STEKLOV REPRESENTATIONS OF GREEN'S FUNCTIONS FOR LAPLACIAN BOUNDARY VALUE PROBLEMS.

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ABSTRACT. This paper describes different representations for solution operators of Laplacian boundary value problems on bounded regions in $\mathbb{R}^N, N \geq 2$ and in exterior regions when $N = 3$. Null Dirichlet, Neumann and Robin boundary conditions are allowed and the results hold for weak solutions in relevant subspaces of Hilbert - Sobolev space associated with the problem. The solutions of these problems are shown to be strong limits of finite rank perturbations of the fundametal solution of the problem. For exterior regions these expressions generalize multipole expansions.

1. INTRODUCTION

This paper will describe some different representations of the Green's functions (or solution operators) for Laplacian boundary value problems. The representations hold when Dirichlet, Neumann or Robin conditions are imposed and for exterior regions as well as on bounded domains with Lipschitz boundaries. The representations involve the fundamental solution of the Laplacian and the Steklov eigenfunctions of the Laplacian and are shown to converge in various Sobolev-type norms and follow from the construction of orthogonal bases of relevant Sobolev-Hilbert spaces.

The results may be related to the methods described in the classic text of Bergman and Schiffer [10]. We concentrate our attention on how the Green's functions differ from the fundamental solution of the differential operator. That is we seek to describe the *boundary correction* (BC) kernel $B(.,.)$ for various operators and boundary conditions such that

$$G(x,y) \;=\; \Gamma(x,y) \,-\, B(x,y) \quad \text{for } (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{1.1}$$

Here G, Γ are the Green's function and the fundamental solution respectively.

The (integral) operator $\mathcal{B}$ associated with this BC kernel is shown to be the limit of finite rank kernels involving the Steklov eigenfunctions and their single and double layer potentials. These approximations converge in H^1-norms and, in general, are not L^2-orthogonal expansions. It is of particular interest to note that this analysis applies to boundary value problems in exterior regions where the standard Green's function may not be represented using eigenfunctions.

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It does not appear to be the same as any of the kernels studied in [10]. Their kernel function for the Laplacian was described in terms of Steklov eigenfunctions in section 8 of Auchmuty [5]. The methods used, and results obtained, also are quite different to those used in the theory of boundary integral methods as described by Folland [17] or McLean [21] amongst others.

After some introductory material a summary of results about Steklov bases for solution spaces of linear homogeneous equations is given in section 4. The original results are primarily due to the author for bounded regions Ω and to joint work with Qi Han on exterior regions when $N \geq 3$. These results are used in sections 5 and 6 to obtain representation theorems for the solution operators as the difference between the fundamental solution and a correction operator that is the strong limit of specific finite rank integral operators. These operators are not symmetric in general - but the limits are.

In sections 7 and 8 the Dirichlet problem for these operators on exterior regions in $\mathbb{R}^3$ is investigated and similar formulae for the solution operators are found. For the Laplacian on the exterior of a ball, the Steklov eigenfunctions are spherical harmonics and the resulting formulae are multipole expansions for the solution. So the results found here provide solutions that have similar properties to multipole expansions but hold for general exterior regions.

2. Definitions and Notation.

This paper treats various Laplacian boundary value problems on regions in $\mathbb{R}^N$. A region is a non-empty, connected, open subset of $\mathbb{R}^N$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. All problems will be posed in a weak, or variational, form as described in the text of Attouch, Buttazzo and Michaille, [1] and the notation of that text will generally be used.

In particular all functions will be regarded as taking values in $\overline{\mathbb{R}} := [-\infty, \infty]$ and the Borel measurable representatives are used. $L^p(\Omega)$ and $L^p(\partial\Omega, d\sigma)$, $1 \leq p \leq \infty$ are the usual spaces with p-norm denoted by $\|u\|_p$ or $\|u\|_{p,\partial\Omega}$ respectively. When $p = 2$ these are real Hilbert spaces with inner products defined by

$$\langle u, v\rangle := \int_\Omega u(x)\ v(x)\ dx \qquad \text{and} \quad \langle u, v\rangle_{\partial\Omega} := \int_{\partial\Omega} u\ v\ \ d\sigma.$$

A function u on Ω is in $W^{1,p}(\Omega)$ provided u and each weak derivative $D_j u$ is in $L^p(\Omega)$. Then $\nabla u(x) := (D_1 u(x), \ldots, D_N u(x))$ is the gradient of u and $H^1(\Omega)$ is the usual real Sobolev space of functions on Ω. It is a real Hilbert space under the standard H^1- inner product

$$[u, v]_1 := \int_\Omega \ [u(x).v(x) \ + \ \nabla u(x) \cdot \nabla v(x)]\ dx. \tag{2.1}$$

and the corresponding norm is denoted $\|u\|_{1,2}$.

The region Ω is said to satisfy *Rellich's theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ where $p_S(n) := 2n/(n-2)$ when $n \geq 3$, or $p_S(2) = \infty$ when $n = 2$.

For our analysis some regularity of the boundary $\partial\Omega$ is required. The boundary should be of Hausdorff dimension $N-1$ and have finite surface area $\sigma(\partial\Omega)$, where σ denotes $(N-1)$-dimensional Hausdorff measure. Assume also that the boundary has a unit outward normal $\nu(z)$ defined σ *a.e.*

Let the boundary trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ be defined and continuous. It is the linear extension of the map restricting Lipschitz continuous functions on $\overline{\Omega}$ to $\partial\Omega$. Often γ is omitted so u is used in place of $\gamma(u)$ for the trace of a function on $\partial\Omega$. The region Ω is said to satisfy a *compact trace theorem* provided the boundary trace mapping $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is compact.

The *Gauss-Green* theorem holds on Ω provided

$$\int_\Omega u(x) D_j v(x)\, dx \;=\; \int_{\partial\Omega} \gamma(u)\, \gamma(v)\, \nu_j\, d\sigma - \int_\Omega v(x)\, D_j u(x)\, dx \quad \text{for } 1 \leq j \leq N. \tag{2.2}$$

for all $u,\ v$ in $H^1(\Omega)$. The requirements on the region will be

Condition B1: *Ω is a bounded region in $\mathbb{R}^N$ with boundary $\partial\Omega$ having a finite number of disjoint closed components, finite surface area and such that the Gauss-Green, Rellich and compact trace theorems hold.*

There is an extensive literature on the description of regions for which these conditions hold. Discussion of general regions for which (B1) holds may be found in Maz'ya and Poborchi [20], especially section 6.3, and also section 3 of Daners [13].

In this paper various equivalent inner products on $H^1(\Omega)$ will be used including

$$[u,v]_\partial := \int_\Omega \nabla u \cdot \nabla v \; dx + \int_{\partial\Omega} u\, v \;\, d\sigma. \tag{2.3}$$

The corresponding norm will be denoted by $\|u\|_\partial$. The proof that this norm is equivalent to the usual $(1,2)-$norm on $H^1(\Omega)$ when (B1) holds is Corollary 6.2 of [2] and also is part of theorem 21A of [24].

A function $u \in H^1(\Omega)$ is said to be *harmonic on* Ω provided it is a solution of Laplace's equation in the usual weak sense. Namely

$$\int_\Omega \nabla u \cdot \nabla \varphi \; dx \;=\; 0 \qquad \text{for all } \varphi \in C_c^1(\Omega). \tag{2.4}$$

Here $C_c^1(\Omega)$ is the set of all C^1-functions on Ω with compact support in Ω.

Define $\mathcal{H}(\Omega)$ to be the space of all such harmonic functions on Ω. When (B1) holds, the closure of $C_c^1(\Omega)$ in the H^1-norm is the usual Sobolev space $H_0^1(\Omega)$. Then (2.4) is equivalent to saying that $\mathcal{H}(\Omega)$ is $\partial-$orthogonal to $H_0^1(\Omega)$. This may be expressed as

$$H^1(\Omega) \;=\; H_0^1(\Omega) \oplus_\partial \mathcal{H}(\Omega), \tag{2.5}$$

where $\oplus_\partial$ indicates that this is a $\partial-$orthogonal decomposition.

Various spaces of continuous and C^1 functions will be used and $\|f\|_b$ will always denote the sup norm of the function f and the subscript c indicates compact support.

3. Laplacian Boundary Value Problems

This paper will treat the classic problem of representing the *solution operators* of regularized Laplacian boundary value problems of the form

$$\mathcal{L}_c\, u(x) \;:=\; c^2\, u(x) \,-\, \Delta\, u(x) \;=\; f(x) \qquad \text{on} \quad \Omega \tag{3.1}$$

Here Δ is the Laplacian; when $c = 0$ this is Poisson's equation. The boundary conditions that will be considered are zero Dirichlet boundary conditions $u \equiv 0$, Robin conditions $D_\nu u + bu \equiv 0$ with $b > 0$ and the Neumann condition $D_\nu u \equiv 0$ on $\partial\Omega$. Here $D_\nu\, u(x) := \nabla u(x) \cdot \nu(x)$ is the unit outward normal derivative of u at a point on the boundary.

A function $u \in H_0^1(\Omega)$ is a weak solution of the *Dirichlet problem* for (3.1) provided it satisfies

$$\int_\Omega \;[\, \nabla u \cdot \nabla \varphi \,+\, (c^2\, u - f)\, \varphi\,]\; dx \;=\; 0 \qquad \text{for all} \quad \varphi \in C_c^1(\Omega). \tag{3.2}$$

The solution operator for this problem is the linear mapping from the data f to the solution $\tilde{u}$. Usually $f \in X$ where X is some Banach space of functions with $X \subset L^1(\Omega)$. Historically, such solution operators have been written as integral operators with a weakly singular kernel called the *Green's function* of the problem. That is the solution of the zero-Dirichlet boundary value problem for (3.2) is written as

$$\tilde{u}(x) \;=\; \mathcal{G}_D(c)(c) f(x) \;=\; \int_\Omega \; G_D(x, y, c)\, f(y)\; dy. \tag{3.3}$$

Here $G_D(.,.,c)$ is a Borel-measurable function that is singular when $x = y$ and symmetric in x, y and there has been extensive study of the solution operator $\mathcal{G}_D(c)$ as a map of different spaces X into $H_0^1(\Omega)$.

A function $u \in H^1(\Omega)$ is a weak solution of the *Robin (Neumann) problem* for (3.1) provided it satisfies

$$\int_\Omega \;[\, \nabla u \cdot \nabla \varphi \,+\, (c^2\, u - f)\, \varphi\,]\; dx \,+\, b \int_{\partial\Omega} u\, \varphi\, d\sigma \;=\; 0 \qquad \text{for all} \quad \varphi \in H^1(\Omega). \tag{3.4}$$

In this case the solution operator is denoted $\mathcal{G}_b(c)$ when $b > 0$ and $G_N(c)$ for the Neumann problem with $b = 0$.

Thus the solution of the zero-Robin boundary value problem for (3.2) will be written as

$$\tilde{u}(x) \;=\; \mathcal{G}_b(c) f(x) \;=\; \int_\Omega \; G_b(x, y, c)\, f(y)\; dy \tag{3.5}$$

Similarly $\mathcal{G}_N(c), G_N$ will be the solution operators and kernels for the Neumann problem for (3.2) with $b = 0$. The functions G_D, G_b, G_N are the *Dirichlet, Robin and Neumann Green's functions* respectively for the operator $\mathcal{L}_c$ on Ω.

Most elementary texts in partial differential equations describe some constructions of Green's functions - with concrete examples for simple regions such as boxes or balls. There is a large literature on the existence of solution operators for these problems with data f in various spaces. For bounded domains there are simple proofs using variational methods. The chapter by Benilan in [15] provides a number of other approaches. Our interest here is in providing representations that have good approximation properties.

The representation of these Green's functions on general regions has been extensively studied when $f \in L^2(\Omega)$ using spectral theory. See Roach [22] for an introduction, Kato [19] for a thorough discussion and Duffy [16] for many explicit examples. However it is well known that the associated finite rank approximations of the solutions generally converge very slowly - and there is little proved about the convergence of their derivatives.

Here a very different approach based on the use of Steklov eigenfunctions and fundamental solutions will be developed. The analysis here is done directly in various subspaces of the Hilbert - Sobolev space $H^1(\Omega)$. Solutions will be sought as perturbations of the solution obtained using the *fundamental solution* Γ_c of $\mathcal{L}_c$. When $c = 0$ the subscript is omitted and they are the familiar functions

$$\Gamma(x) \quad := \quad k_2 \ln |x| \qquad \text{when} \quad N = 2, \quad \text{and} \tag{3.6}$$

$$\Gamma(x) \quad := \quad k_N \, |x|^{2-N} \qquad \text{when} \quad N \geq 3. \tag{3.7}$$

k_N is a constant that depends only on the dimension N. Formulae for these fundamental solutions when $c \neq 0$ are given in Treves [23] chapter 1, section 9 exercises 9.2 to 9.5. When N = 2, 3 the functions are respectively

$$\Gamma_c(x) \;:=\; \frac{1}{2\pi} K_0(c|x|) \qquad \text{or} \qquad \frac{e^{-c|x|}}{4\pi|x|}. \tag{3.8}$$

with K_0 being a modified Bessel function.

Consider the integral operator $V_c : X \to L^1_{loc}(\Omega)$ defined by

$$V_c(f) \;:=\; \int_\Omega \Gamma_c(x-y)\, f(y)\, dy \tag{3.9}$$

where this right hand side is defined as a convolution of distributions. $V_c(f)$ will be called the *potential solution* of equation (3.1).

A requirement will be that X is such that V_c maps X into $H^1(\Omega)$. When Ω is a bounded region in the plane then this condition holds if $X \subset L^p(\Omega)$ for some $p > 1$. When $N \geq 3$, this holds when $X \subset L^p(\Omega)$ for some $p \geq 2N/(N+2)$. These follow from standard estimates for the fundamental solution and Young's inequality for convolutions.

Quite often one is interested in whether the range of V_c is a subspace of the space $C_b(\Omega)$ of bounded continuous functions on Ω. From Young's inequality one has that $V_c : X \to C_b(\Omega)$ is continuous when $X \subset L^p(\Omega)$ with $p > N/2$ and $N \geq 3$.

Attention here is concentrated on the Boundary Correction operator $\mathcal{B}(c)$ and its kernel $B(.,.,c)$ where $\mathcal{B}(c)(f) \;:=\; V_c(f) \,-\, \mathcal{G}(c)(f)$ for $f \in X$ with kernel

$$B(x,y,c) \;=\; \Gamma_c(x-y) \,-\, G(x,y,c) \quad \text{for } (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{3.10}$$

where G is a Green's function for $\mathcal{L}_c$.

This kernel $B(.,.,c)$ depends on both the boundary condition and the region Ω. It will be shown that B is given as the limit of a sequence of finite rank kernels that involve the Steklov eigenfunctions of $\mathcal{L}_c$ on Ω. Convergence of this approximation is studied and other properties of this boundary kernel function $B(.,.,c)$ are found. In particular these methods work when Ω is an exterior region with $N \geq 3$ and these are problems where the associated L^2-operators do not have compact resolvents. Here the analysis for Dirichlet problems on exterior regions in $\mathbb{R}^3$ is described in sections 8 and 9.

4. The c-Harmonic Steklov Eigenproblem

The representations to be described in this paper are obtained using bases of certain Hilbert spaces of solutions of the homogeneous equation $\mathcal{L}_c u = 0$.

Consider the bilinear form $a_c : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$a_c(u,v) \ := \ [u,v]_c \ := \int_\Omega \ [\, c^2\, u\, v \ + \ \nabla u \cdot \nabla v\,] \ dx. \tag{4.1}$$

For $c > 0$ this defines an inner product on $H^1(\Omega)$ that induces an equivalent norm on $H^1(\Omega)$ to the standard norm $(c = 1)$.

A function $u \in H^1(\Omega)$ is said to be a weak solution of the regularized Laplace equation $\mathcal{L}_c u \ = \ 0$ provided it satisfies

$$\int_\Omega \ [\, c^2\, u\, \varphi \ + \ \nabla u \cdot \nabla \varphi\,] \ dx \ = \ 0 \qquad \text{for all} \quad \varphi \in C_c^1(\Omega). \tag{4.2}$$

For $c > 0$ such functions will be called *c-harmonic* or *regularized harmonic* functions. Let $N(\mathcal{L}_c)$ be the class of all H^1-solutions of this problem.

By a density argument (4.2) is equivalent to the decomposition result that

$$H^1(\Omega) \ = \ H_0^1(\Omega) \oplus_c N(\mathcal{L}_c) \tag{4.3}$$

where $\oplus_c$ indicates that these closed subspaces are c-orthogonal.

Many texts on trace theorems prove that the boundary trace space for H^1 functions on a region is isomorphic to the quotient space $H^1(\Omega)/H_0^1(\Omega)$ when $\partial\Omega$ is nice enough. Then the trace space is isomorphic to $N(\mathcal{L}_c)$ as the orthogonal complement in a Hilbert space is isomorphic to the quotient space. This was used by the author in [3] to describe trace spaces using a different inner product on $H^1(\Omega)$.

Our interest here is in describing orthonormal bases of the spaces $N(\mathcal{L}_c), L^2(\partial\Omega,\, d\sigma)$ and $H^{1/2}(\partial\Omega)$. When $c = 0, N(\mathcal{L}_c)$ is the space $\mathcal{H}(\Omega)$ of H^1-harmonic functions on Ω and an analysis of the Steklov basis of $\mathcal{H}(\Omega)$ is described in Auchmuty [3] for the case that Ω is bounded. The Steklov basis for the Laplacian on rectangles in the plane is described in Auchmuty and Cho [6.5] where it is shown that the resulting expansions converge very rapidly. An analysis for exterior regions is provided in Auchmuty and Han [8]. Both were based on Auchmuty [2]. Here the analogous constructions for the Steklov eigenfunctions associated with $\mathcal{L}_c$ for $c > 0$ will be described based on the constructions summarized in theorem 8.2 of Auchmuty [6].

Let Ω be a region in $\mathbb{R}^N$ that satisfies (B1). A non-zero function $s \in H^1(\Omega)$ is said to be a $\mathcal{L}_c$ *−Steklov eigenfunction* on Ω corresponding to the Steklov eigenvalue δ provided s satisfies

$$\int_\Omega [\nabla s \cdot \nabla v + c^2 s\, v]\, dx = \delta \int_{\partial\Omega} s\, v\, d\sigma \qquad \text{for all} \quad v \in H^1(\Omega). \tag{4.4}$$

An eigenfunction is said to be normalized if $\|s\|_c = 1$ and it is boundary normalized if $\|s\|_{2,\partial\Omega} = 1$. (4.4) is the weak form of the boundary value problem

$$\mathcal{L}_c\, s = c^2 s - \Delta\, s = 0 \qquad \text{on } \Omega \text{ with} \quad D_\nu\, s = \delta\, s \quad \text{on } \partial\Omega. \tag{4.5}$$

Let δ_1 be the least Steklov eigenvalue of (4.4) and s_1 be a corresponding normalized c-harmonic Steklov eigenfunction. They exist from theorem 3.1 of [6] and may be found as maximizers of a variational principle for a weakly continuous functional on the unit ball (in the c-norm) of $H^1(\Omega)$. Then an increasing sequence $\Lambda_c := \{\delta_j : j \in \mathbb{N}\}$ and associated normalized c-harmonic Steklov eigenfunctions $\mathcal{S}_c^\star := \{s_j^\star : j \in \mathbb{N}\}$ of this eigenproblem may be constructed as in [6]

Define $s_j := \sqrt{\delta_j}\, s_j^\star$, then the functions s_j will be boundary normalized. Let $\mathcal{S}_c := \{s_j : j \in \mathbb{N}\}$, then the following result holds.

Theorem 4.1. *Assume Ω obeys (B1) and $c > 0$, then there is an increasing sequence of Steklov eigenvalues Λ_c of $\mathcal{L}_c$ with $\delta_j \to \infty$ as j increases. $\mathcal{S}_c^\star$ is a c-orthonormal basis of $N(\mathcal{L}_c)$ and the boundary traces of functions in $\mathcal{S}_c$ are an orthonormal basis of $L^2(\partial\Omega, d\sigma)$.*

Proof. This follows from theorem 8.2 of [6] with Λ_c being the set of Steklov eigenvalues of this problem repeated according to multiplicity. The bilinear forms used there have $A(x) \equiv I_N,\ c(x) = c^2,\ b(z) \equiv 0,\ \rho(z) \equiv 1$ in the notation here; so (B3) - (B5) and (B8) of that theorem all hold. □

This result now provides representation results for solutions of c-harmonic boundary value problems. Consider the problem of finding solutions of equation (4.2) subject to the trace condition $\gamma(u) = g$ on $\partial\Omega$. Since $u \in N(\mathcal{L}_c),\ \mathcal{S}_c^\star$ is an orthonormal basis of $N(\mathcal{L}_c)$ and $\mathcal{S}_c$ is an orthonormal basis of $L^2(\partial\Omega, d\sigma)$. Thus there are coefficients $\hat{u}_j, \hat{g}_j$ such that

$$u(x) = \sum_{j=1}^{\infty} \hat{u}_j\, s_j^\star(x) \qquad \text{on} \quad \Omega \quad \text{and} \quad g(z) = \sum_{j=1}^{\infty} \hat{g}_j\, \gamma(s_j)(x) \qquad \text{on} \quad \partial\Omega. \tag{4.6}$$

Here the circumflexes denote the usual generalized Fourier coefficients, $\hat{u}_j = [u, s_j^\star]_c,\ \hat{g}_j = \langle u, s_j \rangle_{\partial\Omega}$ in these representations. The equation $\gamma(u) = g$ implies that $\hat{u}_j = \sqrt{\delta_j}\ \hat{g}_j$ for each $j \in \mathbb{N}$ so

$$\|u\|_c^2 = \sum_{j=1}^{\infty} \hat{u}_j^2 = \sum_{j=1}^{\infty} \delta_j \hat{g}_j^2$$

That is functions $u \in N(\mathcal{L}_c)$ have boundary traces $\gamma(u) = g$ with $\sum_{j=1}^{\infty} \delta_j \hat{g}_j^2 < \infty$. The class of all functions $g \in L^2(\partial\Omega, d\sigma)$ for which this holds is denoted $H^{1/2}(\partial\Omega)$ and is a real Hilbert space with respect to the inner product

$$\langle f, g\rangle_{1/2,\partial\Omega} \ := \ \sum_{j=1}^{\infty} (1 + \delta_j) \ \hat{f}_j \ \hat{g}_j \tag{4.7}$$

This shows that, when $g \in H^{1/2}(\partial\Omega)$, there is a unique solution of the Dirichlet c-harmonic boundary value problem with $\gamma(u) = g$ on $\partial\Omega$ that is given by the c-extension operator $E_c : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ where

$$u(x) \ = \ (E_c g)(x) \ := \ \sum_{j=1}^{\infty} \sqrt{\delta_j} \ \hat{g}_j \, s_j^\star(x) \ = \ \sum_{j=1}^{\infty} \ \hat{g}_j \, s_j(x) \quad \text{for } x \in \Omega. \tag{4.8}$$

The partial sums of this series are given by

$$u_M(x) \ := \ \int_{\partial\Omega} P_M(x,z,c) \ g(z) \ d\sigma(z) \text{ with } P_M(x,z,c) := \sum_{j=1}^{M} s_j(x)\gamma(s_j)(z). \tag{4.9}$$

Corollary 4.2. *When (B1) holds and $c > 0$, there is a solution $u = E_c g$ of* (4.2) *with $\gamma(u) = g$ on $\partial\Omega$ if and only if $g \in H^{1/2}(\partial\Omega)$. In this case $E_c(g) = \lim_{M\to\infty} u_M(x)$ in the c-norm where u_M is defined by* (4.9) *and E_c is an isometric isomorphism of $H^{1/2}(\partial\Omega)$ and $N(\mathcal{L}_c)$.*

A similar analysis holds for Robin and Neumann boundary value problems for $\mathcal{L}_c$. Consider the problem of finding $u \in H^1(\Omega)$ that satisfies

$$\int_\Omega [\nabla u \cdot \nabla v \, + \, c^2 u\, v] \, dx \, + \, b \int_{\partial\Omega} u\, v \, d\sigma \ = \ \int_{\partial\Omega} g\, v \, d\sigma \qquad \text{for all} \quad v \in H^1(\Omega). \tag{4.10}$$

with the constant $b \geq 0$ and $g \in L^2(\partial\Omega, d\sigma)$. This is the weak form of the equation $\mathcal{L}_c u = 0$ on $\partial\Omega$ subject to the Robin condition $D_\nu u + bu \ = \ g$ on $\partial\Omega$.

Substitute $s_j^\star$ for v here, then use of (4.4) implies that a solution has

$$\hat{u}_j \ = \ [u, s_j^\star]_c \ = \ \frac{\sqrt{\delta_j}\, \hat{g}_j}{b \, + \delta_j} \qquad \text{for all} \quad j \in \mathbb{N}, \tag{4.11}$$

so the unique solution of (4.10) is given by the *Robin solution operator* $\mathcal{R}(b) : H^{-1/2}(\partial\Omega) \to H^1(\Omega)$ defined by

$$\mathcal{R}(b)g(x) \ := \ \sum_{j=1}^{\infty} \ \frac{\hat{g}_j}{b + \delta_j} \, s_j(x) \quad \text{for } x \in \Omega. \tag{4.12}$$

This is valid for all $b \geq 0$ as $\delta_1 > 0$ when $c > 0$. This solution $u = \mathcal{R}(b)g$ has c-norm given by

$$\|u\|_c^2 \ = \ \sum_{j=1}^{\infty} \ \frac{\delta_j \, \hat{g}_j^2}{(b + \delta_j)^2} \ \leq \ C(b) \ \|g\|_{2,\partial\Omega}^2 \tag{4.13}$$

where $C(b) = 1/4b$ if $b \geq \delta_1$ and $C(b) = \delta_1/(b + \delta_1)^2$ if $b \leq \delta_1$.

The partial sums of the solution are represented by finite rank integral kernels

$$u_M(x) \;=\; \int_{\partial\Omega} R_M(x,z,b)\, g(z)\; d\sigma(z) \text{ with } R_M(x,z,b) := \sum_{j=1}^{M} \frac{s_j(x)}{b+\delta_j}\gamma(s_j)(z). \qquad (4.14)$$

Here the dependence of this solution on c is implicit as the Steklov eigenfunctions depend on c. When $g \in L^2(\partial\Omega,\, d\sigma)$, this may be summarized as follows .

Theorem 4.3. *Assume (B1) holds, $c > 0$ and $g \in L^2(\partial\Omega,\, d\sigma)$. Then the unique solution $u \in N(\mathcal{L}_c\,)$ of (4.10) is given by (4.12) and*

$$\mathcal{R}(b)g(x) = \lim_{M\to\infty} \int_{\partial\Omega} R_M(x,z,b)\, g(z)\; d\sigma(z) \qquad \textit{in c-norm.}$$

Proof. Since $\mathcal{S}_c^\star$ is an orthonormal basis of $N(\mathcal{L}_c\,)$, the solution u and the data g have representations of the form (4.6). These series converge in norm from the Riesz-Fisher theorem. Substitute this in the equation (4.10) then the result follows as above with the limit being in H^1-norm. □

5. Representations of Dirichlet Green's Functions

In this section Ω is a bounded region in $\mathbb{R}^N$ with $N \geq 2$ that satisfies (B1). Consider the problem of solving the equation (3.1) subject to zero Dirichlet boundary conditions $u \equiv 0$ on $\partial\Omega$. Let $V_c(f)$ be the potential solution of (3.1) then the difference $w := V_c(f) - u$ is a solution of the regularized harmonic equation

$$\mathcal{L}_c\, w \;=\; 0 \qquad \text{on} \quad \Omega \quad \text{subject to} \quad w \;=\; V_c(f) \qquad \text{on} \quad \partial\Omega. \qquad (5.1)$$

Let $\mathcal{S}_c^\star$ be the orthonormal basis of $N(\mathcal{L}_c\,)$ defined as in the preceding section 4 and $\mathcal{S}_c := \{s_j : j \geq 1\}$ be the boundary normalized family. If the equation $\mathcal{L}_c\, w = 0$ has a solution in $H^1(\Omega)$ then, from the Riesz-Fisher theorem, the solution has the representation

$$w(x) \;=\; \lim_{M\to\infty} \sum_{j=1}^{M} \hat{w}_j\, s_j(x) \qquad \text{with} \quad \hat{w}_j := \langle w, s_j\rangle_{\partial\Omega} \qquad (5.2)$$

with the limit holding in the H^1-norm. From the preceding analysis in 4 of the representation of solutions of Dirichlet problems, the solution of (5.1) is given by (5.2) with coefficients

$$\hat{w}_j \;=\; \int_{\partial\Omega} \int_{\Omega} \Gamma_c(z-y)\, f(y) s_j(z)\, dy\; d\sigma(z). \qquad (5.3)$$

The M-th approximation of this solution is $w_M(x) := \sum_{j=1}^{M} \hat{w}_j\, s_j(x)$ with

$$w_M(x) \;:=\; \mathcal{B}_M(c) f(x) \;:=\; \int_{\Omega} B_M(x,y,c) f(y)\, dy \qquad (5.4)$$

where $B_M(.,.,c)$ is defined by

$$B_M(x,y,c) \ := \ \sum_{j=1}^{M} \ s_j(x)\,(S_c s_j)(y) \text{ where } (S_c g)(y) := \int_{\partial\Omega} \ \Gamma_c(y-z)\,g(z)\ d\sigma(z) \quad (5.5)$$

is the *single layer potential* associated with the operator $\mathcal{L}_c$ and the boundary $\partial\Omega$.

The properties of single layer potentials such as this have been extensively studied and depend on the smoothness of the boundary. The following condition on the boundary will be used here.

Condition B2: $\partial\Omega$ is Lipschitz and S_c is a continuous linear transformation of $C(\partial\Omega)$ to $C(\overline{\Omega})$.

Benilan [15, chapter 2, section 3.3 proposition 10] shows that this holds when Ω is a "regular open set." That is $\partial\Omega$ is a finite union of C^1-manifolds with Ω locally on one side of $\partial\Omega$. His result is proved for $c = 0$ but the proof is easily generalized to $c \geq 0$. The author is not aware of a published proof that the condition on S_c in (B2) holds for all bounded Lipschitz regions.

Lemma 5.1. *Assume Ω is a bounded region in $\mathbb{R}^N$ that satisfies (B1) and (B2) and $c > 0$. Then $B_M(.,.,c)$ defined by (5.5) is continuous and bounded on $\overline{\Omega} \times \overline{\Omega}$ for any $M \in \mathbb{N}$. For each $y \in \Omega$, $B_M(.,y,c)$ is in $N(\mathcal{L}_c)$.*

Proof. When (B1) holds then the Steklov eigenfunctions of $\mathcal{L}_c$ are continuous on $\overline{\Omega}$ from corollary 4.2 of Daners [14]. Hence (B2) implies that the single layer potentials will be continuous on $\overline{\Omega}$. The lemma then follows as B_M is a finite sum of functions with these properties. □

The strong limit of this sequence of finite rank operators provides a formula for the Dirichlet Green's function.

Theorem 5.2. *Suppose Ω is a bounded region in $\mathbb{R}^N$ that satisfies (B1) and (B2), $c > 0$ and $f \in L^1(\Omega)$. If $V_c(f) \in H^1(\Omega)$, then the unique solution of (3.1) in $H_0^1(\Omega)$ is*

$$u(x) \ = \ \mathcal{G}_D(c)f(x) \ = \ V_c f(x) \ - \ \lim_{M\to\infty} \int_{\Omega} \ B_M(x,y,c) f(y)\, dy. \quad (5.6)$$

and this limit holds in the norm of $H^1(\Omega)$.

Proof. When $f \in L^1(\Omega)$ then lemma 5.1 implies that the function w_M defined by (5.5) is a finite sum of functions in $N(\mathcal{L}_c)$ that also are continuous on $\overline{\Omega}$.

The assumption that $V_c(f) \in H^1(\Omega)$ implies that its boundary trace is in $H^{1/2}(\partial\Omega)$, so the problem (5.1) has a unique solution in $H^1(\Omega)$. Since $\mathcal{S}_c$ is an orthonormal basis of $L^2(\partial\Omega,\, d\sigma)$, then (5.2) holds. Thus $\lim_{M\to\infty} w_M(x) = w(x) = V_c f(x) - \mathcal{G}_D(c)f(x)$ which is (5.6). □

Note that, for these regions, $f \in L^p(\Omega)$ with $p > 2N/(N+2)$ implies $V_c f \in H^1(\Omega)$ so (5.6) holds for all such functions. Formally this result can be viewed as saying that the

Dirichlet Green's function has the representation

$$G_D(x,y,c) \;=\; \Gamma_c(x-y) \;-\; \sum_{j=1}^{\infty} \, s_j(x)(S_c s_j)(y) \quad \text{for } (x,y) \in \Omega \times \Omega. \tag{5.7}$$

That is, the Green's function is the sum of the fundamental solution and terms involving the Steklov eigenfunctions and their single layer potentials. Note that the terms in this formula for the Dirichlet Green's function are not in the space $H_0^1(\Omega)$ with respect to either x or y - but they are smooth functions off the diagonal in $\Omega \times \Omega$. It is very different from the usual eigenfunction expansion for this Green's function which is a limit of functions that are in $H_0^1(\Omega)$ in x, y separately.

The sum in (5.7) appears to be non-symmetric in x and y. However, if the single layer potential is a continuous linear transformation of $L^2(\partial\Omega,\, d\sigma)$ to itself then it has the Steklov representation

$$(S_c s_j)(z) \;=\; \sum_{j=1}^{\infty} \, b_{jk} s_k(z) \qquad \text{with} \quad b_{jk} = \langle S_c s_j, s_k \rangle_{\partial\Omega}, \;\; z \in \partial\Omega. \tag{5.8}$$

The coefficients $b_{jk} = b_{kj}$ for all j,k as Γ_c is symmetric and then (5.7) becomes

$$G_D(x,y,c) \;=\; \Gamma_c(x-y) \;-\; \sum_{j,k=1}^{\infty} \, b_{jk} s_j(x) s_k(y) \quad \text{for } (x,y) \in \Omega \times \Omega. \tag{5.9}$$

The partial sums of this series are symmetric in x, y. Criteria on $\partial\Omega$ for S_c to be a self-adjoint linear transformation of $L^2(\partial\Omega,\, d\sigma)$ to itself are known; see Costabel [12] for a discussion of such results when $\partial\Omega$ is Lipschitz.

Often attention is focussed on Green's functions restricted to data $f \in L^2(\Omega)$. Consider the operator $\mathcal{B}(c) : L^2(\Omega) \to N(\mathcal{L}_c)$ defined by

$$\mathcal{B}(c) f(x) \;:=\; E_c(\gamma(V_c f))(x) \;:=\; \lim_{M\to\infty} \int_{\Omega} \; B_M(x,y,c) f(y)\, dy. \tag{5.10}$$

Here γ is the boundary trace operator and E_c is the extension operator of (4.8). From the properties of the individual operators, one observes that $\mathcal{B}(c)$ is a continuous linear mapping of $L^2(\Omega)$ to $H^1(\Omega)$ and $\gamma(V_c(f)) \in H^{1/2}(\partial\Omega)$. Then E_c maps $H^{1/2}(\partial\Omega)$ onto $N(\mathcal{L}_c) \subset H^1(\Omega)$ so from Rellich's theorem, $\mathcal{B}(c)$ is a compact linear mapping of $L^2(\Omega)$ to itself.

This operator $\mathcal{B}(c)$ will be called the *(Dirichlet) boundary correction operator* for $\mathcal{L}_c$ and Ω. The preceding theorem implies the following properties of $\mathcal{B}(c)$ as a linear operator on $L^2(\Omega)$.

Corollary 5.3. *Assume Ω as in theorem 5.2 and $c > 0$. Then $\mathcal{B}(c)$ defined by* (5.10) *is a compact self-adjoint linear map of $L^2(\Omega)$ to itself.*

Proof. The compactness is proved above and from theorem 5.2, $\mathcal{B}(c) = V_C - \mathcal{G}_D(c)$ with this right hand side a self adjoint operator on $L^2(\Omega)$. Thus $\mathcal{B}(c)$ is self-adjoint. □

Note also that if $w = \mathcal{B}(c)(f)$ and w_M is the M-th Steklov approximation, then the maximum principle yields that $\|w - w_M\|_{b,\Omega} \leq \|(I - P_M)\,\Gamma_c f\|_{b,\partial\Omega}$ where P_M is the projection onto the space spanned by the first M Steklov eigenfunctions.

When $c = 0$ there is a similar analysis for the standard zero Dirichlet boundary value problem for Poisson's equation. This is the problem of finding $u \in H_0^1(\Omega)$ obeying

$$\int_\Omega [\nabla u \cdot \nabla v \;-\; f\,v]\;dx \;=\; 0 \qquad \text{for all} \quad v \in H_0^1(\Omega). \tag{5.11}$$

Now $w := \Gamma * f - u$ is a solution of Laplace's equation

$$\Delta\, w \;=\; 0 \qquad \text{on} \quad \Omega \quad \text{subject to} \quad w \;=\; \Gamma * f \qquad \text{on} \quad \partial\Omega. \tag{5.12}$$

A function $s_j \in H^1(\Omega)$ is a *harmonic Steklov eigenfunction* on Ω corresponding to the eigenvalue δ provided it satisfies the equation

$$\int_\Omega \nabla s \cdot \nabla v \;dx \;=\; \delta \int_{\partial\Omega} s\,v\,d\sigma \qquad \text{for all} \quad v \in H^1(\Omega). \tag{5.13}$$

The least eigenvalue of this is $\delta_0 = 0$ corresponding to the constant functions on Ω and there is a infinite increasing sequence $\Lambda := \{\lambda_j : j \geq 0\}$ of positive Steklov eigenvalues each of finite multiplicity and with $\lambda_j \to \infty$ as $j \to \infty$. The algorithms described in [6] construct an associated family of harmonic Steklov eigenfunctions $\mathcal{S}_0 := \{s_j : j \geq 0\}$ that are an orthonormal basis of $L^2(\partial\Omega,\, d\sigma)$.

Let $S := S_0$ be the usual single layer potential for the Laplacian on $\partial\Omega$ and define

$$B_M(x,y) \quad := \quad \sum_{j=0}^{M} \; s_j(x)\,(Ss_j)(y) \qquad \text{on} \quad \Omega\times\Omega \quad \text{and} \tag{5.14}$$

$$\mathcal{B}_M f(x) \quad := \quad \int_\Omega \; B_M(x,y) f(y)\,dy \tag{5.15}$$

The following lemma holds with the same proof as lemma 5.1

Lemma 5.4. *Assume Ω is a bounded region in $\mathbb{R}^N$ that satisfies (B1) and (B2). Then, for any $M \in \mathbb{N}$, $B_M(.,.)$ defined by* (5.14) *is continuous and bounded on $\Omega\times\Omega$ and harmonic in each variable separately. For each $y \in \Omega$, $B_M(.,y)$ is in $\mathcal{H}(\Omega)$.*

As a consequence of this result, each $\mathcal{B}_M$ is a finite rank operator from $L^1(\Omega)$ to $\mathcal{H}(\Omega)$ and the following holds.

Theorem 5.5. *Suppose Ω is a bounded region in $\mathbb{R}^N$ that satisfies (B1) and (B2), $\mathcal{S}_0 := \{s_j : j \geq 0\}$ is an orthogonal class of harmonic Steklov eigenfunctions that is an orthonormal basis of $L^2(\partial\Omega,\, d\sigma)$ and $f \in L^1(\Omega)$. If $\Gamma * f \in H^1(\Omega)$, then the unique solution of* (5.11) *in $H_0^1(\Omega)$ is given by*

$$u(x) \;=\; \mathcal{G}_D(c) f(x) \;=\; (\Gamma * f)(x) \;-\; \lim_{M\to\infty} \mathcal{B}_M f(x). \tag{5.16}$$

Proof. This is proved in the same way as theorem 5.2. The functions $B_M(.,y)$ and $B_M(x,.)$ each are harmonic on Ω and continuous on $\overline{\Omega}$ as the Steklov eigenfunctions and also the single layer potentials of a function have this properties. The limit in (5.16) is in any H^1-norm. □

Define the *harmonic (Dirichlet) correction operator* for Ω to be the operator associated with the strong limit in (5.16) so that

$$\mathcal{B}_D f(x) \;:=\; E(\gamma(Vf))(x) \;:=\; \lim_{M\to\infty} \int_\Omega \; B_M(x,y) f(y)\, dy. \tag{5.17}$$

Here γ is the boundary trace operator and E is the harmonic extension operator similar to that in (4.8). Now V is a continuous linear mapping of $L^2(\Omega)$ to $H^1(\Omega)$, so if $f \in L^2(\Omega)$ then $\gamma(V(f)) \in H^{1/2}(\partial\Omega)$. Also E maps $H^{1/2}(\partial\Omega)$ onto $\mathcal{H}(\Omega) \subset H^1(\Omega)$ so, from Rellich's theorem, $\mathcal{B}_D$ is a compact linear mapping of $L^2(\Omega)$ to itself and the following result holds.

Corollary 5.6. *Assume Ω as in theorem 5.2, then $\mathcal{B}_D$ defined by* (5.17) *is a compact self-adjoint linear map of $L^2(\Omega)$ to itself.*

Proof. The compactness is proved above. From theorem 5.5, $\mathcal{B}_D(f) = V(f) - G_D(f)$ with this right hand side a self adjoint operator on $L^2(\Omega)$. Thus $\mathcal{B}_D$ is self-adjoint. □

An unusual feature of corollaries 5.3 and 5.6 is that the kernels of the finite rank operators $\mathcal{B}_M(c), \mathcal{B}_M$ are neither symmetric nor L^2-orthogonal sums on Ω. Yet these operators converge strongly to the compact, self adjoint boundary correction operators $\mathcal{B}(c)$ and $\mathcal{B}_D$ in the H^1-norm.

It is worth noting that these formulae provide good results for evaluating, and approximating, the *Poisson kernel* $P : \Omega \times \partial\Omega \to [0,\infty]$ for Dirichlet harmonic boundary value problems using the common expression

$$P(x,z) \;:=\; -D_{\nu_z} G_D(x,z)$$

By comparison finite approximations of this formula using eigenfunction expansions give very poor results since the eigenfunctions $e_m(x) \equiv 0$ for all m and $x \in \partial\Omega$.

6. Representations of Robin Green's Functions.

In this section Ω is a bounded region in $\mathbb{R}^N$ with $N \geq 2$ that satisfies (B2). Consider the problem of solving the equation (3.1) subject to zero Robin boundary conditions. The weak version of this problem is to find $u \in H^1(\Omega)$ that satisfies

$$\int_\Omega \left[\nabla u \cdot \nabla v + c^2\, u\, v\right] dx + b \int_{\partial\Omega} u\, v\; d\sigma \;=\; \int_\Omega f\, v\, dx \qquad \text{for all} \quad v \in H^1(\Omega) \tag{6.1}$$

with $b > 0, c \geq 0$ being constants.

Suppose that the potential function $V_c f$ associated with f is in $H^1(\Omega)$, then $w := V_c f - u$ will satisfy

$$\int_\Omega \left[\nabla w \cdot \nabla v \,+\, c^2\, w\, v \right] dx \,+\, b \int_{\partial\Omega} w\, v\ d\sigma \ =\ \int_{\partial\Omega} g\, v\ d\sigma \tag{6.2}$$

for all $v \in H^1(\Omega)$ with $g := D_\nu(V_c f) + b\,(V_c f)$.

The solution of this equation is in $N(\mathcal{L}_c)$, so it has a representation in terms of the Steklov basis of $N(\mathcal{L}_c)$. Substitute $v = s_j$ for each j in (6.2), to see that the solution is

$$w(x) \ =\ \sum_{j=1}^{\infty} \frac{\langle g, s_j\rangle_{\partial\Omega}}{b + \delta_j}\ s_j(x). \tag{6.3}$$

Here the $\{s_j\}$ are boundary L^2-orthogonal.

Let S_c be the single layer potential associated with Γ_c and Ω as in (5.5) and D_c be the double layer potential defined by

$$(D_c g)(y) \ :=\ \int_{\partial\Omega} \left(\nabla_z \Gamma_c(z-y)\cdot \nu(z)\right) g(z)\ d\sigma(z). \tag{6.4}$$

Then, from Fubini's theorem, one has

$$\langle g, s_j\rangle_{\partial\Omega} \ =\ \int_\Omega \left[\, (D_c s_j)(y) \,+\, b\,(S_c s_j)(y) \,\right]\ f(y)\, dy. \tag{6.5}$$

Formally the solution is

$$w(x) \quad = \quad \lim_{M\to\infty} \int_\Omega R_M(x,y,b)\, f(y)\, dy \qquad \text{with} \tag{6.6}$$

$$R_M(x,y,b) \quad := \quad \sum_{j=1}^{M} \frac{s_j(x)}{b+\delta_j}\ \left[\, (D_c s_j)(y) + b\,(S_c s_j)(y) \,\right]. \tag{6.7}$$

To ensure that the integrals in (6.6) are well defined the double layer potentials $D_c s_j$ must be nice operators on $C(\partial\Omega)$. Double layer potentials have been extensively studied and their continuity properties depend on the regularity of the boundary $\partial\Omega$. In particular it is well-known that they are harmonic on Ω but generally not continuous at the boundary. The following boundary regularity condition will be required here.

Condition B3: The region Ω and its boundary $\partial\Omega$ satisfy (B2) and D_c is a continuous linear transformation of $C(\partial\Omega)$ to $L^\infty(\Omega)$.

Benilan [15, chapter 2, section 3.3 proposition 11] shows that this holds when Ω has a boundary of class $C^{1+\epsilon}$. His proof is for the case $c = 0$ but may be extended to fundamental solutions with $c \geq 0$.

Lemma 6.1. *Assume Ω is a bounded region in $\mathbb{R}^N$ that satisfies (B1) and (B3) and $c > 0$. Then $R_M(.,.,b)$ defined by* (6.4) - (6.7) *is bounded on $\Omega \times \Omega$ for any $M \in \mathbb{N},\ b \geq 0$. For each $y \in \Omega,\ R_M(.,y,b)$ is in $N(\mathcal{L}_c)$.*

Proof. When (B1) holds then the Steklov eigenfunctions of $\mathcal{L}_c$ are continuous on $\overline{\Omega}$ from corollary 4.2 of Daners [14]. Hence from (B3), the double layer potentials $D_c s_j$ are essentially bounded on Ω and thus the coefficient of $s_j(x)$ is a bounded function on Ω. The lemma then holds as $R_M(.,y,c)$ is a finite sum of continuous c-harmonic functions with bounded coefficients. □

The Robin Green's function for $\mathcal{L}_c$ on Ω can now be given by the following formula.

Theorem 6.2. *Suppose Ω is a bounded region in $\mathbb{R}^N$ that satisfies (B1) and (B3), $b \geq 0$ and $c > 0$. If $f \in L^1(\Omega)$ and $V_c(f) \in H^1(\Omega)$, then the unique solution of* (6.1) *in $H^1(\Omega)$ is given by*

$$u(x) \ = \ \mathcal{G}_b(c) f(x) \ = \ V_c f(x) \ - \ \lim_{M\to\infty} \int_\Omega \ R_M(x,y,b) f(y)\, dy. \tag{6.8}$$

with $R_M(.,.,b)$ defined by (6.7)*. This limit holds in the norm of $H^1(\Omega)$.*

Proof. The assumption that $V_c(f) \in H^1(\Omega)$ implies that its boundary trace is in $H^{1/2}(\partial\Omega)$ and the function g is in $H^{-1/2}(\partial\Omega)$, so the problem (6.1) has a unique solution in $H^1(\Omega)$. Since $\mathcal{S}_c$ is an orthonormal basis of $L^2(\partial\Omega,\, d\sigma)$, then (6.3) holds. The result then holds from the above analysis upon repeated use of Fubini's theorem and the fact that Ω is bounded. □

Define the *Robin correction operator* for $\mathcal{L}_c$ and Ω to be the operator associated with the strong limit in (6.8) so that

$$\mathcal{B}_R(b) f(x) \ := \ \mathcal{R}(b)(\beta(V_c f))(x) \ := \ \lim_{M\to\infty} \int_\Omega \ R_M(x,y,b)\, f(y)\, dy. \tag{6.9}$$

Here $\beta = D_\nu + b\,\gamma$ is the operator associated with the Robin boundary condition and $\mathcal{R}(b)$ is the Robin solution operator of (4.12). When $f \in L^2(\Omega)$ then $V_c f \in H^1(\Omega)$ so $\beta(V_c(f)) \in H^{-1/2}(\partial\Omega)$. Also $\mathcal{R}(b)$ maps $H^{-1/2}(\partial\Omega)$ into $N(\mathcal{L}_c) \subset H^1(\Omega)$ so, from Rellich's theorem, $\mathcal{B}_R$ is a compact linear mapping of $L^2(\Omega)$ to itself and the following result holds.

Corollary 6.3. *Assume Ω as in theorem 6.4, $b \geq 0, c > 0$ then $\mathcal{B}_R(b)$ defined by* (6.9) *is a compact self-adjoint linear map of $L^2(\Omega)$ to itself.*

Proof. The compactness is proved above. From theorem 6.2, $\mathcal{B}_R = V_c - \mathcal{G}_b(c)$ with this right hand side a self adjoint operator on $L^2(\Omega)$. Thus $\mathcal{B}_R(b)$ is self-adjoint. □

When $c = 0, b > 0$ in the above problem, we have the usual Robin problem for Poisson's equation and similar formulae hold with the convention that the index of the Steklov eigenvalues and eigenfunctions starts with $j = 0$. Thus the solution is

$$w(x) \ = \ \sum_{j=0}^{\infty} \ \frac{\langle g, s_j\rangle_{\partial\Omega}}{b+\delta_j} \ s_j(x) \tag{6.10}$$

with $g(x) := D_\nu(\Gamma * f) + b\,(\Gamma * f)$ on $\partial\Omega$. Here the $\{s_j\}$ are the boundary L^2-orthogonal harmonic Steklov eigenfunctions.

Let $S = S_0, D = D_0$ be the usual single and double layer potentials associated with the Laplacian and $\partial\Omega$ then for $b > 0$ the analog of lemma 6.1 holds and the M-th partial sum here is

$$w_M(x) \;=\; \int_\Omega R_M(x,y,b)\, f(y)\, dy \qquad \text{with} \tag{6.11}$$

$$R_M(x,y,b) \;:=\; \sum_{j=0}^{M} \frac{s_j(x)}{b+\delta_j}\; [\,(Ds_j)(y) + b\,(Ss_j)(y)\,] \tag{6.12}$$

Then the following theorem is proved using the same arguments as for theorem 6.2.

Theorem 6.4. *Suppose Ω is a bounded region in $\mathbb{R}^N$ that satisfies (B1) and (B3). Let $\mathcal{S}_0 := \{s_j : j \geq 0\}$ be an orthogonal class of harmonic Steklov eigenfunctions that is an orthonormal basis of $L^2(\partial\Omega,\, d\sigma)$. If $f \in L^1(\Omega), b > 0$ and $\Gamma * f \in H^1(\Omega)$, then the unique solution of* (6.1) *in $H^1(\Omega)$ with $c = 0$ is*

$$u(x) \;=\; \mathcal{G}_b\, f(x) \;=\; (\Gamma * f)(x) \;-\; \lim_{M\to\infty} \int_\Omega R_M(x,y,b) f(y)\, dy. \tag{6.13}$$

The limit in (6.13) *exists in the $H^1(\Omega)$ norm.*

The last term here is called the *Robin harmonic correction operator* and is a self-adjoint compact linear operator on $L^2(\Omega)$ as before. This result may be regarded as saying that, when $b > 0$, the Robin Green's function for the Laplacian has the representation

$$G_R(x,y,b) \;=\; (\Gamma * f)(x-y) \;-\; R_b(x,y) \qquad \text{on} \quad \Omega\times\Omega \qquad \text{with} \tag{6.14}$$

$$R_b(x,y) \;=\; \sum_{j=0}^{\infty} \frac{s_j(x)}{b+\delta_j}\, [\,(Ds_j)(y) + b\,(Ss_j)(y)\,] \qquad \text{on} \quad \Omega\times\Omega. \tag{6.15}$$

If $b = 0$ the problem has a solution if and only if $\int_\Omega f\, dx = 0$. In this case the coefficient of $s_0(x)$ in (6.13) is zero and the formula (6.13) holds. The sum without this first term is often called a generalized (Neumann) Green's function for this problem.

7. Boundary Value Problems on Exterior Regions

This construction of Green's functions as modifications to fundamental solutions also works for Laplacian boundary value problems on exterior regions when $N \geq 3$.

A region $U \subset \mathbb{R}^N$ is called an *exterior region* provided its complement is non-empty and compact. In the following U will always denote an exterior region. Without loss of generality, assume that $0 \notin U$ and write $R_b := \sup\{|x| : x \notin U\}$. For simplicity the following analysis will just consider the case of most applied importance; namely regions in space with $N = 3$.

$H^1(U)$ is the usual Hilbert-Sobolev space and we will generally use the c-inner product defined as in (4.1) with U in place of Ω. We will use the notation $C_b(U)$ for continuous and bounded functions on U and $C_0(\overline{U})$ for continuous functions on $\overline{U}$ that converge to 0 as $|x| \to \infty$. Given a function f in $C_c^1(\mathbb{R}^N)$, its restriction to $\overline{U}$ is denoted

$R_U f$. The set of all such restrictions is denoted $C^1(\overline{U})$ and is a subspace of $W^{1,\infty}(U)$. Let $G^{1,p}(U)$ be the closure of $C^1(\overline{U})$ in the $W^{1,p}$-norm.

The analysis here is based on the results of Auchmuty and Han from [7] and [8] about Steklov eigenproblems and elliptic boundary value problems on exterior regions. Assume the following boundary regularity condition for exterior regions.

Condition B4: $U \subset \mathbb{R}^3$ is an exterior region with $0 \notin U$. The boundary ∂U is the union of finitely many, disjoint, closed Lipschitz surfaces and $H^1(U) = G^{1,2}(U)$.

When this condition holds, then the Gauss-Green theorem holds and the trace theorem for the mapping $\gamma : H^1(U) \to L^q(\partial U, d\sigma)$ is compact when $1 \leq q < 4$. See section 2 and theorem 3.1 of [7]. In particular the compact trace theorem holds for $q = 2$.

The zero-Dirichlet problem for the regularized Laplacian on U is the problem of finding $u \in H_0^1(U)$ that satisfies

$$\int_U [\, \nabla u \cdot \nabla v \, + \, c^2 \, u \, v \,] \, dx \; = \; \int_U f \, v \, dx \qquad \text{for all} \quad v \in H_0^1(U). \tag{7.1}$$

When $c > 0$ this is a well-posed problem when $f \in L^2(\Omega)$ and there is a continuous linear map $\mathcal{G}_U(c)$ of $L^2(U)$ to $L^2(U)$ with $\|\mathcal{G}_U(c)\|_c \leq c^{-1}$. This is proved in a straightforward manner using the calculus of variations.

Unfortunately when $c = 0$, this is not a well-posed problem and the physically correct solutions often are not in $L^2(U)$. Rather the variational principle for the solution suggests that the problem should be posed in a larger function space. Here the problem will be posed in the space $E^1(U)$ of finite energy functions on U described in Auchmuty and Han [8].

A function $v \in L^1_{loc}(U)$ is said to *decay at infinity* provided, for each $c > 0$,

$$E_c(v) := \{x \in U : |v(x)| \geq c\} \quad \text{has finite Lebesgue measure.}$$

An extended, real-valued function $u \in W^{1,1}_{loc}(U)$ is said to have *finite energy*, provided u decays at infinity, $u \in L^1(U_R)$ for some $R > R_b$ and $|\nabla u| \in L^2(U)$. Define $E^1(U)$ as the class of all finite energy functions on U. It is an exercise to prove that this is a real vector space. $E^1(U)$ is a strictly larger space than $H^1(U)$ when U is an exterior region, as it contains functions that are not in $L^2(U)$.

Let $\gamma : E^1(U) \to L^2(\partial U, d\sigma)$ be the boundary trace map defined as before then the bilinear form

$$\langle u, v \rangle_\partial \; := \; \int_U \nabla u \cdot \nabla v \, dx \, + \, \int_{\partial U} \gamma(u) \, \gamma(v) \; d\sigma. \tag{7.2}$$

is an inner product on $E^1(U)$ and $E^1(U)$ is a real Hilbert space with respect to this inner product. See theorem 3.3 of [8]. For applications this choice of a Hilbert space has advantages over the weighted Sobolev spaces often used by mathematicians defined as in Nedelec [22] Chapter 2 section 5. The norm defined by (7.2) has a physical interpretation; and usually it is much easier to verify that the second integral here is finite in place of the weighted L^2 integral of the other spaces.

Let $E^1_0(U)$ be the null space of the boundary trace operator γ. Then theorem 4.1 of [8] says that

$$E^1(U) \;=\; E^1_0(U) \,\oplus_\partial\, \mathcal{H}(U). \tag{7.3}$$

where $\mathcal{H}(U)$ is the class of all finite energy harmonic functions on U. Note that when $N = 3$, harmonic functions that decay like $1/|x|$ are in $\mathcal{H}(U)$ - but not in $H^1(U)$, so the use of this space $E^1(U)$ is essential if we wish to use this analysis for many physical problems.

The problem of solving (7.1) now may be studied as one of minimizing $\mathcal{E}$ on $E^1_0(U)$ where

$$\mathcal{E}(u) \;:=\; \int_U \big[\, |\nabla u|^2 \,-\, 2\, f\, u \,\big]\, dx \tag{7.4}$$

This variational problem has a unique solution in $E^1_0(U)$ when $f \in L^{6/5}(U)$ upon using the Sobolev imbedding theorem. Then there is continuous solution operator $\mathcal{G}_D(c) : L^{6/5}(U) \to H^1(U)$.

When the boundary ∂U of an exterior region satisfies (B4), then the results of section 4 continue to hold with $c = 0$ when $U, \partial U$ replace $\Omega, \partial\Omega$ and $E^1(U), E^1_0(U), \mathcal{H}(U)$ replace $H^1(\Omega), H^1_0(\Omega).\mathcal{H}(\Omega)$ respectively. See section 8 of [8] for precise statements and proofs in this case. This material will not be repeated here but in the following when a result from section 4 is referenced and $c = 0$, these substitutions should be made.

8. Dirichlet Green's Functions on Exterior Regions

The problem to be studied here is to solve equation (7.1) on an exterior region U that satisfies (B4) with $f \in L^2(U)$. The existence and uniqueness of a solution in $H^1(U)$ of this problem holds from straightforward variational analysis. The solution operator of this problem is a continuous linear map $\mathcal{G}_U(c)$ of $L^2(U)$ to $H^1(U)$ with $\|\mathcal{G}_U(c)\|_2 \le c^{-1}$. and $\|\mathcal{G}_U(c)\|_c \le c^{-1/2}$ for $c > 0$.

Let $V_c f$ be the potential solution defined by (3.6) with the fundamental solution Γ_c from (3.5). From Young's inequality for convolutions one sees that V_c is a continuous linear map of $L^2(U)$ to $L^2(U) \cap C_b(\overline{U}) \cap H^1(U)$. Then $w := V_c f - u$ is a solution of the Dirichlet problem

$$\mathcal{L}_c\, w \;=\; 0 \qquad \text{on} \quad U \qquad \text{subject to} \quad w \;=\; V_c(f) \qquad \text{on} \quad \partial U. \tag{8.1}$$

When the boundary ∂U of an exterior region satisfies (B4), then the results of section 4 continue to hold with $U, \partial U$ replacing $\Omega, \partial\Omega$ throughout. In particular one has that if $N(\mathcal{L}_c(U))$ is the closed subspace of $H^1(U)$ of all solutions of (4.2) (with U replacing Ω) then

$$H^1(U) \;=\; H^1_0(U) \oplus_c N(\mathcal{L}_c(U)) \tag{8.2}$$

where $\oplus_c$ indicates that these closed subspaces are c-orthogonal.

In section 5 of Auchmuty and Han [7], the $\mathcal{L}_c-$Steklov eigenfunctions of U are constructed and it is shown that one can find a family $\mathcal{S}_c := \{s_j \,:\, j \in \mathbb{N}\}$ that is an c-orthogonal and maximal in $H^1(U)$ and an L^2-orthonormal basis of $L^2(\partial U,\, d\sigma)$. See result 5.2 of [7].

Just as in section 5, one finds that the unique solution $w \in N(\mathcal{L}_c(U))$ of (8.1) is given by

$$w(x) \;=\; \lim_{M\to\infty} \mathcal{B}_M(c)f(x), \qquad \text{with} \tag{8.3}$$

$$\mathcal{B}_M(c)f(x) \;:=\; \int_U B_M(x,y,c)f(y)\,dy \quad \text{and} \quad B_M(x,y,c) \;:=\; \sum_{j=1}^{M} s_j(x)\,(S_c s_j)(y). \tag{8.4}$$

Here $S_c g(y) \;:=\; \int_{\partial U} \Gamma_c(y-z)\, g(z)\; d\sigma(z)$ is the exterior single layer potential associated with $\mathcal{L}_c$. For exterior regions, the analog of condition (B2) is

Condition B2e: ∂U is Lipschitz and S_c is a continuous linear transformation of $C(\partial U)$ to $C_b(\overline{U})$.

When this replaces condition (B2), the analog of lemma 5.1 holds - though the proof requires that Daners' regularity result hold for these exterior Steklov problems. This may be proved by approximating the exterior problems by problems on larger but bounded domains that increase to U. Then the following holds.

Theorem 8.1. *Suppose U is an exterior region in $\mathbb{R}^3$ that satisfies (B4) and (B2e), $c > 0$ and $f \in L^2(U)$. Then the unique solution of* (3.1) *in $H_0^1(U)$ is*

$$u(x) \;=\; \mathcal{G}_D(c)f(x) \;=\; V_c f(x) \;-\; \lim_{M\to\infty} \int_U B_M(x,y,c)f(y)\,dy. \tag{8.5}$$

where B_M is defined by (8.3) *and this limit holds in the norm of $H^1(U)$.*

Proof. When $f \in L^2(U)$ then the exterior analog of lemma 5.1 implies that the function $\mathcal{B}_M(c)f$ defined by (8.4) is a finite sum of functions in $N(\mathcal{L}_c(U))$ that also are continuous and bounded on $\overline{U}$.

The function $V_c(f)$ is in $H^1(U_R)$ where $U_R = U \cap B_R$ so its boundary trace is in $H^{1/2}(\partial U)$, and the problem (8.1) has a unique solution in $H^1(U)$. Since $\mathcal{S}_c$ is an orthonormal basis of $L^2(\partial\Omega,\, d\sigma)$, then (8.3) - (8.4) holds. Thus $\lim_{M\to\infty} \mathcal{B}_M f(x) = w(x) = V_c f(x) - \mathcal{G}_D(c)f(x)$ which is (8.5). □

That is the Dirichlet Green's function for $\mathcal{L}_c$ on the exterior region U has the formal representation

$$G_D(x,y,c) \;=\; \Gamma_c(x-y) \;-\; \sum_{j=1}^{\infty} s_j(x)(S_c s_j)(y) \quad \text{for } (x,y) \in U \times U. \tag{8.6}$$

So again the Dirichlet Green's function for these problems differs from the fundamental solution by a boundary correction kernel that involves the $\mathcal{L}_c$ $-$Steklov eigenfunctions and their exterior single layer potentials. Moreover the boundary correction operator is given by the strong limit of this family of finite rank operators.

9. The Laplacian Dirichlet Green's function on Exterior Regions

Unfortunately the analysis of the preceding section does not cover the case $c = 0$ of the standard Poisson equation on exterior regions. The (larger) Hilbert space $E^1(U)$ of finite energy functions on U introduced by Auchmuty and Han in [8] is used instead.

A function $u \in E_0^1(U)$ is said to be a solution of $-\Delta\, u = f$ on u provided it satisfies

$$\int_U \, [\,\nabla u \cdot \nabla v \; - \; f\, v\,] \;\, dx \;=\; 0 \qquad \text{for all} \quad v \in C_c^1(U). \tag{9.1}$$

This is the equation satisfied by the minimizers of the convex, coercive functional $\mathcal{E}$ of (7.4) on $E_0^1(U)$. We will require $f \in L^{6/5}(U)$ to ensure existence of a unique solution of this problem.

When Vf represents the standard Newtonian potential on $\mathbb{R}^3$, the function $w := Vf - u$ will be a harmonic function on U satisfying the boundary condition $w = Vf$ on ∂U. This problem has a unique solution in $\mathcal{H}(U)$ if and only if $\gamma(Vf) \in H^{1/2}(\partial U)$ from theorem 10.1 of [8].

In section 9 of [8], an algorithm for constructing an L^2-orthogonal basis $\mathcal{S}$ of $L^2(\partial U,\, d\sigma)$ consisting of harmonic Steklov eigenfunctions on U is described. For exterior harmonic problems the least Steklov eigenvalue is $\delta_1 > 0$ so the basis is denoted $\mathcal{S} := \{s_j : j \in \mathbb{N}\}$.

Then the unique harmonic function $w \in \mathcal{H}(U)$ obeying this boundary condition is

$$w(x) \;=\; \lim_{M\to\infty} \; \mathcal{B}_M f(x), \qquad \text{with} \tag{9.2}$$

$$\mathcal{B}_M f(x) \;:=\; \int_U \; B_M(x,y) f(y)\, dy \quad \text{and} \quad B_M(x,y) \;:=\; \sum_{j=1}^{M} \; s_j(x)\, (S s_j)(y). \tag{9.3}$$

Here $Sg(y) \;:=\; \int_{\partial U} \; \Gamma(y - z)\, g(z)\; d\sigma(z)$ is the usual Laplacian exterior single layer potential.

Theorem 9.1. *Suppose U is an exterior region in $\mathbb{R}^3$ that satisfies (B4) and (B2e) and $f \in L^{6/5}(U)$. Then the unique solution of* (9.1) *in $E^1(U)$ is*

$$u(x) \;=\; \mathcal{G}_D(c) f(x) \;=\; Vf(x) \;-\; \lim_{M\to\infty} \int_U \; B_M(x,y) f(y)\, dy \tag{9.4}$$

where B_M is defined by (9.3) *and this limit holds in the norm of $E^1(U)$.*

The proof of this is essentially the same as that of theorem 8.1. Thus the Dirichlet Green's function for the Laplacian on the exterior region U has the formal representation

$$G_D(x,y) \;=\; \Gamma(x-y) \;-\; \sum_{j=1}^{\infty} \; s_j(x)(S s_j)(y) \quad \text{for } (x,y) \in U \times U. \tag{9.5}$$

That is Dirichlet Green's function on exterior regions differs from the Newtonian potential by a boundary correction kernel that involves the harmonic Steklov eigenfunctions and their exterior single layer potentials.

When U is the exterior of a ball in $\mathbb{R}^3$ the exterior Steklov eigenfunctions are precisely the exterior spherical harmonics. In physics a representation of harmonic functions via exterior spherical harmonics is called a multipole expansion so the representations obtained here may be regarded as a generalization of multipole expansions to more general regions than the exterior of a ball. In particular multipole expansions often have very rapid convergence properties at a distance from the boundary ∂U so physicist generally use relatively few terms in their approximations.

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