $(7.1)-(7.2)$ if and only if

$$(\gamma\hat{v})(x) \ = \ \sum_{j=1}^{\infty} a_j\ (\gamma s_j)(x) \ = \ \eta(x) \qquad \text{on } \Sigma.$$

Take inner products on Σ of this with g_k, then

$$\langle \gamma\hat{v}, g_k \rangle_\Sigma \ = \ a_k\ (1 + \delta_k)^{-1/2} \ = \ c_k$$

using (9.1) and the orthonormality of $\tilde{\mathcal{S}}$. Thus

$$a_k \ = \ (1 + \delta_k)^{1/2} c_k \qquad \text{for each } k \geq 1. \tag{9.6}$$

Substitute this in (9.5) to obtain (9.4). The uniqueness of this solution was proved in theorem 7.3. □

Define the space $H^{1/2}(\Sigma)$ to be the subspace of $L^2(\Sigma,\ d\tilde{\sigma})$ for which (9.3) holds. This theorem says that $H^{1/2}(\Sigma)$ is the space of allowable traces of $\mathcal{L}\,-$homogeneous functions on Ω that satisfy the no flux condition (7.5) on $\tilde{\Sigma}$ and have finite energy. $H^{1/2}(\Sigma)$ is a real Hilbert space under the inner product

$$\langle u, v\rangle_{1/2,\Sigma} \;:=\; \sum_{j=1}^{\infty} (1 \,+\, \delta_j)\, u_j\, v_j \tag{9.7}$$

where u_j, v_j are the $\Sigma-$Steklov coefficients of u, v. Let $Q_{\mathcal{L}\Sigma}$ be the projection operator of $H^1(\Omega)$ onto the subspaces $\ker \mathcal{L}\,(\Sigma)$ associated with the decompositions of (7.3). Theorems 7.3 and 9.1 may be combined as follows.

Corollary 9.2. *When the conditions of theorem 9.1 hold, the $\Sigma-$trace map $\gamma Q_{\mathcal{L}\Sigma} : H^1(\Omega) \to H^{1/2}(\Sigma)$ is surjective. It is a linear isomorphism of ker$\,\mathcal{L}\,(\Sigma)$ and $H^{1/2}(\Sigma)$. For each $\eta \in H^{1/2}(\Sigma)$ there is a unique solution $\hat{v} \in H^1(\Omega)$ of* (7.1) – (7.2) *obeying* (9.4).

Proof. Theorem 9.1 shows that for each $\eta \in H^{1/2}(\Sigma)$, there is a $v \in \ker \mathcal{L}\,(\Sigma)$ with $\gamma v = \eta$. Hence this map is surjective. Theorem 7.3 says that the mapping is 1-1, so this result holds. □

This corollary implies that a necessary and sufficient condition for there to be a finite energy solution of (7.1) – (7.2) is that $\eta \in H^{1/2}(\Sigma)$.

These weak solutions $\hat{v}$ of (7.1) – (7.2) may be approximated by certain boundary integrals and represented formally by boundary integral operators. Let v_M be the M-th partial sum of the series in (9.4), then

$$v_M(x) \;:=\; \sum_{j=1}^{M} (1+\delta_j)^{1/2}\, s_j(x) \int_{\Sigma} \eta\; g_j\; d\tilde{\sigma} \tag{9.8}$$

$$= \int_{\Sigma} P_M(x,y)\; \eta(y)\; d\tilde{\sigma}(y) \qquad \text{where} \tag{9.9}$$

$$P_M(x,y) \;:=\; \sum_{j=1}^{M} (1+\delta_j)\, s_j(x)\, \gamma s_j(y). \tag{9.10}$$

Each P_M is a well-defined function on $\overline{\Omega} \times \Sigma$ and theorem 9.1 says that

$$\hat{v}(x) \;=\; \lim_{M\to\infty} \int_{\Sigma} P_M(x,y)\; \eta(y)\; d\tilde{\sigma}(y) \qquad \text{for } x \in \Omega. \tag{9.11}$$

This series converges strongly in $H^1(\Omega)$ and (9.10) provides convergent finite rank approximations to the solution $\hat{v}$.

By superposition, this solution together with theorem 6.1 combine to provide a spectral decomposition of the unique solution $\tilde{u}$ of the original problem (1.1) − (1.2). It is

$$\tilde{u}(x) \;=\; \sum_{k=1}^{\infty} \left[{\lambda_k}^{-1} \, (\rho_k \,+\, \eta_k) \;\; w_k(x). \,+\, \; (1+\delta_k)^{1/2} \; c_k \; s_k(x) \right]. \tag{9.12}$$

Here the w_k are L^2- orthonormal DN eigenfunctions on Ω, while the s_k are mixed Steklov eigenfunctions that are $\mathcal{L}\,\Sigma-$ orthonormal on Ω. The coefficients ρ_k, η_k come from (6.3) with η_2 in place of η. c_k is defined by (9.2) with η_1 in place of η. Note that the each solution operator involved here is either continuous or compact so these problems are well-posed under our assumptions provided also that $\eta_1 \in H^{1/2}(\Sigma)$.

10. Mixed Problems on a Finite Cylinder

To illustrate this approach, consider the problem of solving Poisson's equation on a finite circular cylinder with mixed boundary conditions. Take the z-axis to be the axis of symmetry and normalize the radius of a cross-section of the cylinder to be 1. Assume the height of the cylinder is 2h and the plane $z = 0$ is the midplane of the cylinder. Dirichlet conditions are given on the bottom and top plates at $z = \pm h$ and prescribed flux conditions hold on the sides of the cylinder at $r = 1$. Problems such as this arise in the theory of cylindrical capacitors, see [11], Chapter 2, section 2 where references dating back to Kirchoff in 1877 are given.

Cylindrical polar coordinates (r, θ, z) will be used, $B_1 \;:=\; \{(r,\theta) : 0 \le r < 1, \theta \in [-\pi, \pi], \}$ is the open unit disc in the plane and $\Omega \;:=\; B_1 \times (-h, h)$. Write Σ_{-1}, Σ_1 for the bottom and top plates respectively so that $\Sigma \;:=\; \Sigma_{-1} \cup \Sigma_1$. Then $\tilde{\Sigma}$ is the open cylindrical surface $\tilde{\Sigma} \;:=\; \{(1, \theta, z) : \theta \in [-\pi, \pi], -h < z < h\,\}$.

Consider the problem of solving Poisson's equation on this cylinder. The boundary value problem becomes

$$-\Delta u(x) \quad = \quad \rho(x) \quad \text{on } \Omega, \quad \text{subject to} \tag{10.1}$$

$$u(r,\theta,h) \;=\; \eta_1(r,\theta) \qquad \text{and} \qquad u(r,\theta,-h) \;=\; \eta_0(r,\theta) \quad \text{on} \quad B_1, \tag{10.2}$$

$$\frac{\partial u}{\partial r}\,(1,\theta,z) \quad = \quad \eta_2(\theta,z) \quad \text{on } \tilde{\Sigma} \tag{10.3}$$

The conditions (A1) - (A2) and (B1)-(B2) obviously hold. To investigate the first problem associated with this system we require

(A4): ρ is in $L^p(\Omega)$ for some $p \ge 6/5$ and $\eta_2 \in L^q(\partial\Omega, d\sigma)$ for some $q \ge 4/3$.

The component $\hat{u} \in H^1_{\Sigma 0}(\Omega)$ is found by solving the variational problem described in section 4. That is we seek the possible minimizers of the functional

$$\mathcal{D}(u) \;:=\; \int_{\Omega} \left[\, |\nabla u|^2 \right] \,-\, 2\,\rho\, u]\; d^3x \,-\, 2 \int_{\tilde{\Sigma}} \eta_2\, u\; d\sigma. \tag{10.4}$$

on $H^1_{\Sigma 0}(\Omega)$. This is a standard coercive convex quadratic variational principle that has a unique minimizer.

The solutions of this variational problem have a representation in terms of the eigenfunctions of the Laplacian with mixed homogeneous boundary conditions. The mixed eigenvalue problem associated with this is to find the values of λ for which there are non-trivial solutions of

$$\int_\Omega [(\nabla v)\cdot \nabla w \;-\; \lambda\, v\, w]\; d^3x \;=\; 0 \qquad \text{for all } w \in H^1_{\Sigma 0}(\Omega). \tag{10.5}$$

The eigenfunctions of this mixed Laplacian eigenvalue problem may be found using separation of variables in a standard manner. There are two classes of eigenfunctions. The axisymmetric eigenfunctions have the form

$$e_{kl0}(r) \;:=\; J_0(\xi_{0l} r)\, \cos\Big((2k-1)\frac{\pi z}{2h}\Big).$$

There also are eigenfunctions of the forms

$$e_{klm}(r) \quad := \quad J_m(\xi_{ml} r)\, \cos\Big((2k-1)\frac{\pi z}{2h}\Big) \cos m\theta, \qquad \text{and} \tag{10.6}$$

$$f_{klm}(r) \quad := \quad J_m(\xi_{ml} r)\, \cos\Big((2k-1)\frac{\pi z}{2h}\Big) \sin m\theta \tag{10.7}$$

Here $m, k \in \mathbb{N}$, J_m is the usual Bessel function of integer order and ξ_{ml} is the l-th positive zero of J'_m.

These eigenfunctions may be normalized to be an L^2- orthogonal basis of $H^1_{\Sigma 0}(\Omega)$ and the minimizers of the functional $\mathcal{D}$ may be represented as a infinite series involving these eigenfunctions as described in section 6. When (A4) holds this series converges in $H^1_{\Sigma 0}(\Omega)$. The partial sums of this series may be regarded as Galerkin approximations to the solution.

The mixed Steklov eigenproblem associated with this is quite different. The weak form of the system is, from (8.1),

$$\int_\Omega \nabla s\cdot \nabla u\; d^3x \;=\; \delta \int_\Sigma s\, u\; r dr d\theta \qquad \text{for all } u \in H^1(\Omega). \tag{10.8}$$

This is the weak form of the system

$$-\Delta s(x) \;=\; 0 \quad \text{on} \quad \Omega, \quad \text{subject to} \quad \frac{\partial s}{\partial r}(1,\theta,z) \;=\; 0 \quad \text{on } \tilde{\Sigma} \tag{10.9}$$

$$\frac{\partial s}{\partial z} \;=\; \delta s \quad \text{on z = h} \quad \& \qquad \frac{\partial s}{\partial z} \;=\; -\delta s \quad \text{on z = - h.} \tag{10.10}$$

The non-trivial solutions of this problem may be found using separation of variables. Let e_k be the k-th Neumann eigenfunction of the Laplacian on B_1 that satisfies

$$\int_\Omega \nabla e\cdot \nabla v \;=\; {\mu_k}^2 \int_\Omega e\, v\; d^3x \qquad \text{for all } v \in H^1(\Omega) \tag{10.11}$$

The first eigenvalue is $\lambda_0 = \mu_0 = 0$ and the subsequent eigenvalues are $\lambda_k = {\mu_k}^2 > 0$. Normalize the eigenfunctions to be L^2-orthogonal on B_1 so that

$$\int_{B_1} e_k\, e_l \;\; r\, dr\, d\theta \;=\; 0 \quad \text{when } k \neq l \text{ and} \qquad \int_{B_1} {e_k}^2 \;\; r\, dr\, d\theta \;=\; \pi \quad \text{for} \quad k \geq 0. \tag{10.12}$$

Some straightforward analysis yields that the mixed Steklov eigenfunctions of this problem are

$$v_0(r,\theta,z) \equiv 1 \quad \text{corresponding to} \quad \delta_0 = 0, \tag{10.13}$$
$$\tilde{v}_0(r,\theta,z) \equiv z/h \quad \text{corresponding to} \quad \tilde{\delta}_0 = 1/h, \tag{10.14}$$
$$v_k(r,\theta,z) \equiv e_k(r,\theta)\, c_k(z) \quad \text{corresponding to} \quad \delta_k = \mu_k \tanh \mu_k h \tag{10.15}$$
$$\tilde{v}_k(r,\theta,z) \equiv e_k(r,\theta)\, s_k(z) \quad \text{corresponding to} \quad \tilde{\delta}_k = \mu_k \coth \mu_k h \tag{10.16}$$

Here the functions c_k, s_k are defined for $k \geq 1$ by

$$c_k(z) := \frac{\cosh \mu_k z}{\cosh \mu_k h} \quad \text{and} \quad s_k(z) := \frac{\sinh \mu_k z}{\sinh \mu_k h}.$$

The mixed Steklov eigenvalues will be the union of these sequences of $\delta_k, \tilde{\delta}_k$. Note that the two classes of eigenfunctions here correspond to functions that are even or odd respectively about the midplane z=0.

When the functions η_0, η_1 are in $L^2(B_1)$, they will have expansions in terms of the Neumann eigenfunctions of $-\Delta$ on B_1. Suppose that these expansions are

$$\eta_j(r,\theta) = \sum_{k=0}^{\infty} a_k^{(j)}\, e_k(r,\theta) \qquad \text{with } a_k^{(j)} := \pi^{-1} \int_{B_1} \eta_j e_k(r,\theta)\, r\, dr\, d\theta. \tag{10.17}$$

The mixed extension problem is to find the solution w of Laplace's equation on Ω that satisfies the mixed boundary conditions

$$w(r,\theta,h) = \eta_1(r,\theta) \quad \text{and} \quad w(r,\theta,-h) = \eta_0(r,\theta) \quad \text{on} \quad B_1, \tag{10.18}$$
$$\frac{\partial w}{\partial r}(1,\theta,z) = 0 \quad \text{on } \tilde{\Sigma} \tag{10.19}$$

Following the analysis of the previous section, the solution is

$$w(r,\theta,z) = \sum_{k=0}^{\infty} \left[a_k\, v_k(r,\theta,z) + b_k\, \tilde{v}_k(r,\theta,z)\right] \qquad \text{with} \tag{10.20}$$
$$a_k := a_k^{(0)} + a_k^{(1)} \qquad \text{and} \quad b_k := a_k^{(1)} - a_k^{(0)}. \tag{10.21}$$

In particular, from theorem 9.1, this solution will have finite energy provided

$$\sum_{k=0}^{\infty} \left[(1 + \delta_k)\, {a_k}^2 + (1 + \tilde{\delta}_k)\, {b_k}^2\right] < \infty \tag{10.22}$$

Since $\mu_k := \sqrt{\lambda_k}$ and both $\tanh z, \coth z$ converge exponentially to 1 as z increases, this criterion will hold when the functions η_0, η_1 obey the standard criteria to be in the space $H^{1/2}(B_1)$.

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