Notes on Diophantine approximation and aperiodic order

Alan Haynes

June 24, 2016

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0.1 Summary of notation

For sets A and B, the notation $A \times B$ denotes their Cartesian product. If A and B are subsets of the same Abelian group, then $A + B$ denotes their Minkowski sum, which is the collection of all elements of the form $a + b$ with $a \in A$ and $b \in B$. If A and B are any two Abelian groups then $A \oplus B$ denotes their direct sum.

We use $\#A$ to denote the number of elements of a set A, or ∞ if A is not finite. If A is a subset of $\mathbb{R}^k$ then we use $\dim(A)$ to denote its Hausdorff dimension. If $A \subseteq \mathbb{R}^k$ is measurable then, unless otherwise specified, we use $|A|$ to denote its k-dimensional Lebesgue measure. When there is ambiguity about the dimension of the ambient space in which we are viewing A then we will use a subscript to clarify matters. For example, if we want to speak about a d-dimensional measure of a set $A \subseteq \mathbb{R}^k$ which lies in a proper d-dimensional subspace of $\mathbb{R}^k$, then we will write $|A|_d$.

For $x \in \mathbb{R}$, $\{x\}$ denotes the fractional part of x and $\|x\|$ denotes the distance from x to the nearest integer. For $x \in \mathbb{R}^k$, we set $|x| = \max\{|x_1|, \ldots, |x_m|\}$ and $\|x\| = \max\{\|x_1\|, \ldots, \|x_m\|\}$. If $x \in \mathbb{R}^k$ and $r > 0$ then we write $B_r(x)$ for the open Euclidean ball of radius r centered at x.

We use the standard Vinogradov and asymptotic notation, which we now describe. If f and g are complex valued functions which are defined on some domain D then we write

$$f(x) \ll g(x) \text{ for all } x \in D$$

to mean that there exists a constant $C > 0$ with the property that

$$|f(x)| \leq C|g(x)| \text{ for all } x \in D.$$

When it is convenient (e.g. in describing the sizes of error terms in asymptotic formulas) we may also write $f(x) = O(g(x))$ to mean the same thing. In case the domain D is omitted, writing $f(x) \ll g(x)$ is usually taken to mean that this inequality holds on the largest domain common to both f and g. Therefore, to avoid misunderstanding, it is usually best practice to specify the domain under consideration.

The notation $f(x) \gg g(x)$ for all $x \in D$ has the same meaning as $g(x) \ll f(x)$ for all $x \in D$. We write $f(x) \asymp g(x)$ for all $x \in D$ to mean that both $f(x) \ll g(x)$ and $f(x) \gg g(x)$ hold, for all $x \in D$. Finally, the statement that

$$f(x) \sim g(x) \text{ as } x \to x_0 \tag{0.1.1}$$

means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1,$$

while the statement that

$$f(x) = o(g(x)) \text{ as } x \to x_0 \tag{0.1.2}$$

means that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

When f and g are defined on a set which contains arbitrary large real numbers, the value of x_0 is often omitted from (0.1.1) and (0.1.2). In this case it is usually assumed that $x_0 = \infty$.

Chapter 1

Results from Diophantine approximation

In this chapter we review a number of important results from Diophantine approximation, which will be used later in our investigation of aperiodic order. We begin with classical, one dimensional results, including an overview of simple continued fraction expansions and the associated Ostrowksi expansions of both the integers and the real numbers. We move on to discuss probabilistic and dimension theoretic approaches to this subject, which explore the question of how well almost every real number (in the sense of Lebesgue or Hausdorff measure) can be approximated by rationals. Next, we explain how some of these results generalize to higher dimensions and, finally, we conclude with a discussion of a transference principles which connect homogeneous and inhomogeneous approximation.

Our overall goal is not to give a rigorous justification for all of the results in this chapter, and as such we omit most of the proofs. For readers who desire more details, there are a large number of excellent references on Diophantine approximation which can be consulted. For a thorough treatment of classical questions about continued fractions and one dimensional approximation, we refer to the books of Khintchine [20] and Rockett and Szüsz [26]. For the classical theory of approximation by linear forms, inhomogeneous approximation, and transference principles, we refer to Cassels's book [9]. For more modern developments we recommend the books of Kuipers and Niederreiter [21] and Drmota and Tichy [11], and for a good introduction to the probabilistic and dimension theoretic aspects of this subject we refer to Harman's book [16].

1.1 One dimensional approximation and badly approximable numbers

Classical Diophantine approximation is concerned with the study how well real numbers can be approximated by rationals. A basic result in this direction is the following theorem of Dirichlet from 1842.

Theorem 1.1.1. *If α is a real number and N is a positive integer then we can find a rational number a/n with $1 \leq n \leq N$ and*

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{1}{n(N+1)}. \tag{1.1.2}$$

It is not difficult to show that Dirichlet's theorem, in this form, is best possible (see Exercise 1.1.2). It follows from the theorem that, for any irrational real number α, there are infinitely many fractions $a/n \in \mathbb{Q}$ with

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{1}{n^2}. \tag{1.1.3}$$

For rational α this statement is actually false (see Exercise 1.1.2). However, if we are willing to restrict attention to irrational α, then it turns out that we can do better. This is demonstrated by the following theorem, which was proved by Hurwitz in 1891.

Theorem 1.1.4. *For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ there are infinitely many $a/n \in \mathbb{Q}$ with*

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{1}{\sqrt{5}n^2}.$$

Now it can be shown, for example by taking $\alpha = (1+\sqrt{5})/2$, that the constant $1/\sqrt{5}$ which appears in Hurwitz's Theorem cannot be replaced by any smaller number. In fact, there is a countably infinite set of real numbers α for which Hurwitz's theorem is best possible in this sense. However, if we are willing to exclude all of these numbers from our consideration then, for any real number α which remains, it can be shown that there are infinitely many $a/n \in \mathbb{Q}$ for which

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{1}{\sqrt{8}n^2}.$$

Once again there are is a countably infinite set of α for which this is best possible. We can continue in this way, but the story does eventually become a little complicated, so we will return to it after a short detour.

In order to formulate things in a slightly less cumbersome fashion, from here on we make use of the distance to the nearest integer function $\|\cdot\| : \mathbb{R} \to [0, 1/2]$, which is defined by

$$\|x\| = \min_{a\in\mathbb{Z}} |x - a|.$$

For example, multiplying both sides of (1.1.2) by n, we find that Dirichlet's theorem is equivalent to the statement that, for every $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$,

$$\inf_{1\leq n\leq N} \|n\alpha\| \leq \frac{1}{N+1}.$$

Similarly, Hurwitz's Theorem implies that for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we have that

$$\liminf_{n\to\infty} n\|n\alpha\| \leq \frac{1}{\sqrt{5}}.$$

Motivated by our discussion in the previous paragraph, we say that real number α is *badly approximable* if there exists a constant $c(\alpha) > 0$ such that

$$\liminf_{n\to\infty} n\|n\alpha\| \geq c(\alpha).$$

We write $\mathcal{B}$ for the set of all badly approximable numbers. A real number which is not badly approximable is called *well approximable*.

From what we have said before, the golden ratio $\alpha = (1+\sqrt{5})/2$ is badly approximable with the constant $c(\alpha) = 1/\sqrt{5}$, which is therefore largest possible by Hurwitz's theorem. Returning to our previous discussion, the collection of all possible values of the quantities

$$\liminf_{n\to\infty} n\|n\alpha\|, \tag{1.1.5}$$

as α runs over all real numbers, is referred to as the *Lagrange spectrum* (note that some authors consider the reciprocals of these values to be the Lagrange spectrum). The largest number in the Lagrange spectrum is $1/\sqrt{5}$, followed by $1/\sqrt{8}, 5/\sqrt{221}$, and so on. These initial values form a countable subset of the Lagrange spectrum which lies in the interval $(1/3, 1/\sqrt{5}]$, and whose only accumulation point is at $1/3$. The values of α which give rise to this countable subset are also, themselves, a countable subset of the real numbers [9, Chapter II, Section 6]. The part of the Lagrange spectrum which lies in $[0, 1/3]$ is somewhat different in nature. It turns out, for example, that there are uncountably many real numbers α for which (1.1.5) is equal to $1/3$. Furthermore, it is known that the Lagrange spectrum contains an interval of the form $[0, \xi_H]$, with $\xi_H > 0$, which is called Hall's ray [13, 14, 15]. Hopefully all of this gives the reader a more complete impression of what the collection of badly approximable numbers looks like.

EXERCISES

Exercise 1.1.1. *Prove Theorem 1.1.1.*

Exercise 1.1.2. *Prove that, without imposing additional hypotheses in Theorem 1.1.1, inequality* (1.1.2) *cannot be improved.*

Exercise 1.1.3. *Prove that if* $\alpha \in \mathbb{Q}$ *then there are only finitely many rationals* $a/n \in \mathbb{Q}$ *for which* (1.1.3) *holds.*

1.2 Continued fractions and Ostrowski expansions

Continued fractions are a central tool in the study of one-dimensional approximation. They have a long history, with numerous applications both within mathematics and also to real world problems. Our presentation of the material in this chapter, therefore, does not reflect historical sequence but instead aims to prioritize organization of concepts.

Every irrational real number α has an infinite *continued fraction expansion* of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots$ is a sequence of positive integers. For convenience of notation we denote the continued fraction expansion of α by

$[a_0; a_1, a_2, a_3, \ldots]$. The integers a_k are called the *partial quotients* in this expansion and, since α is irrational, they are uniquely determined.

If α is rational then it has two finite expansions of the above form, which can be written as $[a_0; a_1, \ldots, a_m]$ and $[a_0; a_1, \ldots, a_m - 1, 1]$, for an appropriate choice of $k \geq 0, a_0 \in \mathbb{Z}$, and $a_1, \ldots, a_m \in \mathbb{N}$. We may refer to either of these as the continued fraction expansion of α and, unless it is important to make a distinction between these expansions, we will not specify which of the two we are choosing. When we are working with rational α, with continued fraction expansion as above, we will also set $a_k = 0$ for $k > m$.

For $\alpha \in \mathbb{R}$ and $k \geq 0$ the rational numbers

$$\frac{p_k}{q_k} = [a_0; a_1, \ldots, a_k],$$

with p_k and q_k are coprime and $q_k > 0$, are called the *principal convergents* to α. If we also set $p_{-1} = 1$ and $q_{-1} = 0$, then it is not difficult to show that for $k \geq 0$ the numerators and denominators of the principal convergents satisfy the recursive relations

$$p_{k+1} = a_{k+1}p_k + p_{k-1} \quad \text{and} \quad q_{k+1} = a_{k+1}q_k + q_{k-1}. \tag{1.2.1}$$

From this it follows easily that, for $k \geq 0$,

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}. \tag{1.2.2}$$

The principal convergents p_k/q_k converge to α as $k \to \infty$, as seen by the inequality

$$\left|\alpha - \frac{p_k}{q_k}\right| \leq \frac{1}{q_{k+1}q_k}, \tag{1.2.3}$$

for $k \geq 0$. If α is rational then from some point on we actually have that $p_k/q_k = \alpha$. However if α is irrational then the above inequality is close to the truth, as in this case we also have the lower bound

$$\left|\alpha - \frac{p_k}{q_k}\right| \geq \frac{1}{(q_{k+1} + q_k)q_k}. \tag{1.2.4}$$

A primary significance of the principal convergents lies in the fact that, in a strong sense, they provide the best approximations to α. To make this precise, it can be shown that, for any $k \geq 0$,

$$\min_{n < q_{k+1}} \|n\alpha\| = \|q_k \alpha\|. \tag{1.2.5}$$

If α is irrational we have the stronger result that

$$\min\{\|n\alpha\| : 1 \leq n < q_{k+1}, n \neq q_k\} > \|q_k\alpha\|. \tag{1.2.6}$$

In other words, this shows that p_k/q_k is the best rational approximation to α, not only among fractions with denominators up to q_k, but even among all fractions with denominators less than q_{k+1}.

Comparing what we have said so far with the results in the previous section, it is not difficult to show that (1.2.3) implies Theorem 1.1.1. In the other direction, using the recursion (1.2.1) in the lower bound (1.2.4), we obtain that

$$\left|\alpha - \frac{p_k}{q_k}\right| \geq \frac{1}{(a_{k+1}+2)q_k^2}.$$

This together with the best approximation property (1.2.6) implies the well known result that an irrational real number is badly approximable if and only if it has bounded partial quotients in its simple continued fraction expansion.

Moving on, the following lemma describes what we will refer to as the *Ostrowski expansion*, with respect to α, of a positive integer.

Theorem 1.2.7. *Suppose $\alpha \in \mathbb{R}$ is irrational. Then for every $n \in \mathbb{N}$ there is a unique integer $M \geq 0$ and a unique sequence $\{c_{k+1}\}_{k=0}^{\infty}$ of integers such that $q_M \leq n < q_{M+1}$ and*

$$n = \sum_{k=0}^{\infty} c_{k+1}q_k, \tag{1.2.8}$$

$$\text{with} \quad 0 \leq c_1 < a_1, \quad 0 \leq c_{k+1} \leq a_{k+1} \quad \text{for} \quad k \geq 1,$$

$$c_k = 0 \quad \text{whenever} \quad c_{k+1} = a_{k+1} \quad \text{for some } k \geq 1, \qquad \text{and}$$

$$c_{k+1} = 0 \quad \text{for} \quad k > M.$$

A proof of this result can be found in [26, Section II.4]. For convenience we will consider the integer 0 to have the Ostrowski expansion given by taking $c_{k+1} = 0$ for all k. Now, for irrational α and $k \geq 0$ we define

$$D_k = q_k\alpha - p_k.$$

Using the properties of principal convergents, it can be shown that

$$(-1)^k D_k = |q_k\alpha - p_k|, \tag{1.2.9}$$

and that

$$|D_k| = a_{k+2}|D_{k+1}| + |D_{k+2}|. \tag{1.2.10}$$

The following lemma, which follows from [26, Theorem II.4.1], begins to highlight the reason why the Ostrowski expansion is important for problems in Diophantine approximation.

Lemma 1.2.11. *Let α be an irrational number which lies in the interval $[0, 1)$, let n be a positive integer with Ostrowski expansion as above, and let m be the smallest integer such that $c_{m+1} \neq 0$. If $m \geq 2$ then*

$$\|n\alpha\| = \left|\sum_{k=m}^{\infty} c_{k+1}D_k\right| = \operatorname{sgn}(D_m) \cdot \sum_{k=m}^{\infty} c_{k+1}D_k\,. \tag{1.2.12}$$

Also if $m = 1$ and $\{\alpha\} < 1/2$, then (1.2.12) *also holds. In all other cases we have that $\|n\alpha\| \geq |D_2|$.*

The restriction that α lie in $[0, 1)$ is a technical point which makes little difference in practice, since the approximation properties we are interested in only depend on α modulo 1. Lemma 1.2.11 can be used, together with what we know about the quantities D_k, to prove the following accurate upper and lower bounds for $\|n\alpha\|$.

Theorem 1.2.13. *Let $\alpha \in [0, 1)$ be an irrational number, let n be a positive integer, and let m be defined as in Lemma 1.2.11. If $m \geq 2$ then*

$$(c_{m+1}-1)|D_m|+(a_{m+2}-c_{m+2})|D_{m+1}| \;\leq\; \|n\alpha\| \;\leq\; (c_{m+1}+1)|D_m|. \tag{1.2.14}$$

Proof. If $m \geq 2$ then by Lemma 1.2.11 we know that (1.2.12) holds. Using 1.2.9 we then have that

$$\begin{aligned}\|n\alpha\| &= c_{m+1}|D_m| - c_{m+2}|D_{m+1}| + c_{m+3}|D_{m+2}| - c_{m+4}|D_{m+3}| + \cdots \\ &\geq c_{m+1}|D_m| + (a_{m+2} - c_{m+2})|D_{m+1}| \\ &\qquad\qquad - a_{m+2}|D_{m+1}| - a_{m+4}|D_{m+3}| - \cdots .\end{aligned}$$

Now by applying (1.2.10) we find that

$$a_{m+2}|D_{m+1}| + a_{m+4}|D_{m+3}| + \cdots = |D_m|,$$

and substituting this in to our previous equation gives the left hand inequality in (1.2.14).

For the right hand inequality we argue similarly, and we find that

$$\begin{aligned}\|n\alpha\| &\le c_{m+1}|D_m| + c_{m+3}|D_{m+2}| + c_{m+5}|D_{m+4}| + \cdots \\ &\le c_{m+1}|D_m| + |D_{m+1}| \le (c_{m+1}+1)|D_m|,\end{aligned}$$

thus completing the proof. □

In addition to the Ostrowski expansions of the positive integers, there are similar expansions for real numbers which use the D_k's in place of the q_k's. The following result is taken from [26, Theorem II.6.1].

Theorem 1.2.15. *Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ has continued fraction expansion as above. For any $\gamma \in [-\{\alpha\}, 1-\{\alpha\}) \setminus (\alpha\mathbb{Z}+\mathbb{Z})$ there is a unique sequence $\{b_{k+1}\}_{k=0}^{\infty}$ of integers such that*

$$\gamma = \sum_{k=0}^{\infty} b_{k+1} D_k, \tag{1.2.16}$$

$$\text{with } \; 0 \le b_1 < a_1, \quad 0 \le b_{k+1} \le a_{k+1} \;\text{ for }\; k \ge 1, \qquad \text{and}$$
$$b_k = 0 \quad \text{whenever} \quad b_{k+1} = a_{k+1} \;\text{ for some } k \ge 1.$$

We will refer to the expansion provided by this theorem as the Ostrowski expansion, with respect to α, of a real number. Just as the Ostrowski expansion of the positive integers is useful for determining how close multiples of α are to 0, modulo 1, the Ostrowski expansion of a real number γ is useful for determining how close multiples of α are to γ, modulo 1. The following result makes this somewhat precise.

Lemma 1.2.17. *Let α be an irrational number in $[0,1)$ and suppose that $\gamma \in [-\alpha, 1-\alpha) \setminus (\alpha\mathbb{Z}+\mathbb{Z})$. Let $n \in \mathbb{N}$ and, with reference to Ostrowski expansions of α and γ, as denoted above, for each $k \ge 0$ let*

$$\delta_{k+1} = c_{k+1} - b_{k+1},$$

Then there exists a smallest integer $m = m(n, \alpha, \gamma)$ such that $\delta_{m+1} \ne 0$. Setting

$$\Sigma = \sum_{k=m}^{\infty} \delta_{k+1} D_k,$$

we have that this quantity satisfies the equations

$$\|n\alpha - \gamma\| = \|\Sigma\| = \min\Big\{ |\Sigma|\,, 1 - |\Sigma| \Big\}$$

and

$$|\Sigma| = \operatorname{sgn}(\delta_{m+1} D_m)\Sigma\,.$$

This lemma is an inhomogeneous (referring to the fact that $\gamma \neq 0$) analogue of Lemma 1.2.11. There is also an inhomogeneous analogue of Theorem 1.2.13, which is proved in [7, Lemma 4.4], but its statement is quite detailed, and we will not need it in what follows.

EXERCISES

Exercise 1.2.1. *Prove assertions* (1.2.2)*,* (1.2.3)*, and* (1.2.4)*.*

Exercise 1.2.2. *Prove the recursive relation* (1.2.10) *for the D_k's.*

Exercise 1.2.3. *Let $\alpha \in [0, 1/2) \setminus \mathbb{Q}$. Prove that, for any $n \in \mathbb{N}$, we have that $\{n\alpha\} \in [1 - 2\alpha, 1 - \alpha)$ if and only if the digit c_1 in the Ostrowski expansion for α is equal to $a_1 - 1$.*

The next exercise is about the Three Distance Theorem (also known as the Steinhaus Theorem), which says that, for any $\alpha \in \mathbb{R}$ and for any $N \in \mathbb{N}$, the component intervals of the set

$$[0, 1) \setminus \{\{n\alpha\} : 0 \le n \le N\} \tag{1.2.18}$$

take one of at most 3 distinct lengths. Furthermore, if there are 3 distinct lengths then one of them is the sum of the other two. This theorem was first proved by Sós [29] and Świerczkowski [30], and subsequently by many others.

Exercise 1.2.4. *Prove the following refinement of the Three Distance Theorem for irrationals: Let α be an irrational real number and $N \in \mathbb{N}$. Let ℓ and a be the unique integers satisfying $\ell \ge 0,\ 0 \le a < a_{\ell+1}$, and*

$$aq_\ell + q_{\ell-1} \le N < (a+1)q_\ell + q_{\ell-1}.$$

Then the lengths of the component intervals of the set (1.2.18) *take one of the three values*

$$\|q_\ell\alpha\|,\ \|q_{\ell-1}\alpha\|-a\|q_\ell\alpha\|,\quad \textit{or}\quad \|q_{\ell-1}\alpha\|-(a-1)\|q_\ell\alpha\|,$$

which are written, from left to right, in order of increasing magnitude. Furthermore, if $n=(a+1)q_\ell+q_{\ell-1}-1$ *then only the smaller two of these lengths occur.*

1.3 Probabilistic and dimension theoretic results

After the results of the previous sections, a next natural direction is to investigate how well 'typical' numbers can be approximated by rationals. There are various ways to make this precise. For example we might decide to look for results which hold Lebesgue almost everywhere, or we might only require them to hold on a set of large Hausdorff dimension. In this section we will look at results of both of these types, in order to gain a more complete picture of this subject. First we have the following theorem due to Borel (1909) and Bernstein (1912).

Theorem 1.3.1. *For Lebesgue almost every* $\alpha\in\mathbb{R}$ *we have that*

$$\inf_{n\in\mathbb{N}} n\|n\alpha\|=0.$$

It follows immediately from this theorem that $|\mathcal{B}|=0$ (recall that we use the notation $|A|$ to denote the Lebesgue measure of a measurable set). Equivalently, almost every α has unbounded partial quotients in its continued fraction expansion. Therefore, Borel and Bernstein's theorem tells us that badly approximable numbers are not typical, in the sense of Lebesgue measure. However, it turns out that they are typical in the sense of Hausdorff dimension, as demonstrated by the following result of Jarnik (1929).

Theorem 1.3.2. *The set* $\mathcal{B}$ *has Hausdorff dimension one.*

Next, we might ask whether a result stronger than Theorem 1.3.1 holds, for Lebesgue almost every real number. In order to present things in a larger framework, we first make a few definitions.

Given a nonnegative function $\psi : \mathbb{N} \to \mathbb{R}$, which we will call an *approximating function*, we define, for each $n \in \mathbb{N}$, a set $\mathcal{A}_n = \mathcal{A}_n(\psi) \subseteq \mathbb{R}/\mathbb{Z}$ by

$$\mathcal{A}_n = \bigcup_{a=1}^{n} \left[\frac{a}{n} - \frac{\psi(n)}{n}, \frac{a}{n} + \frac{\psi(n)}{n}\right].$$

We then set

$$\mathcal{W}(\psi) = \limsup_{n\to\infty} \mathcal{A}_n(\psi) = \{\alpha \in \mathbb{R}/\mathbb{Z} : \alpha \in \mathcal{A}_n \text{ for infinitely many } n\}.$$

To avoid having to separate out special cases, and because it makes no difference to the main results below, we will always assume that $\psi(n) \leq 1/2$.

Our previous results can be formulated in terms of the sets $\mathcal{W}(\psi)$, by choosing appropriate approximating functions. For example, Borel and Bernstein's theorem is equivalent to the statement that for any $\epsilon > 0$ we have that $|\mathcal{W}(\epsilon/n)| = 1$.

Introducing the sets $\mathcal{W}(\psi)$ emphasizes the fact that we working in a probability space. From this point of view, what should we expect for the Lebesgue measure of these sets? To answer this question first notice that

$$|\mathcal{A}_n| = 2\psi(n).$$

Now we recall the statement of the Borel-Cantelli Lemma.

Lemma 1.3.3. *Suppose that $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of measurable sets in a measure space (X, μ) which satisfies*

$$\sum_{n\in\mathbb{N}} \mu(E_n) < \infty.$$

Then

$$\mu\left(\limsup_{n\to\infty} E_n\right) = 0.$$

Applying this lemma to our sets $\mathcal{A}_n$, we find that if

$$\sum_{n\in\mathbb{N}} \psi(n) < \infty \tag{1.3.4}$$

then $|\mathcal{W}(\psi)| = 0$. In other words, if (1.3.4) holds then almost every real number α has only finitely approximations $a/n \in \mathbb{Q}$ satisfying

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{\psi(n)}{n}.$$

It follows, for example, that for any $\epsilon > 0$ and for almost every α the inequality

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{1}{n^2 \log n (\log \log n)^{1+\epsilon}}$$

is satisfied for at most finitely many $a/n \in \mathbb{Q}$.

The converse of the Borel-Cantelli Lemma, which is what would be needed in order to prove affirmative statements like the Borel and Bernstein Theorem, is not true in general. For example, consider the collection of set $E_n \subseteq \mathbb{R}/\mathbb{Z}$ defined by $E_n = (0, 1/n)$. The sum of the measures of these sets in infinite, but the limsup set is empty. An important problem in probability theory is to determine when the converse of the Borel-Cantelli Lemma holds. It turns out that, when (X, μ) is a probability space, if the sum of the measures of the sets E_n diverges and if the sets are pairwise independent, so that

$$\mu(E_m \cap E_n) = \mu(E_m)\mu(E_n) \quad \text{for all} \quad m \neq n,$$

then $\mu(\limsup E_n) = 1$. This result is due to Erdös and Renyi.

The sets $\mathcal{A}_n$ in our problem are not in general pairwise independent. However, if ψ is monotonic then, after throwing away some overlapping parts of the sets, we can show that the remaining sets are close to being pairwise independent, on average. This idea can be used to prove the following well known theorem of Khintchine (1924).

Theorem 1.3.5. *If ψ is monotonic then $|\mathcal{W}(\psi)| = 1$ if*

$$\sum_{n \in \mathbb{N}} \psi(n) = \infty, \tag{1.3.6}$$

and it equals 0 otherwise.

Duffin and Schaeffer showed in [12] that the assumption of monotonicity in this theorem is crucial. They produced a nonmonotonic approximating function ψ for which (1.3.6) holds and at the same time $|\mathcal{W}(\psi)| = 0$. Duffin and Schaeffer's example works because when two integers m and n have a large common divisor, the sets $\mathcal{A}_m$ and $\mathcal{A}_n$ can have a large overlap. Comments at the end of Duffin and Schaeffer's paper led to the formulation of what is called the Duffin-Schaeffer Conjecture, which proposes to remove the monotonicity assumption in Khintchine's Theorem by restricting attention to reduced fractions (and by appropriately modifying the corresponding divergence condition). This conjecture has a long history, most of which is recorded in [16, Chapter 2]. Recent developments can be found in [1, 6, 17].

1.4 Extensions to higher dimensions and transference principles

Here we turn to the problem of obtaining higher dimensional generalizations of our above results. Some of the arguments used in one-dimensional approximation can be adapted directly to higher dimensions. However, one of the difficulties is that there is no single expansion or multi-dimensional algorithm which does all of the things that the continued fraction expansion does in one dimension. Fortunately, for our applications in later chapters there are still tools which can be used to get around this difficulty.

Let $L : \mathbb{R}^d \to \mathbb{R}^{k-d}$ be a linear map, which is defined by a matrix with entries $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$. For any $N \in \mathbb{N}$, there exists an $n \in \mathbb{Z}^d$ with $|n| \leq N$ and

$$\|L(n)\| \leq \frac{1}{N^{d/(k-d)}}. \tag{1.4.1}$$

This is a multidimensional analogue of Dirichlet's Theorem, which follows from a straightforward application of the pigeonhole principle. We are interested in having an inhomogeneous version of this result, requiring the values taken by $\|L(n) - \gamma\|$ to be small, for all choices of $\gamma \in \mathbb{R}^{k-d}$. For this purpose we will use the following 'transference theorem,' a proof of which can be found in [9, Chapter V, Section 4].

Theorem 1.4.2. *Given a linear map L as above, the following statements are equivalent:*

(T1) There exists a constant $C_1 > 0$ such that

$$\|L(n)\| \geq \frac{C_1}{|n|^{d/(k-d)}},$$

for all $n \in \mathbb{Z}^d \setminus \{0\}$.

(T2) *There exists a constant* $C_2 > 0$ *such that, for all* $\gamma \in \mathbb{R}^{k-d}$, *the inequalities*

$$\|L(n) - \gamma\| \leq \frac{C_2}{N^{d/(k-d)}}, \quad |n| \leq N,$$

are soluble, for all $N \geq 1$, *with* $n \in \mathbb{Z}^d$.

Next, with a view towards applying this theorem, let $\mathcal{B}_{d,k-d}$ denote the collection of numbers $\alpha \in \mathbb{R}^{d(k-d)}$ with the property that there exists a constant $C = C(\alpha) > 0$ such that, for all nonzero integer vectors $n \in \mathbb{Z}^d$,

$$\|L(n)\| \geq \frac{C}{|n|^{d/(k-d)}}.$$

By a slight abuse of notation, we refer to the elements of the set $\mathcal{B}_{d,k-d}$, as well as the systems of linear forms which they define, as collections of *badly approximable systems of linear forms*. The Khintchine-Groshev Theorem, which is a higher dimensional analogue of Khintchine's Theorem (see [8] for a detailed statement and proof) implies that the Lebesgue measure of $\mathcal{B}_{d,k-d}$ is 0 (in fact, like Khintchine's Theorem in one dimension, it implies somewhat more than this). However, in analogy to Jarnik's Theorem, in terms of Hausdorff dimension the sets $\mathcal{B}_{d,k-d}$ are large. Jarnik's Theorem was extended to the multidimensional setting by Wolfgang Schmidt, who showed in [28, Theorem 2] that, for any choices of $1 \leq d < k$,

$$\dim \mathcal{B}_{d,k-d} = d(k-d).$$

Next, in our applications to cut and project sets we will sometimes be working with linear forms $L : \mathbb{R}^d \to \mathbb{R}$ which have the degenerate property that $L(\mathbb{Z}^d) + \mathbb{Z}$ is a periodic subset of $\mathbb{R}/\mathbb{Z}$. If we define $\mathcal{L} : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ by

$$\mathcal{L}(n) = L(n) \bmod 1, \tag{1.4.3}$$

then we can phrase this property by saying that the kernel of the map $\mathcal{L}$ is a nontrivial subgroup of $\mathbb{Z}^d$. For the types of cut and project sets that we want to understand, there is no way to avoid this degeneracy, but we will still want to be able to say something meaningful about the Diophantine approximation properties of L.

To clarify this further, let us consider a simple example. Let $\alpha \in \mathbb{R}$ be a badly approximable real number, and let $L : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$L(x_1, x_2) = \alpha x_1 + x_2.$$

Then we have that

$$\mathcal{L}(n_1, n_2) = n_1\alpha \bmod 1,$$

and it is clear that the kernel of this map is a rank 1 subgroup of $\mathbb{Z}^2$. It follows that L itself is not badly approximable. However, the reason for this is trivial and can be explained by the fact that, modulo 1, it is more appropriate to think of L as a linear form in 1 variable.

Our discussion in the previous paragraph now motivates a definition. Let $S \leqslant \mathbb{Z}^d$ be the kernel of the map $\mathcal{L}$ from (1.4.3), and write $r = \mathrm{rk}(S)$ and $m = d - r$. We say that L is *relatively badly approximable* if $m > 0$ and if there exists a constant $C > 0$ and a group $\Lambda \leqslant \mathbb{Z}^d$ of rank m, with $\Lambda \cap S = \{0\}$ and

$$\|\mathcal{L}(\lambda)\| \geq \frac{C}{|\lambda|^m} \quad \text{for all} \quad \lambda \in \Lambda \setminus \{0\}.$$

Now suppose that L is relatively badly approximable and let Λ be a group satisfying the condition in the definition. Let $F \subseteq \mathbb{Z}^d$ be a complete set of coset representatives for $\mathbb{Z}^d/(\Lambda + S)$. We have the following lemma.

Lemma 1.4.4. *Suppose that L is relatively badly approximable, with Λ and F as above. Then there exists a constant $C' > 0$ such that, for any $\lambda \in \Lambda$ and $f \in F$, with $\mathcal{L}(\lambda + f) \neq 0$, we have that*

$$\|\mathcal{L}(\lambda + f)\| \geq \frac{C'}{1 + |\lambda|^m}.$$

Proof. Any element of F has finite order in $\mathbb{Z}^d/(\Lambda + S)$. Therefore, for each $f \in F$ there is a positive integer u_f, and elements $\lambda_f \in \Lambda$ and $s_f \in S$, for which

$$f = \frac{\lambda_f + s_f}{u_f}.$$

If $\mathcal{L}(\lambda + f) \neq 0$ then either $\lambda + f = s_f/u_f \neq 0$, or $\lambda + u_f^{-1}\lambda_f \neq 0$. The first case only pertains to finitely many possibilities, and in the second case we have that

$$\begin{aligned}\|\mathcal{L}(\lambda + f)\| &\geq u_f^{-1} \cdot \|\mathcal{L}(u_f\lambda + \lambda_f + s_f)\| \\ &= u_f^{-1} \cdot \|\mathcal{L}(u_f\lambda + \lambda_f)\| \\ &\geq \frac{C}{u_f|u_f\lambda + \lambda_f|^m}.\end{aligned}$$

Therefore, replacing C by an appropriate constant $C' > 0$, and using the fact that F is finite, finishes the proof. □

We can also deduce that if L is relatively badly approximable, then the group Λ in the definition may be replaced by any group $\Lambda' \leqslant \mathbb{Z}^d$ which is complementary to S. This is the content of the following lemma.

Lemma 1.4.5. *Suppose that L is relatively badly approximable. Then, for any group $\Lambda' \leqslant \mathbb{Z}^d$ of rank m, with $\Lambda' \cap S = \{0\}$, there exists a constant $C' > 0$ such that*

$$\|\mathcal{L}(\lambda')\| \geq \frac{C'}{|\lambda'|^m} \quad \textit{for all} \quad \lambda' \in \Lambda' \setminus \{0\}.$$

Proof. Let Λ be the group in the definition of relatively badly approximable. Choose a basis $v_1, \ldots, v_m$ for Λ', and for each $1 \leq j \leq m$ write

$$v_j = \frac{\lambda_j + s_j}{u_j},$$

with $\lambda_j \in \Lambda, s_j \in S$, and $u_j \in \mathbb{N}$.

Each $\lambda' \in \Lambda'$ can be written in the form

$$\lambda' = \sum_{j=1}^{m} a_j v_j,$$

with integers $a_1, \ldots, a_m$, and we have that

$$\|\mathcal{L}(\lambda')\| \geq (u_1 \cdots u_m)^{-1} \left\| \mathcal{L}\left(\sum_{j=1}^{m} b_j \lambda_j \right) \right\|,$$

with $b_j = a_j u_1 \cdots u_m / u_j \in \mathbb{Z}$ for each j. If the integers a_j are not identically 0 then, since $\Lambda' \cap S = \{0\}$, it follows that

$$\lambda := \sum_{j=1}^{m} b_j \lambda_j \neq 0.$$

Using the relatively badly approximable hypothesis gives that

$$\|\mathcal{L}(\lambda')\| \geq \frac{C}{u_1 \cdots u_m \cdot |\lambda|^m}.$$

Finally since $|\lambda| \ll |\lambda'|$, we have that

$$\frac{C}{u_1 \cdots u_m \cdot |\lambda|^m} \geq \frac{C'}{|\lambda'|^m},$$

for some constant $C' > 0$. $\square$

Chapter 2

Shift spaces and Sturmian words

This chapter contains a collection of background material from symbolic dynamics. The goal is to introduce, in the relatively simple setting of Sturmian words, many of the ideas which we will encounter later in higher dimensions. Most of the material in this chapter has been well understood since work of Morse and Hedlund in the 1930's and 1940's. The exposition which we give here was influenced mainly by Morse and Hedlund's original papers on symbolic dynamics and Sturmian trajectories [22, 23], and by Baake and Grimm's book "Aperiodic Order, Vol. 1" [5].

2.1 Bi-infinite words and shift spaces

A *word* is an ordered sequence of symbols, taken from a set called an *alphabet.* A word can be finite, one-sided infinite, or bi-infinite, and we also consider the empty word to be a word. The *length* of a (finite or infinite) word u, which we will denote by $|u|$, is the number of elements in the sequence which defines it (which is 0 in the case of the empty word). If $u = u_1 \ldots u_m$ and $v = v_1 \ldots v_n$ are finite words then we write

$$uv = u_1 \ldots u_m v_1 \ldots v_n$$

to denote their *concatenation.* We say that v is a *subword* of u if there are words u_1 and u_2, either of which may be the empty word, for which $u = u_1 v u_2$. If $u = v u_2$ then we say that v is a *prefix* of u, while if $u = u_1 v$

then we say that v is a *suffix* of u. These definitions extend in the obvious ways to one-sided infinite and bi-infinite words.

To avoid any ambiguity in what follows, let us give a precise definition of the collection of objects that we are going to focus on. Let $\mathcal{A}$ be a finite alphabet, and let $\mathcal{S} = \mathcal{S}(\mathcal{A})$ denote the collection of all functions from $\mathbb{Z}$ to $\mathcal{A}$. We identify $\mathcal{S}$ with $\mathcal{A}^{\mathbb{Z}}$ in the natural way, by bijectively mapping each function in $\mathcal{S}$ to the sequence of its values. The elements of $\mathcal{S}$ can then be written in the form

$$w = (w_i)_{i\in\mathbb{Z}} = \ldots w_{-2}w_{-1}.w_0w_1\ldots,$$

where each $w_i \in \mathcal{A}$ and, as denoted by the dot, w_0 is a distinguished point. We refer to $\mathcal{A}$ as the *alphabet* and to $\mathcal{S}$ as the set of all *bi-infinite words* in $\mathcal{A}$. We will take $\mathcal{A}$ with the discrete topology and $\mathcal{S}$ with the corresponding product topology. Note that, by Tychonoff's theorem, $\mathcal{S}$ is compact.

Next we define $\sigma : \mathcal{S} \to \mathcal{S}$, the (two-sided, left) *shift map* on $\mathcal{S}$, by

$$\sigma(\ldots w_{-2}w_{-1}.w_0w_1\ldots) = \ldots w_{-1}w_0.w_1w_2\ldots.$$

It is clear that σ is invertible and, by composition, it thus defines a $\mathbb{Z}$-action on $\mathcal{S}$. Given a word $w \in \mathcal{S}$, if there exists an integer $\pi \in \mathbb{N}$ with $\sigma^{\pi}(w) = w$ then we say that w is *periodic*, and we call the smallest such π its *period*. If no such integer exists then, following [5], we say that w is *nonperiodic*.

A subset $\Sigma \subseteq \mathcal{S}$ is called a *shift space* if it is closed and invariant under both σ and σ^{-1}, so that $\sigma(\Sigma) = \Sigma$. The set $\mathcal{S}$ itself is a shift space, called the *full shift*. For any element $w \in \mathcal{S}$, there is a smallest shift space containing w, which we call the *hull* of w. Denoting the hull of w by $\mathbb{X}(w)$, we have that

$$\mathbb{X}(w) = \overline{\{\sigma^n(w) : n \in \mathbb{Z}\}}.$$

The hull of an element of $\mathcal{S}$ may be the full shift. However, as we will see, in general there are many other interesting possibilities. If $w \in \mathcal{S}$ has the property that every element of $\mathbb{X}(w)$ is nonperiodic, then we say that w is *aperiodic*.

If Σ is a shift space with the property that $\mathbb{X}(w) = \Sigma$, for all $w \in \Sigma$, then we say that the $\mathbb{Z}$-action of σ on Σ is *minimal*. In this case we also refer to Σ itself as a minimal shift space.

Returning to basic notation and terminology, for any subset $\Sigma \subseteq S$, the *language* of Σ, denoted $\mathcal{L}(\Sigma)$, is the collection of all finite words (including the empty word) which occur as subwords of any element of Σ. For simplicity, if $\Sigma = \{w\}$ is a singleton set then we write $\mathcal{L}(w)$ instead of $\mathcal{L}(\{w\})$. We say that two words $w, w' \in \mathcal{S}$ are *locally indistinguishable (LI)* if $\mathcal{L}(w) = \mathcal{L}(w')$. In other words, w and w' are LI if and only if every finite word which is a subword of one of them is also a subword of the other. It is clear that local indistinguishability is an equivalence relation, and we will denote the equivalence class of a word $w \in \mathcal{S}$ by $\mathrm{LI}(w)$.

Finally, we say that a word $w \in \mathcal{S}$ is *repetitive* if, for every finite subword u of w, there exists an integer $C(u) \in \mathbb{N}$ with the property that every $v \in \mathcal{L}(w)$ with $|v| = C(u)$ contains u as a subword. If w is repetitive then we define its *repetitivity function* $R : \mathbb{N} \to \mathbb{N}$ by taking $R(n)$ to be the smallest positive integer with the property that every $u \in \mathcal{L}$ with $|u| = n$ is a subword of every $v \in \mathcal{L}$ with $|v| = R(n)$.

The following important theorem connects several of the concepts which we have introduced.

Theorem 2.1.1. *For any $w \in \mathcal{S}(\mathcal{A})$, the following are equivalent:*

(i) $\mathbb{X}(w)$ *is minimal*

(ii) $\mathbb{X}(w) = \mathrm{LI}(w)$

(iii) $\mathrm{LI}(w)$ *is closed*

The statement of this result, as well as its proof, can be found in [5, Proposition 4.1]. However, as it is a good exercise in understanding definitions, we encourage the reader to attempt the proof themselves.

EXERCISES

Exercise 2.1.1. *Let $\nu : \mathcal{S} \times \mathcal{S} \to \mathbb{Z}_{\geq 0}$ be defined by the rule that $\nu(w, w') = 0$ if $w_0 \neq w'_0$, and otherwise*

$$\nu(w, w') = \sup\{n \geq 1 : w_m = w'_m \text{ whenever } |m| \leq n - 1\},$$

and let $d : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ be defined by

$$d(w, w') = 2^{-\nu(w,w')}.$$

Prove that d is a metric on $\mathcal{S}$, and that it induces the product topology described above.

Exercise 2.1.2. *Prove that σ and σ^{-1} are continuous maps.*

Exercise 2.1.3. *Give an example of a word $w \in \mathcal{S}$ which is nonperiodic but not aperiodic.*

Exercise 2.1.4. *Prove that for any $w \in \mathcal{S}$, if $\mathbb{X}(w)$ is minimal then w is nonperiodic if and only if it is aperiodic.*

Exercise 2.1.5. *Give an example of a nonperiodic word $w \in \mathcal{S}$ for which $\mathbb{X}(w)$ is not minimal.*

Exercise 2.1.6. *Suppose that w and w' are repetitive words with the property that, for any $n \in \mathbb{N}$, there exists a word $u \in \mathcal{L}(w) \cap \mathcal{L}(w')$ with $|u| = n$. Prove that w and w' are in the same LI equivalence class.*

Exercise 2.1.7. *Prove that a word $w \in \mathcal{S}$ is repetitive if and only if $\mathbb{X}(w)$ is minimal.*

2.2 Complexity and Sturmian words

We now begin our discussion of subword complexity for bi-infinite words. Given any $w \in \mathcal{S}$, we define its complexity function $p : \mathbb{N} \to \mathbb{N}$ by setting $p(n)$ equal to the number of words $u \in \mathcal{L}(w)$ with $|u| = n$. It is clear that p is a nondecreasing function and that

$$p(n) \leq n^{\#\mathcal{A}}.$$

It is also not difficult to show that w is periodic if and only if there exists a constant C with the property that $p(n) \leq C$ for all n (see Exercises 2.2.1 and 2.2.2). This implies that if w is nonperiodic then

$$\lim_{n\to\infty} p(n) = \infty.$$

However it turns out that more is true, as demonstrated by the following well known theorem of Morse and Hedlund [22, Theorem 7.3].

Theorem 2.2.1. *For any word $w \in \mathcal{S}$, if $p(n_0) < n_0 + 1$ for some $n_0 \in \mathbb{N}$, then w is periodic and $p(n) = p(n_0)$ for all $n \geq n_0$.*

Proof. Suppose that $p(n_0) < n_0 + 1$ for some n_0, and that n_0 has been chosen as the smallest positive integer with this property. Without loss of generality, we may assume that $n_0 \geq 2$, otherwise what we are trying to prove is trivial.

By our choice of n_0, we have that $p(n_0 - 1) \geq n_0$. However, since $p(n_0 - 1) \leq p(n_0)$, it must in fact be the case that $p(n_0 - 1) = n_0$, and thus that $p(n_0) = n_0$. Now suppose that u and v are distinct words of length $n_0 - 1$ in $\mathcal{L}(w)$. Any words u' and v' in $\mathcal{L}(w)$ which contain u and v (respectively) as prefixes must also be distinct, so it follows that every word of length $n_0 - 1$ in $\mathcal{L}(w)$ is a prefix of exactly one word of length n_0 in $\mathcal{L}(w)$.

Finally, every word of length n_0 in $\mathcal{L}(w)$ can be written in the form au, for some $a \in \mathcal{A}$ and for some word u of length $n_0 - 1$. It therefore follows from our conclusion in the previous paragraph that every word of length n_0 in $\mathcal{L}(w)$ is a prefix of exactly one word of length $n_0 + 1$ in $\mathcal{L}(w)$. By induction, we have that $p(n) = p(n_0)$ for all $n \geq n_0$, and w is thus seen to be periodic. $\square$

It follows immediately from the statement of this theorem that any nonperiodic bi-infinite word w must have $p(n) \geq n + 1$ for all n. This raises the question of whether or not there are nonperiodic words which attain the minimum possible complexity allowed by the theorem, i.e. words with $p(n) = n + 1$ for all n. Such words do exist, and they are called *Sturmian words*. It is clear that any Sturmian word must only use two symbols from its alphabet. Therefore, without loss of generality, we will restrict our discussion of Sturmian words to the case when $\mathcal{A} = \{0, 1\}$.

One example of a Stumian word is given by

$$\ldots 00.100 \ldots . \tag{2.2.2}$$

Words of this type form a countably infinite collection of nonrepetitive Sturmian words. Of considerably more interest, however, is the collection of repetitive Sturmian words which, as we will see in the next section, has a dynamical description in terms of circle rotations.

EXERCISES

Exercise 2.2.1. *Show that if $w \in \mathcal{S}$ is periodic with period π then $p(n) \leq \pi$ for all n, with equality for $n \geq \pi$.*

Exercise 2.2.2. *Prove that, for any $w \in \mathcal{S}$, if there exists a constant $C > 0$ such that $p(n) \leq C$ for all n, then w is periodic.*

2.3 Dynamical characterization of Sturmian words

In this section we present a dynamical characterization of Sturmian words, in terms of irrational rotations of the circle. The theorems in this section are originally due to Morse and Hedlund [22]. The first direction of this characterization shows how irrational rotations can be used to construct uncountably many LI classes of Sturmian words.

Theorem 2.3.1. *For $I \subseteq \mathbb{R}$, let $\chi_I : \mathbb{R}/\mathbb{Z} \to \{0,1\}$ denote the indicator function of the set $I + \mathbb{Z}$. Then, for any $\alpha, \gamma \in \mathbb{R}$ with $\alpha \notin \mathbb{Q}$, the bi-infinite words*

$$(\chi_{[1-\alpha,1)}(i\alpha+\gamma))_{i\in\mathbb{Z}} \quad \textit{and} \quad (\chi_{(1-\alpha,1]}(i\alpha+\gamma))_{i\in\mathbb{Z}}, \tag{2.3.2}$$

are Sturmian and repetitive.

Proof. We consider only the case when the interval defining the word is $[1-\alpha, 1)$, as both cases follow from the same argument. With α and γ as above, let

$$w_i = \chi_{[1-\alpha,1)}(i\alpha+\gamma) \ \text{ for } \ i \in \mathbb{Z},$$

and let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by

$$T(x) = x + \alpha \text{ mod } 1.$$

For each $n \in \mathbb{N}$ let

$$x^{(n)} = \{\{-i\alpha\} : 0 \leq i \leq n\} \subseteq [0,1),$$

and suppose that the elements of $x^{(n)}$ are written in increasing order as

$$x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)}.$$

Set $x_{n+1}^{(n)} = 1$ and, for each $1 \leq i \leq n+1$ define

$$I_i^{(n)} = [x_{i-1}^{(n)}, x_i^{(n)}).$$

Now suppose that $x \in I_i^{(n)}$ and that $y \in I_j^{(n)}$. We claim that

$$\chi_{[1-\alpha,1)}(T^m x) = \chi_{[1-\alpha,1)}(T^m y) \quad \text{for all} \quad 0 \le m \le n-1,$$

if and only if $i = j$. To prove one direction of the claim suppose, without loss of generality, that $T^m(x) \in [0, 1-\alpha)$ and $T^m(y) \in [1-\alpha, 1)$, for some $0 \le m \le n-1$. Then, modulo $\mathbb{Z}$, we have that

$$x \in [-m\alpha, 1-(m+1)\alpha) \quad \text{and} \quad y \in [1-(m+1)\alpha, 1-m\alpha).$$

Thus x and y are contained in disjoint intervals whose endpoints are elements of $x^{(n)}$, from which we conclude that $i \ne j$. To prove the other direction of the claim suppose, by interchanging the roles of x and y if necessary, that $x < y$. In this case we may also suppose, without loss of generality, that n is the smallest positive integer with the property that x and y lie in different intervals $I_i^{(n)}$ and $I_j^{(n)}$, respectively. Then we must have that $j = i+1$ and that $x_i^{(n)} = \{-n\alpha\}$. Furthermore, by dividing into cases depending on whether or not $\{\alpha\} < 1/2$, we can show that

$$|I_i^{(n)}| \le 1 - \{\alpha\} \quad \text{and} \quad |I_j^{(n)}| \le \{\alpha\}.$$

It follows that $T^{n-1}(x) \in [0, 1-\{\alpha\})$ and $T^{n-1}(y) \in [1-\{\alpha\}, 1)$, thus establishing the claim.

It follows from our claim that, for any $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, the word $w_m w_{m+1} \ldots w_{m+n-1}$ is uniquely determined by which interval $I_i^{(n)}$, $1 \le i \le n+1$, the number $\{m\alpha+\gamma\}$ belongs to. The fact that α is irrational implies that the sequence $\{m\alpha+\gamma\}_{m\in\mathbb{Z}}$ is dense in $\mathbb{R}/\mathbb{Z}$, therefore we have that $p(n) = n+1$.

In our dynamical formulation, the fact that w is repetitive is equivalent to the statement that, for any $n \in \mathbb{N}$ and for any $1 \le i \le n+1$,

$$\sup\left\{\min\{j \in \mathbb{N} : T^j(x) \in I_i^{(n)}\} : x \in I_i^{(n)}\right\} < \infty.$$

In other words, it is equivalent to the statement that the orbit under T of every point in $I_i^{(n)}$, returns to $I_i^{(n)}$ in a (uniformly) finite amount of time. This is quite easy to verify, and we leave it as an exercise. $\square$

The next result shows that all repetitive Sturmian words can be realized using the construction from the previous theorem. This is quite remarkable, considering that they were introduced as seemingly combinatorial objects.

Theorem 2.3.3. *Any repetitive Sturmian word is given by a bi-infinite sequence of one of the forms in* (2.3.2)*, for some* $\alpha, \gamma \in \mathbb{R}$ *with* $\alpha \notin \mathbb{Q}$*.*

Proof. Assume w is a repetitive Sturmian word. Then for all $n \in \mathbb{N}$ there is a unique word $u^{(n)} \in \mathcal{L}(w)$ which has $|u^{(n)}| = n$ and for which both words $1u^{(n)}$ and $0u^{(n)}$ are in $\mathcal{L}(w)$. It follows that for all $m \le n$, the word $u^{(m)}$ is a prefix of $u^{(n)}$, and so all of the words $u^{(n)}$ are prefixes of the one-sided infinite word $u^{(\infty)}$ defined by $u = u_1 u_2 \ldots$, with $u_i = u_i^{(i)}$.

Let us assume, with little loss of generality, that $u_1^{(1)} = 0$. Then, since w is repetitive, there is a smallest integer $m_1 \in \mathbb{N}$ with the property that $u_{m_1+1} = 1$. In other words, if we write B_1 for the prefix of u of length $m_1 + 1$, then we have that

$$B_1 = \overbrace{0\ldots0}^{m_1}1.$$

For the sake of what is to come, let us also write $B_0 = 0$ (which is clearly the prefix of $u^{(\infty)}$ of length 1). Now we claim that, anywhere in $u^{(\infty)}$ where B_1 occurs as a subword, it must be immediately followed either by another occurrence of B_1, or by the word B_0B_1. To see why this is true, consider the following sequence of observations:

(i) Any occurrence of the letter 1 must be followed immediately by a 0, otherwise both 11 and 01 would belong to $\mathcal{L}(w)$, contradicting the definition of $u^{(\infty)}$ and the fact that it begins with $B_0 = 0$.

(ii) If a block of m consecutive zeros immediately follows a B_1 block, then it must be the case that $m \le m_1 + 1$. If this were not true then the word

$$v = \overbrace{0\ldots0}^{m_1+1}$$

would have the property that $0v$ and $1v$ are both in $\mathcal{L}(w)$. However this would contradict that fact that B_1 is the unique word of length $m_1 + 1$ with this property.

(iii) If a block of m consecutive zeros immediately follows a B_1 block, and then is immediately followed by a 1, then it must be the case that $m \ge m_1$. Again, if this were not the case then the word

$$\overbrace{0\ldots0}^{m}1$$

would be a suffix of two distinct words of length $m+2$ in $\mathcal{L}(w)$, but this would contradict the fact that

$$\overbrace{0\ldots 0}^{m+1}$$

is the unique word with this property.

This sequence of observations clearly establishes our claim. This means that we can rewrite the word $u^{(\infty)}$ as an infinite concatenation of B_1 and B_0 blocks.

Again using repetitivity, there is an integer $m_2 \in \mathbb{N}$ with the property that the word

$$B_2 = \overbrace{B_1 \cdots B_1}^{m_2} B_0$$

is a prefix of $u^{(\infty)}$. Then, by essentially the same argument as before, we can show that any occurrence of the word B_2 in $u^{(\infty)}$ must be immediately followed either by the word B_2 or by the word B_1B_2. Continuing in this way, there is a sequence of integers $m_k \in \mathbb{N}$, with $k \geq 1$, and a sequence of words B_k, defined recursively by

$$B_k = \overbrace{B_{k-1} \cdots B_{k-1}}^{m_k} B_{k-2} \quad \text{for} \quad k \geq 2,$$

with the property that each B_k is a prefix of $u^{(\infty)}$.

Now this is where it pays to have done the exercises. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be defined by

$$\alpha = [0; m_1 + 1, m_2, \ldots]. \tag{2.3.4}$$

Then it follows from Exercise 1.2.3, together with basic properties of the Ostrowski expansion, that

$$u_i = \chi_{[1-\alpha,1)}(i\alpha) \quad \text{for} \quad i \in \mathbb{N}. \tag{2.3.5}$$

Now, since both w and the word

$$w' = (\chi_{[1-\alpha,1)}(i\alpha))_{i\in\mathbb{Z}}$$

are repetitive, it follows from Exercise 2.1.6 that these two words are in the same LI equivalence class. Finally, it follows from Exercise 2.1.7 together with Theorem 2.1.1 that $w \in \mathbb{X}(w')$.

The final statement, that $w \in \mathbb{X}(w')$, means that there is a sequence of words $\{w^{(n)}\}_{n\in\mathbb{N}}$, each of the form

$$w^{(n)} = (\chi_{[1-\alpha,1)}((i + i_n)\alpha))_{i\in\mathbb{Z}} \quad \text{for some} \quad i_n \in \mathbb{Z},$$

with the property that $w_n \to w$ as $n \to \infty$, in the topology of $\mathcal{S}$. By the definition of convergence in this topology, together with the argument used in the proof of Theorem 2.3.1, there is a real number γ with the property that $\{i_n\alpha\} \to \gamma$ as $n \to \infty$. If $\gamma \notin \alpha\mathbb{Z}+\mathbb{Z}$ then both of the words in (2.3.2) are equal to w. Otherwise only one of them is, depending on whether or not the point γ itself is counted as a 0 or a 1. This final technical detail, which is only relevant when γ lies in the orbit of 0 under rotation by α modulo 1, is determined by whether the numbers $i_n\alpha$ mod 1 are eventually all equal to γ, or whether they approach γ from the left. $\square$

Theorems 2.3.1 and 2.3.3 give us a complete classification of the collection of all repetitive Sturmian words. Every Sturmian word w takes one of the forms given in 2.3.2. The number α is then referred to as the *slope* of w, and γ as the *intercept*.

EXERCISES

Exercise 2.3.1. *Prove that there are uncountably many LI classes of Sturmian words.*

Exercise 2.3.2. *Complete the proof that any bi-infinite sequence of one of the forms in* (2.3.2)*, for $\alpha, \gamma \in \mathbb{R}$ and $\alpha \notin \mathbb{Q}$, is repetitive.*

Exercise 2.3.3. *Verify the statement, in the proof of Theorem 2.3.3, that* (2.3.5) *holds, for the number α defined by* (2.3.4).

2.4 Calculating the repetitivity function

In this section we prove one more remarkable result about Sturmian words, which originates from a second paper of Morse and Hedlund [23]. The main result shows that there is a formula for the repetitivity function $R(n)$ of a repetitive Sturmian words, which depends on the continued fraction expansion of the slope. Our exposition of the proof closely follows Alessandri and Berthé's proof in [2].

Theorem 2.4.1. *Suppose that w is a repetitive Sturmian word with slope α, and let $n \in \mathbb{N}$. Then, referring to the continued fraction expansion of α,* *if* $q_k \leq n < q_{k+1}$*, we have that*

$$R(n) = q_{k+1} + q_k + n - 1.$$

Proof. First we will reformulate the definition of $R(n)$ in terms of the dynamics of rotation by α on $\mathbb{R}/\mathbb{Z}$. Then the proof will be a simple application of Diophantine approximation.

Let w be a repetitive Sturmian word with slope α and intercept γ and recall that, for $n \in \mathbb{N}$, $R(n)$ is defined to be the smallest integer with the property that every word of length n in $\mathcal{L}(w)$ is a subword of every word of length $R(n)$ in $\mathcal{L}(w)$. Now let

$$u = w_m w_{m+1} \dots w_{m+R(n)-1}$$

be word of length $R(n)$ in $\mathcal{L}(w)$. The subwords of u of length n are

$$w_{m'} w_{m'+1} \dots w_{m'+n-1},$$

for $m \leq m' \leq m+R(n)-n$. Using the notation and proof of Theorem 2.3.1, each of these subwords is uniquely determined by which of the intervals $I_i^{(n)}$, with $1 \leq i \leq n+1$, the point $\{m'\alpha+\gamma\}$ lies in. Working modulo 1 we have that

$$\{m'\alpha+\gamma : m \leq m' \leq m+R(n)-1\} = \{m\alpha\}+\{i\alpha+\gamma : 0 \leq i \leq R(n)-1\}.$$

For $R(n)$ to be the value of the repetitivity function, it must be the smallest integer with the property that, for any choice of $m \in \mathbb{Z}$, the collection of points above intersects every interval $I_i^{(n)}$ with $1 \leq i \leq n+1$. Since m is arbitrary and $m\alpha$ is dense modulo 1, we deduce that $R(n)$ must be the smallest integer with the property that the longest among the intervals $I_i^{(R(n)-n)}$, with $1 \leq i \leq R(n)-n+1$, is at least as short as the shortest among the intervals $I_i^{(n)}$, with $1 \leq i \leq n+1$.

Now by Exercise 1.2.4, the shortest length of an interval of the form $I_i^{(n)}$ is $\|q_k\alpha\|$ and, by the same exercise again, we must have that $R(n) - n = q_{k+1} + q_k - 1$. $\square$

A final word about the repetitivity function. An important class of Sturmian words is the collection of words whose repetitivity function is bounded above by a linear function. Accordingly, we say that a repetitive bi-infinite word is *linearly repetitive* if there exists a constant $C > 0$ with the property that $R(n) \leq Cn$ for all n. In light of what we know about continued fractions and Diophantine approximation, we immediately obtain the following corollary to the previous theorem.

Corollary 2.4.2. *A repetitive Sturmian word is linearly repetitive if and only if its slope is a badly approximable real number.*

The ideas used in the proofs in this chapter will return later, in our discussion of complexity and repetitivity for cut and project sets.

Chapter 3

Point sets in Euclidean space

Following Baake and Grimm [5], we call any countable subset of $\mathbb{R}^k$ a *point set* in $\mathbb{R}^k$. In this chapter we will introduce several collections of point sets, which will be the primary objects of study in later chapters.

3.1 Delone set, lattices, and crystallographic point sets

A set $Y \subseteq \mathbb{R}^k$ is *uniformly discrete* if there is a constant $r > 0$ with the property that, for every $y \in Y$,

$$B_r(y) \cap Y = \{y\}.$$

It is clear that any uniformly discrete set must be a point set. If Y is uniformly discrete then the supremum of the set of all constants r which satisfy the above condition is called the *packing radius* of Y.

We say that a set $Y \subseteq \mathbb{R}^k$ is *relatively dense* if there is a constant $R > 0$ with the property that, for any $x \in \mathbb{R}^k$,

$$\overline{B_R(x)} \cap Y \neq \emptyset.$$

If Y is relatively dense then the infimum of the set of all constants R which satisfy the above condition is called the *covering radius* of Y.

A set $Y \subseteq \mathbb{R}^k$ which is both uniformly discrete and relatively dense is called a *Delone set*. For any pair of positive constants (r, R), we let $\mathcal{D}_k(r, R)$

denote the collection of all Delone sets in $\mathbb{R}^k$ with packing radius at most r and covering radius at least R.

Among the simplest examples of Delone sets are lattices. A *lattice* in $\mathbb{R}^k$ is a discrete subgroup $\Lambda \leqslant \mathbb{R}^k$ with the property that the quotient space $\mathbb{R}^k/\Lambda$ has finite co-volume (i.e. it has a Lebesgue measurable fundamental domain with finite volume). Of course, this is equivalent to asking that Λ be discrete and co-compact (i.e. so that $\mathbb{R}^k/\Lambda$ is compact). It is an easy exercise to check that a discrete subgroup of $\mathbb{R}^k$ will be a lattice if and only if it has rank k. Lattices themselves are completely periodic and well structured objects. However, they will also be a key ingredient in our constructions of examples of ordered point sets which are not periodic.

If $Y \subseteq \mathbb{R}^k$ is a point set, then a point $x \in \mathbb{R}^k$ with the property that $Y + x = Y$ is called a *period* of Y. The collection of all periods of Y forms a group, called its *group of periods*. We say that Y is *nonperiodic* if its group of periods is $\{0\}$. On the other extreme, we say that Y is a *crystallographic point set* if its group of periods is a lattice in $\mathbb{R}^k$. The following result, which follows from [5, Proposition 3.1], gives an important alternative characterization of crystallographic point sets.

Lemma 3.1.1. *A uniformly discrete point set $Y \subseteq \mathbb{R}^k$ is a crystallographic point set if and only if there exists a lattice $\Lambda \subseteq \mathbb{R}^k$ and a finite set $F \subseteq \mathbb{R}^k$ with the property that, for any $y \in Y$, there are unique elements $\lambda \in \Lambda$ and $f \in F$ such that*

$$y = \lambda + f.$$

3.2 Cut and project sets

Cut and project sets are point sets which are obtained by projecting the collection of lattice points in a strip in some total space, to a lower-dimensional subspace. Generally speaking, these sets have a great amount of structure, imposed by the fact that they are constructed from lattices, but they are also typically aperiodic. Furthermore, many problems in mathematics involve manifestations of aperiodic order which can be described using cut and project sets. A prototypical example of this, which may already convince the reader of the fundamental importance of these sets, is that all Sturmian words can be defined using cut and project sets.

There are other many other examples which illustrate the importance of cut and project sets. First of all, they arise naturally in dynamical systems, as they are the collections of return times, to prescribed regions, of linear actions on higher dimensional tori. They are also an abundant source of aperiodic tilings of Euclidean space (which, at this point, we have not defined), and can be used to construct famous tilings such as the Penrose and Ammann-Beenker tilings. Finally, cut and project sets are used as a mathematical model for physical materials known as quasicrystals.

We proceed with more rigorous definitions. First of all, we will say that subspaces V_1 and V_2 of R^k are *complementary* if $V_1 \cap V_2 = \emptyset$ and if we have the Minkowski sum decomposition

$$\mathbb{R}^k = V_1 + V_2.$$

This implies that $\dim(V_1) + \dim(V_2) = k$ and that every point in $\mathbb{R}^k$ has a unique representation as the sum of an element of V_1 with an element of V_2.

Cut and project sets are defined as follows. Let $1 \leq d < k$ be integers, let E be a d-dimensional subspace of $\mathbb{R}^k$, and $F_\pi \subseteq \mathbb{R}^k$ a subspace complementary to E. The subspaces E and F_π are referred to as the *physical space* and *internal space*, respectively, and $\mathbb{R}^k$ is called the *total space*. Write π for the projection onto E with respect to the decomposition $\mathbb{R}^k = E + F_\pi$. Choose a set $\mathcal{W}_\pi \subseteq F_\pi$, and define $\mathcal{S} = \mathcal{W}_\pi + E$. The set $\mathcal{W}_\pi$ is referred to as the *window*, and $\mathcal{S}$ as the *strip*. Given this data, for each $s \in \mathbb{R}^k/\mathbb{Z}^k$, we define the *cut and project set* $Y_s \subseteq E$ by

$$Y_s = \pi(\mathcal{S} \cap (\mathbb{Z}^k + s)). \tag{3.2.1}$$

In this situation we refer to Y_s as a *k to d cut and project set*.

This definition, as it stands, is too general to be able to say anything meaningful about the collection of all cut and project sets. Therefore, let us take a moment to explore three phenomena which we will attempt to justify in excluding from further consideration. To help keep track, the first of these phenomena is a pathology, the second is more of a degeneracy, and the third is simply a minor nuisance. Actually, there are two nuisances, but we will encounter the second one later on. To aid in the discussion, let us write $\pi^* : \mathbb{R}^k \to F_\pi$ for the projection onto F_π, according to the above decomposition.

First the pathology. For simplicity, consider the case when $k = 2$ and $d = 1$. Suppose that E is a line in $\mathbb{R}^2$ with irrational slope, take $s =$

0, and let $F_\pi = E^\perp$ be the line perpendicular to E, which is clearly a complementary subspace. It is not difficult to verify in this case that π is injective and that $\pi(\mathbb{Z}^2)$ is a dense subset of E. Now, let $Y' \subseteq \pi(\mathbb{Z}^k)$ be any subset at all which we would like to obtain as a cut and project set, and then choose

$$\mathcal{W} = \pi^*(\mathbb{Z}^2 \cap \pi^{-1}(Y')).$$

Using again the fact that the slope of E is irrational, it is clear from the definitions that $Y_s = Y'$. In order to remedy this pathology we will assume in all of what follows, unless otherwise specified, that the window is a relatively compact set and that closure of $\mathcal{W}$ is equal to the closure of its interior.

Next, to see what we call the degeneracy, consider first the case when $k = 2$ and $d = 1$, and when E is a subspace with rational slope. In this case $E \cap \mathbb{Z}^2$ contains a rank 1 subgroup of $\mathbb{Z}^2$. Depending on how the window is chosen, there are two possibilities. Either $\mathcal{S} \cap (\mathbb{Z}^2 + s) = \emptyset$, in which case $Y_s = \emptyset$, or $\mathcal{S} \cap (\mathbb{Z}^2 + s) \neq \emptyset$, in which case Y_s has a nontrivial group of periods. Neither of these outcomes produces an interesting point set which we don't already understand so, since $k = 2$, we could choose to eliminate both of them by requiring that E always have irrational slope. However, in higher dimensions the situation is slightly more complicated, as we now describe.

For $k > 2$ and $d > 1$, to avoid situations where our cut and project set may be empty, we require that $E + \mathbb{Z}^k$ be dense in $\mathbb{R}^k$. There are various ways of describing subspaces of $\mathbb{R}^k$ with this property. We choose to say that such a subspace E *acts minimally* on the k-dimensional torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, and we describe the corresponding sets Y_s as *minimal* cut and project sets. This simply expresses the fact that the natural linear $\mathbb{R}^d$ action on $\mathbb{T}^k$ defined by E (i.e. translation by elements of E, modulo $\mathbb{Z}^k$) is a minimal dynamical system (i.e. all orbits under this action are dense in $\mathbb{T}^k$). Some authors refer to subspaces E which act minimally on $\mathbb{R}^k$ as *totally irrational* subspaces. This terminology is justified by the fact that a totally irrational subspace is one which is not contained in any proper rational subspace of $\mathbb{R}^k$ (see Exercise 3.2.4). However, the confusion which often arises is that there are totally irrational subspaces, in dimensions greater than 2, which contain rational points.

Now, continuing our discussion of what we termed the degenerate behaviors, we may solve the problem of having empty cut and project sets by requiring that our subspace E act minimally on $\mathbb{R}^k$. However, in light of the final comment of the previous paragraph, this does not in general guarantee that the resulting cut and project set will have a trivial group of periods. Therefore, in what follows we will also sometimes impose as an additional condition that Y_s be *aperiodic*, which means that its group of periods is $\{0\}$. Note that in this case the notions of aperiodic and nonperiodic, which we usually distinguish from one another, coincide.

Finally, it would become a minor technical nuisance in many of our arguments below if restriction of the map π to $\mathbb{Z}^k$ were many to one. Therefore we adopt the conventional assumption that it is injective. This is the same as assuming that the internal space F_π does not contain a rational point. It is possible to relax this assumption, if necessary, but we will not do so.

In the last few paragraphs we have narrowed down the collection of what we consider to be nondegenerate cut and project sets which are not too badly behaved. Now there is some satisfaction in presenting the following the lemma.

Lemma 3.2.2. *If E acts minimally on $\mathbb{R}^k$ and if $\mathcal{W}$ is a relatively compact set with non-empty interior, then the cut and project set Y_s is a Delone set in E.*

If we think of the physical space E as being identified with $\mathbb{R}^d$, which we often do, then for point sets in $\mathbb{R}^d$ we have the hierarchy

$$\begin{aligned}&\{\text{lattices}\} \subseteq \{\text{uniformly discrete crystallographic point sets}\}\\ &\subseteq \{\text{cut and project sets}\} \overset{Lem 3.2.2}{\subseteq} \{\text{Delone sets}\},\end{aligned}$$

where the last inclusion is only valid under the hypotheses of Lemma 3.2.2. The proof of the second inclusion is Exercise 3.2.1 below.

Moving on, for the problems we are going to study, the s in the definition of Y_s plays only a minor role. Points s for which $\mathbb{Z}^k+s$ does (resp. does not) intersect the boundary of $\mathcal{S}$ are called *singular* (resp. *nonsingular*) points, and the corresponding sets Y_s are called *singular* (resp. *nonsingular*) cut and project sets. We have already seen the issues that arise when dealing with singular points, at the end of the proof of Theorem 2.3.3 about Sturmian words. For all practical purposes which we will encounter, there is little difference between singular and nonsingular cut and project sets. However, we can avoid special technical cases by only working with nonsingular s, and this is what we often choose to do. Furthermore, if E acts minimally then, since $E + \mathbb{Z}^k$ is dense in $\mathbb{R}^k$, the value of s is usually not important. For this reason in much of what follows we will suppress the dependence on s and write Y instead of Y_s.

EXERCISES

Exercise 3.2.1. *Prove that every uniformly discrete crystallographic point set can be obtained as a cut and project set.*

Exercise 3.2.2. *Give an example of a subspace E which acts minimally, and a cut and project set formed from E which is a Delone set and which has a nontrivial group of periods.*

Exercise 3.2.3. *Give an example of an aperiodic cut and project set formed from a subspace E which does not act minimally on $\mathbb{R}^k$.*

Exercise 3.2.4. *Suppose that E is a subspace of $\mathbb{R}^k$ which does not act minimally. Prove that E is contained in a proper subspace R of $\mathbb{R}^k$ with the property that $\mathbb{Z}^k \cap R$ is a lattice in R.*

Exercise 3.2.5. *Suppose that Y is a cut and project set formed from a physical space which acts minimally, and using a bounded window with non-empty interior. Prove that $Y - Y$ is also a cut and project set, and that it is a Delone set.*

3.3 Parameterizations and special windows

Let E be a d-dimensional subspace of $\mathbb{R}^k$, and let us assume that E can be written as

$$E = \{(x, L(x)) : x \in \mathbb{R}^d\}, \tag{3.3.1}$$

where $L : \mathbb{R}^d \to \mathbb{R}^{k-d}$ is a linear function. This can always be achieved by a relabelling of the standard basis vectors, so for simplicity we will only work with subspaces E which can be written this way. For each $1 \leq i \leq k - d$, we define the linear form $L_i : \mathbb{R}^d \to \mathbb{R}$ by

$$L_i(x) = L(x)_i = \sum_{j=1}^{d} \alpha_{ij} x_j, \tag{3.3.2}$$

and we use the points $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$ to parametrize the choice of E.

As a reference point, when allowing E to vary, we also make use of the fixed $(k-d)$-dimensional subspace F_ρ of $\mathbb{R}^k$ defined by

$$F_\rho = \{(0, \ldots, 0, y) : y \in \mathbb{R}^{k-d}\}, \tag{3.3.3}$$

and we let $\rho : \mathbb{R}^k \to E$ and $\rho^* : \mathbb{R}^k \to F_\rho$ be the projections onto E and F_ρ with respect to the decomposition $\mathbb{R}^k = E + F_\rho$ (note that E and F_ρ are complementary subspaces of $\mathbb{R}^k$). Our notational use of π and ρ is intended to be suggestive of the fact that F_π is the subspace which gives the *projection* defining Y (hence the letter π), while F_ρ is the subspace with which we *reference* E (hence the letter ρ). We write $\mathcal{W} = \mathcal{S} \cap F_\rho$, and for convenience we also refer to this set, in addition to $\mathcal{W}_\pi$, as the *window* defining Y. This slight ambiguity should not cause any confusion in the arguments below.

In our investigations of patterns in cut and project sets, we will focus on two special types of windows $\mathcal{W}$. The first is what we will call the *cubical* window, which we define as

$$\mathcal{W} = \left\{ \sum_{i=d+1}^{k} t_i e_i : 0 \le t_i < 1 \right\}.$$

The second is what is called the *canonical* window, which is given by

$$\mathcal{W} = \rho^* \left(\left\{ \sum_{i=1}^{k} t_i e_i : 0 \le t_i < 1 \right\} \right).$$

As we will see, the cubical window is a natural choice if we want to use Diophantine approximation properties of E to say something about the corresponding cut and project sets Y. The canonical window is also important as it arises in many well known constructions, such as the Penrose and Ammann-Beenker tilings.

Finally, bringing together all of our disclaimers and simplifying assumptions under one label, we say that Y is a *cubical* (resp. *canonical*) *cut and project set* if it is nonsingular, minimal, and aperiodic, and if $\mathcal{W}$ is a cubical (resp. canonical) window.

EXERCISES

Exercise 3.3.1. *Let w be a repetitive Sturmian word with slope α and intercept $\gamma \notin \alpha\mathbb{Z}+\mathbb{Z}$. Explain how w can be constructed using a cubical cut and project set. Conversely, explain why every such set corresponds to such a Sturmian word. Finally, what happens if $\gamma \in \alpha\mathbb{Z}+\mathbb{Z}$?*

3.4 Patches in cut and project sets

In analogy with the definition of 'subword of length n' for a bi-infinite word, we now consider 'patches of size r' in a cut and project set Y. It turns out that there is more than one reasonable choice for how to define patches of size r in Y. We will work with two definitions, moving back and forth between them.

Assume that we are given a bounded convex set $\Omega \subseteq E$ which contains a neighborhood of 0 in E. For $y \in Y$ and $r \geq 0$ define $P_1(y, r)$, the *type* 1 *patch* of size r at y, by

$$P_1(y,r) := \{y' \in Y : y' - y \in r\Omega\}.$$

Writing $\tilde{y}$ for the point in $\mathcal{S} \cap (\mathbb{Z}^k + s)$ with $\pi(\tilde{y}) = y$, we define $P_2(y, r)$, the *type* 2 *patch* of size r at y, by

$$P_2(y,r) := \{y' \in Y : \rho(\tilde{y'} - \tilde{y}) \in r\Omega\}.$$

Note that the point $\tilde{y}$ is uniquely determined by y because of our standing assumption that $\pi|_{\mathbb{Z}^k}$ is injective.

To rephrase the definitions, a type 1 patch consists of all points of Y in a certain neighborhood of y in E, while a type 2 patch consists of the projections of all points of $\mathcal{S} \cap \mathbb{Z}^k$ whose first d coordinates are in a certain neighborhood of the first d coordinates of $\tilde{y}$. Type 1 patches are more natural from the point of view of working within E, but the behavior of type 2 patches is more closely tied to the Diophantine properties of L.

Since the window $\mathcal{W}$ is assumed to be relatively compact, it follows that type 1 and 2 patches of size r at y differ at most within a constant neighborhood of the boundary of $y + r\Omega$. However, it turns out to be substantially easier from a technical point of view to work with type 2 patches, so our strategy will be to prove results for type 2 patches and then estimate the error when converting to type 1 patches.

For $i = 1$ or 2 and $y_1, y_2 \in Y$, we say that $P_i(y_1, r)$ and $P_i(y_2, r)$ are equivalent if

$$P_i(y_1, r) = P_i(y_2, r) + y_1 - y_2.$$

This defines an equivalence relation on the collection of type i patches of size r, and we denote the equivalence class of $P_i(y, r)$ by $\mathcal{P}_i(y, r)$.

In analogy with our study of bi-infinite words, for any cut and project set Y, and for $i = 1$ or 2, we define the *complexity function* $p_i : [0, \infty) \to \mathbb{N} \cup \{\infty\}$ of Y by setting $p_i(r)$ to be the number of equivalence classes of type i patches of size r in Y. It is not difficult to show that, if $\mathcal{W}$ satisfies the assumption from Lemma 3.2.2, then $p_i(r)$ is always finite. If in addition E acts minimally, then for any $r > 0$ and $i \in \{1, 2\}$, the value of $p_i(r)$ will be the same for any nonsingular cut and project set Y formed from E.

Next, we say that a cut and project set Y is *repetitive* if, for every patch $P_i(y, r)$ of size r in Y, there exists a constant $R > 0$ with the property that every ball of radius R in E contains a point $y' \in Y$ with the property that

$$P_i(y', r) \in \mathcal{P}_i(y, r).$$

The property of being repetitive does not depend on whether we are considering $i = 1$ or 2 and, not surprisingly, all nonsingular cut and project sets with windows satisfying the hypotheses of Lemma 3.2.2 are repetitive. Therefore, it is natural to speak about the *repetitivity function* $R_i : [0, \infty) \to [0, \infty)$ of a cut and project set. This is defined, for $i = 1$ or 2, by setting $R_i(r)$ to be the smallest real number with the property that, for every type i equivalence class $\mathcal{P}$ of patches of size r in Y, every ball of radius $R(r)$ in Y contains a point $y' \in Y$ with the property that

$$P_i(y', r) \in \mathcal{P}.$$

Of special significance is the case when $R_i(r)$ is bounded from above by a linear function, for large enough values of r. Therefore, we say that a repetitive cut and project set Y is *linearly repetitive*, which we abbreviate as *LR*, if there exists a $C > 0$ with the property that, for all $r \geq 1$, we have that $R_i(r) \leq Cr$.

EXERCISES

Exercise 3.4.1. *Prove that the property of being LR does not depend on whether we are considering type* 1 *or type* 2 *patches.*

Exercise 3.4.2. *Prove that the property of being LR does not depend on what convex patch shape* Ω *we use.*

3.5 Cut and project sets with rotational symmetry

In this section we discuss the problem of constructing cut and project sets with prescribed rotational symmetry. In keeping with convention, identify the *group of rotations* of $\mathbb{R}^k$ with the special orthogonal group $\mathrm{SO}_k(\mathbb{R})$, the group of $k \times k$ orthogonal matrices with determinant 1, which acts on $\mathbb{R}^k$ by left multiplication. We say that a point set $Y \in \mathbb{R}^k$ has *n-fold symmetry* if there is an element $A \in \mathrm{SO}_k(\mathbb{R})$ of order n which stabilizes Y (i.e. with the property that $AY = Y$). Furthermore, we will say that a rotation $A \in \mathrm{SO}_k(\mathbb{R})$ is an *irreducible rotation of order n* if $A^n = \mathrm{I}$ and if, for any $1 \leq m < n$ the only element of $\mathbb{R}^k$ which is fixed by A^m is $\{0\}$. If a point set $Y \subseteq \mathbb{R}^k$ is stabilized by an irreducible rotation of $\mathbb{R}^k$ of order n then we say that Y has has *irreducible n-fold symmetry*.

For the purposes of orientation, suppose first that Λ is a lattice in $\mathbb{R}^k$, and that there is an irreducible rotation $A \in \mathrm{SO}_k(\mathbb{R})$ of order n, which maps Λ into itself. Since A is a bijective, area preserving transformation of $\mathbb{R}^k$ to itself, it is not difficult to see that $A\Lambda = \Lambda$. By choosing a basis for Λ we may write it as $B\mathbb{Z}^k$, where $B \in \mathrm{GL}_k(\mathbb{R})$. Then we have that

$$A(B\mathbb{Z}^k) = B\mathbb{Z}^k \;\Rightarrow\; (B^{-1}AB)\mathbb{Z}^k = \mathbb{Z}^k,$$

from which it follows that $B^{-1}AB \in \mathrm{GL}_k(\mathbb{Z})$. Since similar matrices share the same characteristic polynomial, this shows that the characteristic polynomial of A is an element of $\mathbb{Z}[x]$.

Furthermore, since A is irreducible, all of its eigenvalues must be primitive nth roots of unity. If ζ_n is any primitive nth root of unity, then all primitive nth roots of unity are roots of the nth cyclotomic polynomial

$$\Phi_n(x) = \prod_{\substack{a=1 \\ (a,n)=1}}^{n} (x - \zeta_n^a),$$

which is an irreducible polynomial with integer coefficients. Since $\mathbb{Z}[x]$ is a unique factorization domain, the characteristic polynomial of the matrix A must be a power of $\Phi_n(x)$. This implies in particular that k must be divisible by $\varphi(n)$, where $\varphi(n)$ is the Euler phi function. Using a slight variant of this argument, we deduce the following classical version of the crystallographic restriction theorem.

Theorem 3.5.1. *A lattice in* 2 *or* 3 *dimensional Euclidean space can have n-fold symmetry only if $n = 1, 2, 3, 4,$ or* 6*.*

Proof. For irreducible rotations, the statement of the theorem follows from the observations of the previous paragraph, together with the following basic properties of the Euler phi function:

(i) The function φ is multiplicative. In other words, for any $m, n \in \mathbb{N}$ with $(m, n) = 1$, we have that $\varphi(mn) = \varphi(m)\varphi(n)$.

(ii) For any prime p and $a \in \mathbb{N}$, we have that $\varphi(p^a) = p^{a-1}(p - 1)$.

A simple calculation now shows that the only values of n for which $\varphi(n) \leq 3$ are $n = 1, 2, 3, 4,$ or 6 and, furthermore, that $\varphi(n) = 2$ for all of these values. Therefore if A is an irreducible rotation of order n which fixes a lattice in $\mathbb{R}^2$ or $\mathbb{R}^3$, then it must be the case that n takes one of these values.

For the general case, suppose that Λ is a lattice in $\mathbb{R}^3$, that $A \in \mathrm{SO}_3(\mathbb{R})$ satisfies $A\Lambda = \Lambda$, and that $A^n = \mathrm{I}$ (note that the two-dimensional case also follows from this level of generality). The matrix A is a root of the polynomial $f(x) = x^n - 1$, which implies that the minimal polynomial of A over $\mathbb{Z}[x]$ must divide $f(x)$. Writing $e(x) = \exp(2\pi i x)$, we have that

$$f(x) = \prod_{a=1}^{n} \left(x - e\left(\frac{a}{n}\right) \right) = \prod_{m|n} \Phi_m(x).$$

The right hand side of this equation is the factorization of f into irreducible elements over $\mathbb{Z}[x]$.

Now, by the Cayley-Hamilton Theorem, the minimal polynomial of A divides the characteristic polynomial of A, which has degree 3. All the roots of the characteristic polynomial must be nth roots of unity, although not necessarily primitive. Therefore its only possible divisors are $\Phi_m(x)$, where $m = 1, 2, 3, 4$, or 6, since in all other cases $\varphi(m) > 3$. By degree considerations there are now finitely many possibilities and, in all cases, the order of the resulting matrix A must be a divisor of 4 or 6, which completes the proof. □

It is an easy exercise to check that all of the symmetries permitted in the statement of this theorem are actually possible to realize, for lattices in $\mathbb{R}^2$.

The problem of determining which rotations of order n can stabilize lattices, in Euclidean spaces of varying dimensions, can be reduced to the analogous problem for irreducible rotations. We have seen that, if there exists a lattice $\Lambda \in \mathbb{R}^k$ with irreducible n-fold symmetry, then it is necessary that $\varphi(n)|k$. By using basic results from algebraic number theory we can also show, in a completely constructive manner, that this is a sufficient condition for the existence of such a lattice.

Theorem 3.5.2. *Choose $n \in \mathbb{N}$ and suppose that $\varphi(n)|k$. Then there is a lattice in $\mathbb{R}^k$ with irreducible n-fold symmetry.*

Proof. It is sufficient to verify the statement of the theorem when $k = \varphi(n)$. Then, for $k = k'\varphi(n)$, we can always write

$$\mathbb{R}^k = \mathbb{R}^{\varphi(n)} + \cdots + \mathbb{R}^{\varphi(n)} \ (k'\text{-times}),$$

and embed our solution for $\mathbb{R}^{\varphi(n)}$ into each of the components of this decomposition. For readers who are familiar with algebraic number theory, the entire proof can be summarized in the single observation that, for $n > 2$, the Minkowski embedding of the ring of integers of $\mathbb{Q}(\zeta_n)$ into $\mathbb{C}^{\varphi(n)/2}$ is a lattice, with irreducible n-fold symmetry given as (the image of) multiplication by ζ_n. For those who are less familiar with these ideas, or want to see the details of the calculation, we now explain further.

Let $k = \varphi(n)$ and, without loss of generality, assume that $n > 2$. Let ζ_n be a root of the polynomial $\Phi_n(x)$ from above and let $K = \mathbb{Q}(\zeta_n)$ be the cyclotomic field of nth roots of unity. Then we have that

$$[K : \mathbb{Q}] = \deg(\Phi_n) = \varphi(n).$$

Note that in our construction so far we have treated ζ_n as a purely algebraic object, the root of a polynomial with coefficients in some field, without talking about an algebraic completion or specifying which element of $\mathbb{C}$ the number ζ_n actually is. Now, it turns out that there are exactly $\varphi(n)$ different ways of embedding the field K into $\mathbb{C}$, which preserve the algebraic structure. The embeddings are determined by their values at ζ_n, which must be other roots of $\Phi_n(x)$. Therefore, they are the $\varphi(n)$ homomorphisms from K to $\mathbb{C}$ determined, for each $1 \leq a \leq n$ with $(a, n) = 1$, by the requirement that

$$\zeta_n \mapsto e\left(\frac{a}{n}\right).$$

These maps are all non-real complex embeddings, and the map determined by an integer a is the complex conjugate of the one determined by $n - a$. Therefore we can choose a collection of distinct, non-pairwise conjugate embeddings $\{\sigma_1, \ldots, \sigma_r\}$, with $r = \varphi(n)/2$.

The collection of algebraic integers which lie in K form a subring of K, which, in this instance, happens to be the ring $\mathbb{Z}[\zeta_n]$ (this is not obvious, but in our application we actually only need to know that this is a finite index subring of the ring of integers). Now consider the map $\sigma : \mathbb{Z}[\zeta_n] \to \mathbb{C}^{\varphi(n)/2}$ defined by

$$\sigma(\alpha) = (\sigma_1(\alpha), \ldots, \sigma_r(\alpha)). \tag{3.5.3}$$

The map σ is a homomorphism with trivial kernel, and it follows that its image is a rank k subgroup of $\mathbb{C}^{\varphi(n)/2}$. Furthermore, we leave it as an exercise to check that the image of σ is a discrete subgroup, therefore (identifying $\mathbb{C}^{\varphi(n)/2}$ with $\mathbb{R}^k$) it is a lattice in $\mathbb{R}^k$.

Finally, write $\Lambda = \sigma(\mathbb{Z}[\zeta_n])$ and consider the action of the cyclic group

$$C_n = \langle \tau : \tau^n = 1 \rangle$$

on Λ defined by

$$\tau(\sigma(\alpha)) = \sigma(\zeta_n \alpha).$$

This action is an invertible linear transformation of Λ to itself. For each $1 \leq i \leq r$, choose an integer a_i such that $\sigma_i(\zeta_n) = \zeta_n^{a_i}$. Then, again identifying $\mathbb{C}^{\varphi(n)/2}$ with $\mathbb{R}^k$, we find that the action of τ on $\mathbb{R}^k$ is realized as left multiplication by the block diagonal matrix

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix} \in \mathrm{GL}_k(\mathbb{R}),$$

with

$$A_i = \begin{pmatrix} \mathrm{Re}(\zeta_n^{a_i}) & -\mathrm{Im}(\zeta_n^{a_i}) \\ \mathrm{Im}(\zeta_n^{a_i}) & \mathrm{Re}(\zeta_n^{a_i}) \end{pmatrix}, \quad 1 \le i \le r.$$

Since each of the matrices A_i is an orthogonal matrix with determinant 1, we have that $A \in \mathrm{SO}_k(\mathbb{R})$. Finally, for each i, the eigenvalues of A_i are $\zeta_n^{a_i}$ and $\overline{\zeta_n^{a_i}}$, so the characteristic polynomial of A is the irreducible polynomial $\Phi_n(x) \in \mathbb{Z}[x]$. It follows that A is an irreducible rotation of order n. $\square$

The lattices from the previous proof can also be used to construct cut and project sets with rotational symmetry. Strictly speaking, a cut and project set Y lives in a d-dimensional subspace E of the total space $\mathbb{R}^k$. From the point of view of working in E, the definition of rotational symmetry depends on our choice of basis for E. To see what we mean, suppose that we have two different $\mathbb{R}$-bases for E, say $\{b_1, \ldots, b_d\}$, and $\{b'_1, \ldots, b'_d\}$. Let $B \in \mathrm{GL}_d(\mathbb{R})$ be the change of basis matrix which takes coordinates in the first basis to coordinates in the second. Suppose that Y is a point set in E, expressed with coordinates in the first basis, and let $Y' = BY$ be the same point set, expressed with coordinates in the second basis. If $A \in \mathrm{SO}_d(\mathbb{R})$ stabilizes Y then we have that

$$A(B^{-1}Y') = B^{-1}Y' \;\Rightarrow\; (BAB^{-1})Y' = Y'.$$

Now the matrix BAB^{-1} still has determinant 1, but it may no longer be orthogonal. However, this at least shows that the property of being stabilized by an element in the conjugacy class of $\mathrm{SO}_d(\mathbb{R})$ is not dependent on the choice of basis we are using. For clarity, it may be useful to keep this in mind during the following discussion.

Now, with the caveat of the previous paragraph, we are prepared to speak about cut and project sets with rotational symmetry. Actually, the example we have already seen in the proof of Theorem 3.5.2 is essentially all we need to complete our exposition. Choose $n > 2$, let the map σ be defined as in the proof, and suppose that σ_1 is the map determined by $\sigma_1(\zeta_n) = \zeta_n$. Let $\Gamma = \sigma(\mathbb{Z}[\zeta_n]) \subseteq \mathbb{R}^k$, with $k = \varphi(n)$. As already mentioned, Γ is a lattice in $\mathbb{R}^k$ which we may identify, after a linear change of variables, with the standard integer lattice. We will not actually make this change of variables because, although it would be straightforward to do so, it would only complicate notation. We take the physical space to be the space

$$E = \{(z_1, 0, \ldots, 0) : z_1 \in \mathbb{C}\} \cong \mathbb{R}^2,$$

and the internal space F_π to be the space orthogonal to E. Then we take the window to be the set

$$\mathcal{W}_\pi = \{(0, z_2, \ldots, z_r) : z_i \in \mathbb{C}, |z_i| \leq 1\},$$

which of course we identify with a subset of $F_\pi \cong \mathbb{R}^{k-2}$. With π and π^* having the usual meanings, we define a cut and project set $Y \subseteq E$ by

$$Y = \pi(\{\gamma \in \Gamma : \pi^*(\gamma) \in \mathcal{W}_\pi\}).$$

The set $\pi(\mathbb{Z}[\zeta_n])$ is dense in E. Therefore, since $\mathcal{W}_\pi$ is bounded and has non-empty interior, it is not difficult to show that Y is a Delone set in E. Suppose that $y \in Y$ is given by $\sigma_1(\alpha)$, for some $\alpha \in \mathbb{Z}[\zeta_n]$ with

$$\pi^*(\sigma(\alpha)) \in \mathcal{W}_\pi.$$

Then it follows from the construction $\mathcal{W}_\pi$ that

$$\pi^*(\sigma(\zeta_n \alpha)) \in \mathcal{W}_\pi,$$

allowing us to conclude that

$$\sigma_1(\zeta_n \alpha) = \zeta_n y \in Y.$$

Therefore Y has n-fold rotational symmetry.

EXERCISES

Exercise 3.5.1. *Give an example of a lattice $\Lambda \subseteq \mathbb{R}^6$ with 15-fold rotational symmetry.*

Exercise 3.5.2. *Let* $Y \subseteq \mathbb{R}^k$ *be a crystallographic point set with group of periods* Λ*. Prove that if* A *is a rotation of* $\mathbb{R}^k$ *which maps* Y *into itself, then it must satisfy* $A\Lambda = \Lambda$*.*

Exercise 3.5.3. *Verify the claim that the image of the map* σ *from* (3.5.3) *is a lattice in* $\mathbb{R}^k \cong \mathbb{C}^{\varphi(n)/2}$*.*

Exercise 3.5.4. *Verify the claim in the construction at the end of this section, that* $\pi(\mathbb{Z}[\zeta_n])$ *is dense in* E*.*

Chapter 4

Complexity and repetitivity for cut and project sets

4.1 Dynamical coding of patches

In the proof of Theorem 2.3.1, we saw how every word of length n in the language of a Sturmian word corresponds naturally to a subinterval of the circle $\mathbb{R}/\mathbb{Z}$ (which we identify with the half open unit interval). If the slope of the Sturmian word is α, then this subinterval is a component interval of the partition of $\mathbb{R}/\mathbb{Z}$ obtained by removing the first $n+1$ points in the orbit of 0 under rotation by α. Whether or not a subword $w_m \dots w_{m+n-1}$ of w will be equal to the word we have selected, is determined by whether or not $\{m\alpha+\gamma\}$ falls into the distinguished interval. Now we will see how this carries over, at least in principle, to cut and project sets in higher dimensions.

Recalling the conventions set out in the previous chapter, $E \subseteq \mathbb{R}^k$ is the physical space and F_π is the internal space, E is parametrized by linear forms as in (3.3.1) and (3.3.2), and the F_ρ is the reference subspace given by (3.3.3). We make the standard assumption that the window is a relatively compact subset of F_π whose closure is the closure of its interior, and we identify the window with its image $\mathcal{W} \subseteq F_\rho$ under the map ρ^*, and we also assume that $\pi|_{\mathbb{Z}^k}$ is injective. Furthermore, for any $y \in Y$, we let $\tilde{y} \in \mathbb{Z}^k$ be the (unique) point given by

$$\tilde{y} = \mathbb{Z}^k \cap \pi^{-1}(y).$$

There is a natural action of $\mathbb{Z}^k$ on F_ρ, given by

$$n.w = \rho^*(n) + w = w + (0, n_2 - L(n_1)),$$

for $n = (n_1, n_2) \in \mathbb{Z}^k = \mathbb{Z}^d \times \mathbb{Z}^{k-d}$ and $w \in F_\rho$. For each $r \geq 0$ we define the *r-singular points of type* 1 by

$$\mathrm{sing}_1(r) := \mathcal{W} \cap \left((-\pi^{-1}(r\Omega) \cap \mathbb{Z}^k).\partial\mathcal{W}\right),$$

and, similarly, the *r-singular points of type* 2 by

$$\mathrm{sing}_2(r) := \mathcal{W} \cap \left((-\rho^{-1}(r\Omega) \cap \mathbb{Z}^k).\partial\mathcal{W}\right).$$

Then, for $i = 1$ or 2 we define the *r-nonsingular points of type i* by

$$\mathrm{nsing}_i(r) := \mathcal{W} \setminus \mathrm{sing}_i(r).$$

Guided by our study of Sturmian words, we might expect that there should be a connection between equivalence classes of patches and connected components of the collections of nonsingular points. In this direction, we begin with the following observation, which is motivated by work of Antoine Julien [19].

Lemma 4.1.1. *Let $i = 1$ or 2, suppose that E acts minimally on $\mathbb{T}^k$, and suppose that $Y = Y_s$ is nonsingular. Suppose that U is any connected component of* $\mathrm{nsing}_i(r)$. *Then for any points $y, y' \in Y$,*

$$\text{if} \quad \rho^*(\tilde{y}), \rho^*(\tilde{y'}) \in U \quad \text{then} \quad \mathcal{P}_i(y, r) = \mathcal{P}_i(y', r). \tag{4.1.2}$$

Proof. For each $y \in Y$, let $y^* = \rho^*(\tilde{y}) \in \mathcal{W}$. The point y^* determines the pattern around y, as follows. Each point $y' \in Y$ lifts to a point $\tilde{y}' = \tilde{y} + n$. But such a point is in $\mathcal{S}$ if and only if $\pi^*(\tilde{y}') = n.y^*$ lies in $\mathcal{W}$. As we vary y^*, the pattern around y can only change when some $n.y^*$ passes through $\partial\mathcal{W}$, that is when y^* passes from one connected component of $\mathrm{nsing}_i(r)$ to another. The only difference between $i = 1$ and $i = 2$ is the set of n's being considered. In both cases, each connected component of $\mathrm{nsing}_i(r)$ corresponds to a single equivalence class of patches. □

A word of caution, the converse of Lemma 4.1.1 is not true, in general. In other words, although each connected component determines an equivalence class of patches via (4.1.2), it is possible that two or more components could correspond to the same patch. For convex windows the converse of Lemma 4.1.1 is true when $k-d=1$ (which is the reason why this issue did not arise when dealing with Sturmian words), but it is still not true in general when $k-d>1$. However if $\mathcal{W}$ is a parallelotope generated by integer vectors then, for type 2 patches, things are much simpler.

Lemma 4.1.3. *Suppose that E acts minimally on $\mathbb{T}^k$, that Y is non-singular, and that $\mathcal{W}$ is a parallelotope generated by integer vectors. Then for every equivalence class $\mathcal{P}_2 = \mathcal{P}_2(y,r)$ of type 2 patches, there is a unique connected component U of $\mathrm{nsing}_2(r)$ with the property that, for any $y' \in Y$,*

$$\mathcal{P}_2(y',r) = \mathcal{P}_2(y,r) \quad \textit{if and only if} \quad \rho^*(\tilde{y'}) \in U.$$

Proof. We follow the proof of Lemma 4.1.1. Suppose that y_1 and $y_2 \in Y$, and that $P_2(y_1,r)$ is equivalent to $P_2(y_2,r)$. Imagine varying y^* in a straight line from y_1^* to y_2^*. In moving y^* from one connected component to another, the patch $P_2(y,r)$ gains and/or loses points whenever y^* crosses from one component to another. We will show that none of the points of $P_2(y_1,r)$ may be removed in going from y_1^* to y_2^*, and that no points may be added without removing other points. Combining these observations, no points can be added or removed, so y_1^* and y_2^* must lie in the same component.

To see that no points may be removed, note that $\mathcal{W}$ is convex. Thus, for each n for which $\pi(\tilde{y}_i+n)$ is in $P_2(y_i,r)$, the set of points y^* satisfying $n.y^* \in \mathcal{W}$ is convex. Since $n.y_1^*$ and $n.y_2^*$ are in $\mathcal{W}$, all points on the line segment connecting them must also be in $\mathcal{W}$. Thus all points y^* on the line segment correspond to patches that contain a translate of $P_2(y_i,r)$.

Next notice that, since $\mathcal{W}$ is a parallelotope generated by integer vectors, after possibly modifying a subset of its boundary it is a (strict) fundamental domain for a sublattice of $\mathbb{Z}^k \cap F_\rho \cong \mathbb{Z}^{k-d}$, of some index I. This implies that for each $n_1 \in \mathbb{Z}^d$, and each $\tilde{y}$, there are exactly I points $n_2 \in \mathbb{Z}^{k-d}$ such that $\tilde{y}+(n_1,n_2) \in \mathcal{S}$. In other words, as we cross a boundary between connected components, a point is removed from $P_2(y,r)$ for each point added. We have already shown that no points can be removed, so no points can be added. □

The first part of the proof of Lemma 4.1.3 applies equally well to the more geometric type 1 patches. If the patches associated to y_1^* and y_2^* are equivalent, then any patch associated to $ty_1^*+(1-t)y_2^*$ must contain all the points of $P_1(y_i,r)$. However, the final part of the argument does not work.

Since $\pi(\tilde{y} + (n_1, n_2))$ depends on both n_1 and n_2, some points associated with n_1 might have images in $r\Omega + y$, while others might not. As we change the (fixed number) of points associated with n_1, points can jump in and out of a patch, so in going from y_1^* to y_2^*, we could gain a point, then have it leave, leading to the same patch that we started with.

In general, even for parallelotope windows, we do not know how to guarantee that there is only a bounded number of connected components corresponding to each equivalence class of type 1 patch. However, if we impose additional Diophantine approximation hypotheses on the subspace E, then we can limit this bad behavior. This is our next topic of discussion.

Let us suppose that $\mathcal{W} \subseteq F_\rho$ is a parallelotope generated by $k-d$ linearly independent vectors $w_1, \ldots, w_{k-d} \in F_\rho \cap \mathbb{Z}^k$. Define an integer linear transformation $B : F_\rho \to F_\rho$ such that $\mathcal{W}$ is the image under B of the unit hypercube in F_ρ (i.e. the cubical window). Applying the linear transformation B^{-1} converts the window to the unit hypercube in F_ρ, converts the integer lattice $\mathbb{Z}^k$ to $\mathbb{Z}^d \times \Lambda$, where Λ is a finite-index extension of $\mathbb{Z}^{k-d}$, and converts E to the graph of the linear transformation $L' = B^{-1}L$. Let $\{\beta_{ij}\}$ be the matrix elements of L' and for each $1 \leq i \leq k-d$ let $L'_i : \mathbb{R}^d \to \mathbb{R}$ be the linear form defined by

$$L'_i(x) = \sum_{j=1}^{d} \beta_{ij} x_j.$$

We emphasize that β depends on both $\mathcal{W}$ (i.e. on B) and on α.

Now we demonstrate how assumptions on the Diophantine approximation properties of the numbers β can be used to control the complexity of the sets $\mathrm{nsing}_1(r)$. At this point the reader may want to review some of the definitions from Section 1.4.

Lemma 4.1.4. *With notation as above, suppose that $(\beta_{ij})_{j=1}^d \in \mathcal{B}_{d,1}$ for each $1 \leq i \leq k-d$. Then there exist constants $c_1, c_2 > 0$ such that, for all $r > 0$, every element connected component of* $\mathrm{nsing}_1(r)$ *is a union of at most c_1 connected components of* $\mathrm{nsing}_2(r + c_2)$.

Proof. First, from the definitions of type 1 and type 2 patches (as well as the convexity of Ω) we can choose $c_2 > 0$ so that, for all $y \in Y$ and all sufficiently large r,

$$P_2(y, r - c_2) \subseteq P_1(y, r) \subseteq P_2(y, r + c_2). \tag{4.1.5}$$

Actually this is not completely obvious, but it follows from standard facts about Hausdorff distance and dilations of convex sets.

Therefore, each connected component of $\mathrm{nsing}_1(r)$ is contained in a single connected component of $\mathrm{nsing}_2(r-c_2)$, and it is a union of connected components of $\mathrm{nsing}_2(r+c_2)$. We will show that, under our Diophantine hypotheses, we can choose c_1 so that each component of $\mathrm{nsing}_2(r-c_2)$ is the union of at most c_1 components of $\mathrm{nsing}_2(r+c_2)$.

Let Ω' be the projection of Ω on $\mathbb{R}^d$ along F_ρ. That is, Ω' is the set of first d coordinates of the points in Ω. Let R_1 and R_2 be closed hypercubes in $\mathbb{R}^d$, each containing a neighborhood of 0, and satisfying $R_1 \subseteq \Omega' \subseteq R_2$. Also, for $1 \le i \le k-d$, let H_i be the hyperplane in F_ρ orthogonal to e_{d+i}. Let

$$\mathrm{sing}_2'(r) := B^{-1}\mathrm{sing}_2(r) \quad \text{and} \quad \mathrm{nsing}_2'(r) := B^{-1}\mathrm{nsing}_2(r).$$

The set $\mathrm{sing}_2'(r)$ is composed of the intersection of the unit hypercube with translates of the faces of the hypercube by points of the form $(-n_1, -\lambda) \in \mathbb{Z}^d \times \Lambda$ with $n_1 \in r\Omega'$. The action of $\{0\} \times \mathbb{Z}^{k-d}$ maps the faces of the unit hypercube into the set

$$\bigcup (H_i + \mathbb{Z}e_{d+i}),$$

so, for the purposes of studying $\mathrm{sing}_2'(r)$, we can restrict our attention to the action of $\mathbb{Z}^d \times (\Lambda/\mathbb{Z}^{k-d})$.

For any $r > 0$, the set $\mathrm{sing}_2'(r)$ is therefore given by

$$[0,1]^{k-d} \cap \left(\bigcup_{n \in r\Omega' \cap \mathbb{Z}^d} \bigcup_{i=1}^{k-d} (H_i + L'(n) + \Lambda) \right).$$

When considering translates of H_i, all that matters is the $(d+i)$th coordinate of the offset. The $(d+i)$th coordinates of Λ form a group of the form $m_i^{-1}\mathbb{Z}$, for some $m_i \in \mathbb{Z}$, so we have that

$$H_i + L'(n) + \Lambda = H_i + (L_i'(n) + m_i^{-1}\mathbb{Z})e_{d+i}.$$

The assumption that $(\beta_{ij})_{j=1}^d \in \mathcal{B}_{d,1}$ implies that $(m_i\beta_{ij})_{j=1}^d \in \mathcal{B}_{d,1}$ for each i, and it follows from Theorem 1.4.2 that there is a constant $c_3 > 0$ such that, for each $1 \le i \le k-d$, the set

$$\{L_i'(n) \bmod m_i^{-1} : n \in rR_1 \cap \mathbb{Z}^d\}$$

is c_3/r^d-dense in $\mathbb{R}/m_i^{-1}\mathbb{Z}$. From this we conclude that if U is any connected component of $\mathrm{nsing}_2'(r)$ then U is a rectangle of the form

$$\{t \in W : \ell_i < t_i < r_i\},$$

with

$$r_i - \ell_i \leq \frac{c_3}{m_i r^d} \text{ for each } 1 \leq i \leq k-d.$$

Now observe that the number of connected components of $\mathrm{nsing}_2'(r+2c_2)$ which intersect U is equal to

$$\prod_{i=1}^{k-d}\left(1 + \#\{n \in ((r+2c_2)\Omega' \setminus r\Omega') \cap \mathbb{Z}^d : L_i'(n) \in (\ell_i, r_i) \bmod m_i^{-1}\}\right).$$

This is bounded above by

$$\prod_{i=1}^{k-d}\left(1 + \#\{n \in (r+2c_2)R_2 \cap \mathbb{Z}^d : L_i'(n) \in (\ell_i, r_i) \bmod m_i^{-1}\}\right),$$

and, again using our hypotheses on β, we see that the final quantity is bounded above by a constant $c_1 > 0$. We have shown that every connected component of $\mathrm{nsing}_2'(r)$ is a union of at most c_1 connected components of $\mathrm{nsing}_2'(r+2c_2)$. After applying the linear map B this, together with the observations in the first paragraph, completes the proof of the lemma. □

EXERCISES

Exercise 4.1.1. *Give an example to show that the statement of Lemma 4.1.3 is not true, in general, for type* 1 *patches.*

Exercise 4.1.2 (Open problem)**.** *Give an example of a cut and project set satisfying the hypotheses of Lemma 4.1.3, but for which there is no uniform bound on the number of connected components corresponding to each equivalence class of type* 1 *patches.*

Exercise 4.1.3. *Verify that there is a constant* $c_2 > 0$ *for which* (4.1.5) *holds.*

4.2 Linear repetitivity, cubical case

In this section we focus on cubical cut and project sets. Recall that these sets are defined to be nonsingular, minimal, and aperiodic, and they are formed using the cubical window

$$\mathcal{W} = \left\{ \sum_{i=d+1}^{k} t_i e_i : 0 \leq t_i < 1 \right\}. \tag{4.2.1}$$

Since we are discussing repetitivity, the role of the parameter s in the definition of the cut and project set is irrelevant. Therefore we will suppress the notational dependence on s as much as possible. The main theorem we would like to present, which is an extension of our results about Sturmian words (see Corollary 2.4.2), is the following classification of the collection of LR cubical cut and project sets.

Theorem 4.2.2. *A k to d cubical cut and project set defined by linear forms $\{L_i\}_{i=1}^{k-d}$ is LR if and only if*

(LR1) The sum of the ranks of the kernels of the maps $\mathcal{L}_i : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ defined by

$$\mathcal{L}_i(n) = L_i(n) \bmod 1$$

is equal to $d(k-d-1)$, and

(LR2) Each L_i is relatively badly approximable.

In the statement of the theorem, condition (LR1) is necessary and sufficient for Y to have minimal patch complexity. Condition (LR2) is clearly a Diophantine approximation condition, which places a restriction on how well the subspace defining Y can be approximated by rationals. Note that in the special case when $k-d=1$, condition (LR1) is automatically satisfied, and condition (LR2) requires the linear form defining Y to be badly approximable in the usual sense. This immediately implies the following corollary, which is a direct extension of Corollary 2.4.2 for Sturmian words.

Corollary 4.2.3. *A k to $k-1$ cubical cut and project set defined by a linear form L is LR if and only if L is badly approximable.*

Before we begin the proof of Theorem 4.2.2, let us tidy up a few more technical points. First of all, if Y is LR with respect to one convex patch shape Ω, then it is LR with respect to all convex patch shapes (see Exercise 3.4.2). The precise shape Ω which we will use will be specified later in the proof, but until then everything we say will apply to any fixed choice of such a shape. Secondly, by Exercise 3.4.1, it does not matter in our proof whether we consider type 1 or type 2 patches. Therefore, since we have Lemma 4.1.3 at our disposal, we will be in a much better position if we choose to work with type 2 patches. Of course, this is what we do and, for convenience of notation, we suppress the corresponding subscript 2's which are attached to all related objects.

Proof of Theorem 4.2.2. If Y is LR then there must exist a constant $C > 0$ with the property that $p(r)$ is bounded above by Cr^d, for all $r > 0$. For the first part of the proof of Theorem 4.2.2 we will show that condition (LR1) is necessary and sufficient for a bound of this type to hold.

For each $1 \leq i \leq k-d$, let $S_i \leqslant \mathbb{Z}^d$ denote the kernel of the map $\mathcal{L}_i$, and let r_i be the rank of S_i. Furthermore, for each subset $I \subseteq \{1, \ldots, k-d\}$ let

$$S_I = \bigcap_{i \in I} S_i,$$

and let r_I be the rank of S_I. For convenience, set $S_\emptyset = \mathbb{Z}^d$ and $r_\emptyset = d$. For any pair $I, J \subseteq \{1, \ldots, k-d\}$, the sum set $S_I + S_J$ is a subgroup of $\mathbb{Z}^d$, and it therefore has rank at most d. On the other hand we have that

$$\mathrm{rk}(S_I + S_J) = \mathrm{rk}(S_I) + \mathrm{rk}(S_J) - \mathrm{rk}(S_I \cap S_J),$$

which gives the inequality

$$r_I + r_J \leq d + r_{I \cup J}. \tag{4.2.4}$$

As one application of this inequality we see immediately that

$$\begin{aligned} r_1 + r_2 + \cdots + r_{k-d} &\leq d + r_{12} + r_3 + \cdots + r_{k-d} \\ &\leq 2d + r_{123} + r_4 + \cdots + r_{k-d} \\ &\vdots \\ &\leq d(k-d-1) + r_{12\ldots(k-d)} \end{aligned}$$

$$= d(k-d-1). \tag{4.2.5}$$

The last equality here uses the assumption that Y is aperiodic.

From Lemma 4.1.3, we know that $p(r)$ is equal to the number of connected components of $\mathrm{nsing}(r)$. Let the map $\mathcal{C} : \mathbb{Z}^{d(k-d)} \to \mathcal{W}$ be defined by

$$\mathcal{C}(n^{(1)}, \ldots, n^{(k-d)}) = (\{L_1(n^{(1)})\}, \ldots, \{L_{k-d}(n^{(k-d)})\}),$$

for $n^{(1)}, \ldots, n^{(k-d)} \in \mathbb{Z}^d$. Identify $\mathbb{Z}^d$ with the set $\mathcal{Z} = \mathbb{Z}^k \cap \langle e_1, \ldots, e_d\rangle_{\mathbb{R}}$, and for each $r > 0$ let $\mathcal{Z}_r \subseteq \mathbb{Z}^d$ be defined by

$$\mathcal{Z}_r = -\rho^{-1}(r\Omega) \cap \mathcal{Z}.$$

Since our window $\mathcal{W}$ is a fundamental domain for the integer lattice in F_ρ, there is a one to one correspondence between points of Y and elements of $\mathcal{Z}$. This correspondence is given explicitly by mapping a point $y \in Y$ to the vector in $\mathcal{Z}$ given by the first d coordinates of $\tilde{y}$. Also, notice that if $n \in \mathbb{Z}^k$ and $-n.0 \in \mathcal{W}$, then it follows that

$$-n.0 = (\{L_1(n_1, \ldots, n_d)\}, \ldots, \{L_{k-d}(n_1, \ldots, n_d)\}).$$

These observations together imply that the collection of all vertices of connected components of $\mathrm{nsing}(r)$ is precisely the set $\mathcal{C}(\mathcal{Z}_r^{k-d})$, which in turn implies that

$$p(r) \asymp \#\mathcal{C}(\mathcal{Z}_r^{k-d}).$$

The values of the function $\mathcal{C}$ define a natural $\mathbb{Z}^{d(k-d)}$ action on $\mathcal{W}$. Therefore we may regard the set $\mathcal{C}(\mathbb{Z}^{d(k-d)})$ as a group, isomorphic to

$$\mathbb{Z}^{d(k-d)}/\ker(\mathcal{C}) \cong \mathbb{Z}^d/S_1 \oplus \cdots \oplus \mathbb{Z}^d/S_{k-d}.$$

If (LR1) holds then we have that

$$\mathrm{rk}(\mathcal{C}(\mathbb{Z}^{d(k-d)})) = \ d(k-d) - \sum_{i=1}^{k-d} r_i = \ d,$$

and from this it follows that

$$\#\mathcal{C}(\mathcal{Z}_r^{k-d}) \asymp r^d.$$

On the other hand, if (LR1) does not hold then by (4.2.5) we have that

$$\mathrm{rk}(\mathcal{C}(\mathbb{Z}^{d(k-d)})) > d,$$

which implies that

$$\#\mathcal{C}(\mathcal{Z}_r^{k-d}) \gg r^{d+1}.$$

We conclude that $p(r) \ll r^d$ if and only if condition (LR1) holds, so (LR1) is a necessary condition for linear repetitivity.

Next we assume that (LR1) holds and we prove that, under this assumption, condition (LR2) is necessary and sufficient in order for Y to be LR. First of all, suppose that I and J were disjoint, nonempty subsets of $\{1, \ldots, k-d\}$ for which

$$r_I + r_J < d + r_{I\cup J}.$$

Then, by the same argument used in (4.2.5), we would have that

$$\sum_{i=1}^{k-d} r_i \ \leq \ d(k-d-3) + r_{(I\cup J)^c} + r_I + r_J \ < \ d(k-d-1).$$

This clearly contradicts (LR1). Therefore if (LR1) holds then, by (4.2.4), we have that

$$r_I + r_J = d + r_{I\cup J},$$

whenever I and J are disjoint and nonempty.

For each $1 \leq i \leq k-d$, define $J_i = \{1, \ldots, k-d\} \setminus \{i\}$, and let $\Lambda_i = S_{J_i}$. Write $m_i = r_{J_i}$ for the rank of Λ_i. Then, by what was established in the previous paragraph, we have that

$$m_i + r_i = d.$$

If n is any nonzero vector in Λ_i, then n is in S_j for all $j \neq i$. Since Y is aperiodic, this means that $n \notin S_i$, which gives that

$$\mathrm{rk}(\Lambda_i + S_i) = m_i + r_i - \mathrm{rk}(\Lambda_i \cap S_i) = d.$$

Furthermore, for any $j \neq i$, the fact that $\Lambda_j \subseteq S_i$ implies that $\Lambda_j \cap \Lambda_i = \{0\}$, so

$$\mathrm{rk}(\Lambda_1 + \cdots + \Lambda_{k-d}) = \sum_{i=1}^{k-d} \mathrm{rk}(\Lambda_i) = \sum_{i=1}^{k-d} (d - r_i) = d.$$

For each i, let $F_i \subseteq \mathbb{Z}^d$ be a complete set of coset representatives for $\mathbb{Z}^d/(\Lambda_i + S_i)$. Also, write $\Lambda = \Lambda_1 + \cdots + \Lambda_{k-d}$, and let $F \subseteq \mathbb{Z}^d$ be a complete set of representatives for $\mathbb{Z}^d/\Lambda$. What we have shown so far implies that all of the sets $F_1, \ldots, F_{k-d}$, and F are finite.

Again thinking of $\mathbb{Z}^d$ as being identified with the set $\mathcal{Z}$, let

$$\mathcal{Z}_{r,\Lambda} = \mathcal{Z}_r \cap \Lambda, \quad \mathcal{Z}_{r,\Lambda_i} = \mathcal{Z}_r \cap \Lambda_i, \quad \text{and} \quad \mathcal{Z}_{r,S_i} = \mathcal{Z}_r \cap S_i.$$

For each i, choose a basis $\{v_j^{(i)}\}_{j=1}^{m_i}$ for Λ_i, and define

$$\Omega_i' = \left\{ \sum_{j=1}^{m_i} t_i v_j^{(i)} : -1/2 \le t_i < 1/2 \right\},$$

and

$$\Omega' = \Omega_1' + \cdots + \Omega_{k-d}',$$

so that Ω' is a fundamental domain for $\mathbb{R}^d/\Lambda$. We now specify Ω to be the subset of points in E whose first d coordinates lie in Ω'. In other words,

$$\Omega = E \cap \rho^{-1}(\Omega').$$

Notice that every $n \in \Lambda$ has a unique representation of the form

$$n = \sum_{i=1}^{k-d} \sum_{j=1}^{m_i} a_{ij} v_j^{(i)}, \quad a_{ij} \in \mathbb{Z}.$$

Using this representation, we have that

$$\mathcal{L}(n) = \mathcal{C}((n^{(i)})_{i=1}^{m_i}),$$

where, for each i, the vector $n^{(i)} \in \mathbb{Z}^d$ is given by

$$n^{(i)} = \sum_{j=1}^{m_i} a_{ij} v_j^{(i)}.$$

This gives a one to one correspondence between elements of $\mathcal{L}(\Lambda)$ and elements of the set

$$\mathcal{C}(\Lambda_1 \times \cdots \times \Lambda_{k-d}) = \mathcal{L}_1(\Lambda_1) \times \cdots \times \mathcal{L}_{k-d}(\Lambda_{k-d}).$$

We will combine this observation with the facts that

$$\mathcal{L}(\mathbb{Z}^d) = \mathcal{L}(\Lambda + F)$$

and

$$\mathcal{C}(\mathbb{Z}^{d(k-d)}) = \mathcal{C}((\Lambda_1 + F_1) \times \cdots \times (\Lambda_{k-d} + F_{k-d})),$$

in order to study the spacings between points of the sets $\mathcal{L}(\mathcal{Z}_r)$ and $\mathcal{C}(\mathcal{Z}_r^{k-d})$.

First of all, it is clear that

$$\mathcal{L}(\mathcal{Z}_r) \supseteq \mathcal{L}_1(\mathcal{Z}_{r,\Lambda_1}) \times \cdots \times \mathcal{L}_{k-d}(\mathcal{Z}_{r,\Lambda_{k-d}}), \tag{4.2.6}$$

and that

$$\mathcal{C}(\mathcal{Z}_r^{k-d}) \supseteq \mathcal{L}_1(\mathcal{Z}_{r,\Lambda_1}) \times \cdots \times \mathcal{L}_{k-d}(\mathcal{Z}_{r,\Lambda_{k-d}}). \tag{4.2.7}$$

Since all of the sets $F_1, \ldots, F_{k-d}$, and F are finite, there is a constant $\kappa > 0$ with the property that, for all sufficiently large r,

$$\begin{aligned} \mathcal{Z}_r &\subseteq \mathcal{Z}_{r+\kappa,\Lambda} + F, \quad \text{and} \\ \mathcal{Z}_r &\subseteq \mathcal{Z}_{r+\kappa,\Lambda_i} + \mathcal{Z}_{r+\kappa,S_i} + F_i, \end{aligned}$$

for each $1 \leq i \leq k-d$. For the second inclusion here we are using the definition of Ω and the fact that $\Lambda_j \subseteq S_i$ for all $j \neq i$. These inclusions imply that

$$\begin{aligned} \mathcal{L}(\mathcal{Z}_r) &\subseteq \mathcal{L}(\mathcal{Z}_{r+\kappa,\Lambda}) + \mathcal{L}(F) \\ &\subseteq \mathcal{L}_1(\mathcal{Z}_{r+\kappa,\Lambda_1} + F) \times \cdots \times \mathcal{L}_{k-d}(\mathcal{Z}_{r+\kappa,\Lambda_{k-d}} + F), \end{aligned} \tag{4.2.8}$$

and that

$$\begin{aligned} \mathcal{C}(\mathcal{Z}_r^{k-d}) &\subseteq \mathcal{C}\left((\mathcal{Z}_{r+\kappa,\Lambda_1} + F_1) \times \cdots \times (\mathcal{Z}_{r+\kappa,\Lambda_{k-d}} + F_{k-d})\right) \\ &= \mathcal{L}_1(\mathcal{Z}_{r+\kappa,\Lambda_1} + F_1) \times \cdots \times \mathcal{L}_{k-d}(\mathcal{Z}_{r+\kappa,\Lambda_{k-d}} + F_{k-d}). \end{aligned} \tag{4.2.9}$$

Now we are positioned to make our final arguments.

Suppose first of all that (LR2) holds. Let U be any connected component of nsing(r). Then U is a $(k-d)$-dimensional box, with faces parallel to the coordinate hyperplanes, and with vertices in the set $\mathcal{C}(\mathcal{Z}_r^{k-d})$. Therefore we can write U in the form

$$U = \{x \in \mathcal{W} : \ell_i < x_i < r_i\}, \tag{4.2.10}$$

where for each i, the values of ℓ_i and r_i are elements of the set $\mathcal{L}_i(\mathcal{Z}_r)$. By equation (4.2.9), together with Lemma 1.4.4, there is a constant $c_1 > 0$ such that, for every i,

$$r_i - \ell_i \geq \frac{c_1}{r^{m_i}}.$$

Next we will show that there is a constant $c_2 > 0$ such that, for all sufficiently large r, the orbit of every point in $F_\rho/\mathbb{Z}^{k-d}$ under the action of $\mathcal{Z}_{c_2 r}$ intersects every connected component of $\mathrm{nsing}(r)$. Then Lemma 4.1.3 will imply that Y is LR. To show that there is such a constant c_2, we use (4.2.6) and Theorem 1.4.2. Each one of the linear forms L_i is a badly approximable linear form in m_i variables, when restricted to Λ_i. Therefore, by (T2) of Theorem 1.4.2, there is a constant $\eta > 0$ with the property that, for all sufficiently large r and for each i, the collection of points $\mathcal{L}_i(\mathcal{Z}_{c_2 r, \Lambda_i})$ is $\eta/(c_2 r^{m_i})$-dense in $\mathbb{R}/\mathbb{Z}$. Choosing $c_2 > 3c_1/\eta$ completes the proof of this part of the theorem, verifying that (LR1) and (LR2) together imply linear repetitivity.

For the final part, suppose that (LR1) holds and (LR2) does not. Then one of the linear forms L_i is not relatively badly approximable, and we assume without loss of generality that it is L_1. Let c_2 be any positive constant, and consider the collection of points $\mathcal{L}(\mathcal{Z}_{c_2 r})$. By (4.2.8), the first coordinates of these points are a subset of

$$\mathcal{L}_1(\mathcal{Z}_{c_2 r + \kappa, \Lambda_1} + F).$$

There are at most $c_2 \delta r^{m_1} - 1$ points in the latter set, for some constant δ depending on Λ_1. Therefore, thinking of the points as being arranged in increasing order in $[0, 1)$, there must be two consecutive points which are at least $1/(c_2 \delta r^{m_1})$ apart. On the other hand, by (4.2.7) and our hypothesis on L_1, we can choose r large enough so that there is a connected component U of $\mathrm{nsing}(r)$, given as in (4.2.10), with

$$r_1 - \ell_1 < \frac{1}{c_2 \delta r^{m_1}}.$$

From these two observations it is clear that there is some point in $F_\rho/\mathbb{Z}^{k-d}$ whose orbit under $\mathcal{Z}_{c_2 r}$ does not intersect U. Since $c_2 > 0$ was arbitrary, this means that Y is not LR. Therefore, (LR1) and (LR2) are necessary conditions for linear repetitivity, and the proof of Theorem 4.2.2 is complete. $\square$

4.3 Linear repetitivity, canonical case

Now we explore the same questions as in the previous section, but for canonical cut and project sets. Recall that canonical cut and project sets are defined to be nonsingular, minimal, and aperiodic, and they are formed using the canonical window

$$\mathcal{W} = \rho^* \left(\left\{ \sum_{i=1}^{k} t_i e_i : 0 \leq t_i < 1 \right\} \right).$$

As in the previous section, we ignore the dependence of our cut and project sets on s, and we work only with type 2 patches, suppressing the notational dependence on the subscript 2.

To begin our discussion, we will show below that if a canonical cut and project set is LR, then so are the associated cubical cut and project sets. For the converse direction, in many specific examples and cases which are commonly cited in the literature (e.g. the case when $k-d$=1), the proofs we have given in the previous section do apply. However, it turns out that there are examples of LR cubical cut and project sets which are no longer LR when their windows are replaced by canonical ones.

As we will see, there are at least two seemingly different sources for this type of behavior. The first is geometric, and arises in the situation when at least two of the linear forms defining the physical space have co-kernels with different ranks (we will explain what this means in an example below). The second (which can occur even in the absence of the geometric situation just described) is Diophantine, and is related to the fact that any number can be written as a product of two badly approximable numbers (this follows from continued fraction Cantor set arguments, see [15]). We summarize what has just been mentioned so far the following theorem.

Theorem 4.3.1. *If Y is a cubical cut and project set which is not LR, then the cut and project set formed from the same data as Y, but with the canonical window in place of the cubical one, is also not LR. However, the converse of this statement is not true, in general.*

It should be pointed out that many canonical cut and project sets of specific interest in the literature arise from subspaces defined by linear forms with coefficients in a fixed algebraic number field. In such a case the Diophantine behavior alluded to at the end of the paragraph before Theorem

4.3.1 cannot occur. To illustrate this point, we will explain in a later section how to prove that Penrose and Ammann-Beenker tilings are LR. This in itself is not a new result, and in fact it is fairly obvious from descriptions of these tilings using substitution rules. What is new is that our proof uses only their descriptions as cut and project sets.

In order to gain a broader perspective on our results, we introduce the notion of local derivability, which originated in a paper of Baake, Schlottmann, and Jarvis [3]. Suppose that Y_1 and Y_2 are two cut and project sets formed from a common physical space E, and suppose (without loss of generality for the purposes of all of our results) that Y_1 and Y_2 are both Delone sets. We say that Y_1 *is locally derivable from* Y_2 if there exists a constant $c > 0$ with the property that, for all $x \in E$ and for all sufficiently large r, the equivalence class of the patch of size r centered at x in Y_2 uniquely determines the patch of size $r - c$ centered at x in Y_1. There is a minor technical issue here, that x may not belong to Y_1 or Y_2. However, since Y_1 and Y_2 are relatively dense, this can be rectified by requiring that x be moved, when necessary in the definition above, to a nearby point of the relevant cut and project set. Finally, we say that Y_1 and Y_2 are *mutually locally derivable*, which we abbreviate as (MLD), if each set is locally derivable from the other.

The argument in [3, Appendix] (see also [4] and [5, Remark 7.6]) provides us with the following characterization of MLD cut and project sets Y_1 and Y_2 as above.

Lemma 4.3.2. *Let Y_1 and Y_2 be nonsingular, minimal, aperiodic k to d cut and project sets, constructed with the same physical and internal spaces and with windows $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively. Then Y_1 is locally derivable from Y_2 if and only if $\mathcal{W}_1$ is a finite union of sets each of which is a finite intersection of translates of $\mathcal{W}_2$ (or of its complement), with translations taken from $\rho^*(\mathbb{Z}^k)$.*

From this lemma we immediately deduce the following result relating cubical and canonical cut and project sets.

Corollary 4.3.3. *Let Y_1 be a cubical cut and project set, and let Y_2 be the cut and project set formed from the same data as Y_1, but with the canonical window. Then Y_1 is locally derivable from Y_2. Furthermore, Y_2 is locally derivable from Y_1 if and only if, for each $1 \leq i \leq d$, the point $\rho^*(e_i)$ lies on a line of the form $\mathbb{R}e_j$, for some $d + 1 \leq j \leq k$.*

This corollary, together with Exercise 4.3.1, implies the first part of Theorem 4.3.1, that if a cubical cut and project set is not LR, then neither is the corresponding canonical cut and project set. Furthermore, it also implies the following result, as a corollary of Theorem 4.2.2.

Corollary 4.3.4. *A k to $k-1$ canonical cut and project set defined by a linear form L is LR if and only if L is badly approximable.*

The second part of Theorem 4.3.1 is not quite as obvious. To understand the issue, note that it is easy, using Theorem 4.2.2 and Corollary 4.3.3, to come up with cut and project sets Y_1 and Y_2, as in the statement of Corollary 4.3.3, for which Y_1 is LR but Y_2 is not locally derivable from Y_1. For example, the subspace used to define the Ammann-Beenker tiling in Section 4.4.2 provides us with such sets. On the other hand, as can be seen in the Ammann-Beenker example, in general this does not imply that Y_2 is not LR.

We will demonstrate two different constructions for producing cubical cut and project sets which are LR, but for which their canonical counterparts are not. Our first construction is based on elementary geometric considerations.

Lemma 4.3.5. *Suppose that α_1, α_2, and β are positive real numbers with $(\alpha_1, \alpha_2) \in \mathcal{B}_{2,1}$ and $\beta \in \mathcal{B}_{1,1}$, and let E be the three dimensional subspace of $\mathbb{R}^5$ defined by*

$$E = \{(x, -\alpha_1 x_1 - \alpha_2 x_2 - x_3, -\beta x_3) : x \in \mathbb{R}^3\}.$$

Then every cubical cut and project set defined using E is LR, but no canonical cut and project sets defined using E are LR.

Proof. Observe, as is implicit in the statement of the lemma, that E acts minimally on $\mathbb{T}^k$, and that any cubical or canonical cut and project set defined using E is aperiodic. In the notation of the proof of Theorem 4.2.2, the kernels of the linear forms $\mathcal{L}_1$ and $\mathcal{L}_2$ have ranks $r_1 = 1$ and $r_2 = 2$. Therefore Theorem 4.2.2 allows us to conclude that any cubical cut and project set formed using E is LR.

Let $\mathcal{W}$ be the canonical window in F_ρ. The window is a hexagon with vertices at

$$e_5,\ e_4 + (1+\beta)e_5,\ (2+\alpha_1+\alpha_2)e_4 + (1+\beta)e_5,$$

$$(2+\alpha_1+\alpha_2)e_4+\beta e_5,\ (1+\alpha_1+\alpha_2)e_4,\ \text{and } 0.$$

For $r>0$, let us consider the orbit of the line segment from e_5 to $e_4+(1+\beta)e_5$, under the action of the collection of integers

$$-\rho^{-1}(r\Omega)\cap\mathbb{Z}^k \tag{4.3.6}$$

used to define sing(r). By our Diophantine hypotheses on the subspace E (using the transference principles in the same way as we have in the proof of Theorem 4.2.2), there is a constant $C>0$ with the property that, for all sufficiently large r, there is an integer point n in the set (4.3.6) satisfying

$$n.e_5=(y_1,y_2),$$

with

$$1-\frac{C}{r^2}<y_1<1 \ \text{ and } \ \left|y_2-\frac{1}{2}\right|<\frac{C}{r}. \tag{4.3.7}$$

Provided r is large enough, the line segment from $n.e_5$ to $n.(e_4+(1+\beta)e_5)$, together with the lines $n.e_5+\mathbb{R}e_4$ and $e_4+\mathbb{R}e_5$, bound a triangle $\mathcal{T}_r$ with

$$|\mathcal{T}_r|\ll_\beta \frac{1}{r^4}. \tag{4.3.8}$$

Now we claim that if y and y' are two points in the canonical cut and project set formed using $\mathcal{W}$, and if $\rho^*(\tilde{y})\in\mathcal{T}_r$ but $\rho^*(\tilde{y'})\notin\mathcal{T}_r$, then $P(y,r)$ and $P(y',r)$ are not in the same equivalence class of patches of size r. Note that we cannot appeal directly to Lemma 4.1.3 in this case, since the window we are using does not satisfy its hypotheses. In this case we argue directly as follows. Write $y^*=\rho^*(\tilde{y})$ and $y'^*=\rho^*(\tilde{y'})$, suppose that $y^*\in\mathcal{T}_r$ and $y'^*\notin\mathcal{T}_r$, and write ℓ_n for the line segment from $n.e_5$ to $n.(e_4+(1+\beta)e_5)$. Consider the following three cases:

Case 1: If y'^* lies in the half-plane above the line containing ℓ_n then the point $\tilde{y}-n$ lies in $\mathcal{S}$, while $\tilde{y'}-n$ does not. Therefore $y-\pi(n)\in P(y,r)$ but $y'-\pi(n)\notin P(y',r)$.

Case 2: If y'^* lies in the half-plane below the line $n.e_5+\mathbb{R}e_4$ then the point $\tilde{y}-n-e_5$ lies in $\mathcal{S}$, while $\tilde{y'}-n-e_5$ does not. As in the previous case, $y-\pi(n+e_5)\in P(y,r)$ but $y'-\pi(n+e_5)\notin P(y',r)$.

Case 3: If neither Case 1 nor Case 2 applies, then y'^* lies to the right of the line $e_4+\mathbb{R}e_5$, in the cone below the line containing ℓ_n and above the line

$n.e_5+\mathbb{R}e_4$. It is clear that $\tilde{y}-e_4 \notin \mathcal{S}$, and if $\tilde{y'}-e_4 \in \mathcal{S}$ then the argument is the same as before. Otherwise, if $\tilde{y'}-e_4 \notin \mathcal{S}$, then we would have to have that $y'^* \notin \mathcal{W}+e_4$. Since we have arranged in (4.3.7) for y_2 to be close to $1/2$, this would imply that $|y^*-y'^*| \gg 1$. As long as r is sufficiently large, we could then conclude, by the same types of arguments used in Cases 1 and 2, that $P(y,r)$ contains a point which does not appear in $P(y',r)$.

Since these cases cover all possibilities, we have verified our claim above, that the connected component $\mathcal{T}_r$ corresponds to a unique equivalence class of patches of size r. This, together with (4.3.8) and a simple application of the Birkhoff Ergodic Theorem, shows that for all sufficiently large r, there are equivalence classes of patches of size r which occur with frequency $\ll r^{-4}$. Since $d=3$, we conclude that canonical cut and project sets formed using E cannot be LR. $\square$

Although we do not attempt to generalize Lemma 4.3.5, we remark that a similar construction would likely work whenever $d>k-d>1$, to show that some canonical cut and project sets are not LR (and even with the extra requirement that their cubical counterparts are LR). We posit that the analogous conditions necessary to draw this conclusion from the argument just given should be, in the notation of the proof of Theorem 4.2.2, that there are integers $1 \le j < j' \le k-d$ satisfying

(i) $r_j \neq r_{j'}$, and

(ii) there exists an integer $1 \le i \le d$ such that the orthogonal projection (with respect to the standard basis vectors) of the vector $\rho^*(e_i) \in F_\rho$, onto the plane in F_ρ spanned by e_j and $e_{j'}$, does not lie on either of the lines $\mathbb{R}e_j$ or $\mathbb{R}e_{j'}$.

Condition (i) is simply the requirement that the kernels of two of the linear forms defining E, considered modulo 1, have different ranks. The second condition is that there is a 'slant' in the window, when projected orthogonally to the $e_je_{j'}$-plane.

Interestingly, there is a different type of behavior which can cause canonical cut and project sets to fail to be LR. This behavior is related to Diophantine approximation, and occurs because of the fact that two subspaces defined by relatively badly approximable linear forms can still intersect in a subspace which is not definable using relatively badly approximable forms. A one dimensional realization of this fact is the famous theorem of Marshall

Hall [15, Theorem 3.2], which implies that every nonzero real number can be written as a product of two badly approximable numbers. This provides the basis for the following example with $k = 4$ and $d = 2$, the smallest possible choices of k and d for which 'cubical LR but canonical not' can occur.

Lemma 4.3.9. *Suppose that α and β are positive badly approximable real numbers with the property that*

$$\inf_{n\in\mathbb{N}} \; n \cdot \left\{\frac{5\alpha\beta}{2}n\right\} = 0. \tag{4.3.10}$$

If E is the two dimensional subspace of $\mathbb{R}^4$ defined by

$$E = \{(x, -(2/5)x_1 - \alpha x_2, -\beta(x_1 + (5/2)x_2)) : x \in \mathbb{R}^2\},$$

then every cubical cut and project set defined using E is LR, but no canonical cut and project sets defined using E are LR.

Proof. First of all we remark that, by a general version of Khintchine's Theorem (see [27, Theorem 1]), almost every real number γ has the property that

$$\inf_{n\in\mathbb{N}} n\{n\gamma\} = 0.$$

For such a γ, it follows from Hall's Theorem that there are badly approximable α and β satisfying $5\alpha\beta/2 = \gamma$, and therefore (assuming $\gamma > 0$) the hypotheses of the lemma.

It is easy to see that E acts minimally on $\mathbb{T}^k$ and that cubical and canonical cut and project sets formed using E will be aperiodic. By Theorem 4.2.2, cubical cut and project sets formed using E will be LR.

The canonical window in F_ρ is an octagon which includes, on its boundary, the line segment from e_4 to $(2/5)e_3 + (1+\beta)e_4$. Each integer $n \in \mathbb{Z}^4$ acts on this line segment, moving it to a line segment which we denote by ℓ_n. The initial point of ℓ_n is the point

$$((2/5)n_1 + \alpha n_2 + n_3, \beta n_1 + (5\beta/2)n_2 + 1 + n_4)$$

in the e_3e_4-plane. For any choice of n_2, n_3, and n_4, there is a unique choice of n_1 with the property that ℓ_n intersects the line $e_3 + \mathbb{R}e_4$, and it is clear that $|n_1|$ is bounded above by a constant (depending at most on α and

β) times the maximum of $|n_2|, |n_3|$, and $|n_4|$. The intersection point just described is $e_3 + ye_4$, where $y = y(n_2, n_3, n_4)$ is given by

$$y = \frac{-5\alpha\beta}{2}n_2 + 1 + n_4 + \frac{5\beta}{2}(1 + n_2 - n_3).$$

Since (4.3.10) is satisfied, for any $\epsilon > 0$ there is a number $r > 1$ and integers n_2 and n_4 with $|n_2|, |n_4| \leq r$, such that

$$\left|\frac{5\alpha\beta}{2}n_2 - (1 + n_4) + 1\right| < \frac{\epsilon}{r}.$$

For such a choice of n_2 and n_4, and with n_1 selected as above, we take $n_3 = 1 + n_2$. Then the line segment ℓ_n, together with the lines $e_4 + \mathbb{R}e_3$ and $e_3 + \mathbb{R}e_4$, bound a triangle of area $\ll \epsilon r^{-2}$. Since ϵ can be taken arbitrarily small, the remainder of the proof follows from the same type of argument as the one used at the end of the proof of Lemma 4.3.5. $\square$

To bring us to our concluding remarks for this section, we first of all mention that the proof we have just given is somewhat biased towards one particular point of view. In fact, there is a conceptually easier proof (which is instructive in a different way), which we now explain. If we reparameterize the subspace E in the statement of the lemma, with respect to the 'reference' subspace generated by e_3 and e_4, then it is easy to see that condition (LR1) from Theorem 4.2.2 is not satisfied, so the corresponding cubical cut and project sets are not LR. Therefore canonical cut and project sets obtained using E (which of course do not depend on the choice of reference subspace) are also not LR. This simple consideration leads naturally to an open question, which we pose as Exercise 4.3.2 below. At the moment we do not know the answer to this question. However, if the answer is yes, it means that Theorem 4.2.2 gives a complete characterization of all canonical as well as cubical cut and project sets.

EXERCISES

Exercise 4.3.1. *Prove that if a Delone set Y_1 is locally derivable from Y_2, and if Y_2 is LR, then Y_1 is also LR.*

Exercise 4.3.2 (Open problem)**.** *Is it true that a canonical cut and project set will be LR if and only if all of the cubical cut and project sets obtained from taking different parameterizations of E, with respect to different orderings of the standard basis vectors, are also LR?*

4.4 Examples of linearly repetitive cut and project sets

Our proof of Theorem 4.2.2 gave an explicit correspondence between the collection of k to d LR cubical cut and project sets, and the Cartesian product of the following two sets:

(S1) The set of all $(k-d)$-tuples $(L_1,\ldots,L_{k-d})$, where each L_i is a badly approximable linear form in $m_i \geq 1$ variables, with the integers m_i satisfying $m_1+\cdots+m_{k-d}=d$, and

(S2) The set of all $d\times d$ integer matrices with nonzero determinant.

The fact that the set (S1) is empty when $d<k/2$ implies that, for this range of k and d values, there are no LR cubical (or canonical) cut and project sets. On the other hand, for $d\geq k/2$, there are uncountably many, as implied by the following corollary to our main result.

Corollary 4.4.1. *For $d<k/2$, there are no LR cubical cut and project sets. For $d\geq k/2$, the collection of $\{\alpha_{ij}\}\in\mathbb{R}^{d(k-d)}$ which define LR cubical cut and project sets is a set with Lebesgue measure* 0 *and Hausdorff dimension d. Furthermore, these statements also apply to canonical cut and project sets.*

Proof. The proof of Theorem 4.2.2 demonstrates how, to each LR cubical cut and project set, we may associate a subgroup $\Lambda\leqslant\mathbb{Z}^d$ of finite index, with decomposition

$$\Lambda=\Lambda_1+\cdots+\Lambda_{k-d},$$

so that each L_i is badly approximable, when viewed as a linear form in m_i variables, restricted to Λ_i. The first part of Corollary 4.4.1 clearly follows from the fact that the integers $m_i\geq 1$ have sum equal to d.

In the other direction, suppose that $d\geq k-d$. If we start with $k-d$ positive integers m_i, with sum equal to d, and a collection of badly approximable linear forms $L_i : \mathbb{R}^{m_i} \to \mathbb{R}$ then, thinking of

$$\mathbb{R}^d = \mathbb{R}^{m_1} + \cdots + \mathbb{R}^{m_{k-d}},$$

any cubical cut and project set arising from the subspace

$$E = \{(x, L_1(x), \ldots, L_{k-d}(x)) : x \in \mathbb{R}^d\}$$

is LR, by the proof of Theorem 4.2.2. It follows that the collection of $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$ which define LR cubical cut and project sets is a countable union (over all allowable choices of Λ_i above) of sets of Lebesgue measure 0 and Hausdorff dimension at most

$$\dim \mathcal{B}_{m_1,1} + \cdots + \dim \mathcal{B}_{m_{k-d},1} = m_1 + \cdots + m_{k-d} = d.$$

Since the cubical cut and project sets corresponding to $\Lambda_i = \mathbb{Z}^{m_i}$ are all LR, the Hausdorff dimension of this set is equal to d.

The part of Corollary 4.4.1 about canonical cut and project sets follows from the same arguments just given, together with Corollary 4.3.3. □

Now we explain how to explicitly construct examples of LR cut and project sets, and we also give some numerical examples to show how the machinery we have developed in this chapter can be used to verify that vertex sets of Penrose and Ammann-Beenker tilings are LR.

4.4.1 Explicit examples for all $d \geq k - d$

For $d \geq k/2$ it is easy to construct examples of subspaces E satisfying the hypotheses of Theorem 4.2.2. Write $d = m_1 + \cdots + m_{k-d}$, with positive integers m_i, and for each i let K_i be an algebraic number field, of degree $m_i + 1$ over $\mathbb{Q}$. Suppose that the numbers $1, \alpha_{i1}, \ldots, \alpha_{im_i}$ form a $\mathbb{Q}$-basis for K_i, and define $L_i : \mathbb{R}^{m_i} \to \mathbb{R}$ to be the linear form with coefficients $\alpha_{i1}, \ldots, \alpha_{im_i}$. Then, using the decomposition $\mathbb{R}^d = \mathbb{R}^{m_1} + \cdots + \mathbb{R}^{m_{k-d}}$, let

$$E = \{(x, L_1(x), \ldots, L_{k-d}(x)) : x \in \mathbb{R}^d\}.$$

The collection of points

$$\{(L_1(n), \ldots, L_{k-d}(n)) : n \in \mathbb{Z}^d\}$$

is dense in $\mathbb{R}^{k-d}/\mathbb{Z}^{k-d}$, and it follows that the subspace E acts minimally on $\mathbb{T}^k$. The intersection of the kernels of the corresponding maps $\mathcal{L}_i$ is $\{0\}$, so any cubical cut and project set formed from E will be aperiodic. Condition (LR1) of Theorem 4.2.2 is clearly satisfied. Furthermore, by a result of Perron [24], each of the linear forms L_i is badly approximable. Therefore (LR2) is also satisfied, and any cubical cut and project set formed from E is LR. Furthermore, the hypotheses in the second part of Corollary 4.3.3 are also satisfied, so any canonical cut and project set formed using E is also LR.

4.4.2 Ammann-Beenker tilings

Collections of vertices of Ammann-Beenker tilings can be obtained as canonical cut and project sets, using the two dimensional subspace E of $\mathbb{R}^4$ defined by

$$E = \{(x, L_1(x), L_2(x)) : x \in \mathbb{R}^2\},$$

with

$$L_1(x) = \frac{\sqrt{2}}{2}(x_1 + x_2) \quad \text{and} \quad L_2(x) = \frac{\sqrt{2}}{2}(x_1 - x_2).$$

Although we cannot directly appeal to either Theorem 4.2.2 or Corollary 4.3.3, we will explain how the machinery we have developed can be used to easily show that these sets are LR.

The canonical window $\mathcal{W}$ in F_ρ is the regular octagon with vertices at

$$\left(\frac{1+\sqrt{2}}{2} \pm \frac{1+\sqrt{2}}{2}\ ,\ \frac{1}{2} \pm \frac{1}{2}\right) \text{ and } \left(\frac{1+\sqrt{2}}{2} \pm \frac{1}{2}\ ,\ \frac{1}{2} \pm \frac{1+\sqrt{2}}{2}\right).$$

By Lemma 4.1.1, every patch of size r corresponds to a finite collection of connected components of $\mathrm{nsing}(r)$. Therefore to demonstrate that a canonical cut and project set formed using E is LR, it is enough to show that the there is a constant $C > 0$ with the property that, for all sufficiently large r, the orbit of any nonsingular point $w \in F_\rho$, under the action of the collection of integers

$$\rho^{-1}(Cr\Omega) \cap \mathbb{Z}^k,$$

intersects every connected component of $\mathrm{nsing}(r)$.

We claim that every connected component of $\mathrm{nsing}(r)$ contains a square with side length $\gg r^{-1}$. This follows from elementary considerations, by writing down the equations of the line segments that form the boundary of $\mathcal{W}$, considering the action of

$$-\rho^{-1}(r\Omega) \cap \mathbb{Z}^k$$

on these line segments, and then computing all possible intersection points. Since all of our algebraic operations take place in the field $\mathbb{Q}(\sqrt{2})$, it is not difficult to show that every connected component of $\mathrm{nsing}(r)$ must contain a right isosceles triangle of side length $\gg r^{-1}$. The claim about squares follows immediately.

Finally, the linear forms L_1 and L_2 are relatively badly approximable, and the sum of the ranks of $\mathcal{L}_1$ and $\mathcal{L}_2$ is equal to 2. Therefore our study of the orbits of points towards the end of the proof of Theorem 4.2.2 applies as before, allowing us to conclude that the Cr-orbit of any nonsingular point in F_ρ intersects every connected component of $\mathrm{nsing}(r)$.

EXERCISES

Exercise 4.4.1. *Rigorously verify the claim above, that every connected component of* $\mathrm{nsing}(r)$ *contains a square with side length* $\gg r^{-1}$.

4.4.3 Penrose tilings

This example is similar to the previous one, but it also gives an indication of how to apply our techniques in cases when the physical space does not act minimally on $\mathbb{T}^k$. Let $\zeta = \exp(2\pi i/5)$ and let Y be a canonical cut and project set defined using the two dimensional subspace E of $\mathbb{R}^5$ generated by the vectors

$$(1, \mathrm{Re}(\zeta), \mathrm{Re}(\zeta^2), \mathrm{Re}(\zeta^3), \mathrm{Re}(\zeta^4))$$

and

$$(0, \mathrm{Im}(\zeta), \mathrm{Im}(\zeta^2), \mathrm{Im}(\zeta^3), \mathrm{Im}(\zeta^4)).$$

Well known results of de Bruijn [10] and Robinson [25] show that the set Y is the image under a linear transformation of the collection of vertices of a Penrose tiling, and in fact that all Penrose tilings can be obtained in a similar way from cut and project sets. The fact that Y is LR can be deduced directly from the definition of the Penrose tiling as a primitive substitution. However, as in the previous example, we will show how to prove this starting from the definition of Y as a cut and project set.

The subspace E is contained in the rational subspace orthogonal to $(1, 1, 1, 1, 1)$. In this case Theorem 4.2.2 does not apply directly, but our proof is still robust enough to allow us to draw the desired conclusions. Set

$$\alpha_1 = \cos(2\pi/5), \alpha_2 = \cos(4\pi/5), \beta_1 = \sin(2\pi/5), \text{ and } \beta_2 = \sin(4\pi/5),$$

so that

$$E = \{(x, x\alpha_1 + y\beta_1, x\alpha_2 + y\beta_2, x\alpha_2 - y\beta_2, x\alpha_1 - y\beta_1) : x, y \in \mathbb{R}\}.$$

After making the change of variables $x_1 = x$ and $x_2 = x\alpha_1 + y\beta_1$, we can write E as

$$E = \{(x, L_1(x), L_2(x), L_3(x)) : x = (x_1, x_2) \in \mathbb{R}^2\}.$$

The functions L_i are linear forms which (using the fact that $4\alpha_1^2 + 2\alpha_1 - 1 = 0$) are given by

$$\begin{aligned} L_1(x) &= -x_1 + 2\alpha_1 x_2, \\ L_2(x) &= -2\alpha_1 x_1 - 2\alpha_1 x_2, \text{ and} \\ L_3(x) &= 2\alpha_1 x_1 - x_2. \end{aligned}$$

Write $\mathcal{L}_i : \mathbb{Z}^2 \to \mathbb{R}/\mathbb{Z}$ for the restriction of L_i to $\mathbb{Z}^2$, modulo 1, and notice that $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 = 0$. This means that the orbit of 0 under the natural $\mathbb{Z}^2$-action of E on $F_\rho/\mathbb{Z}^3$ is contained in the two dimensional rational subtorus with equation $x + y + z = 0$. The kernels of the forms $\mathcal{L}_i$ are all rank 1 subgroups of $\mathbb{Z}^2$, and it follows that the number of connected components of $\text{nsing}(r)$ which intersect the rational subtorus is $\asymp r^2$.

Since the forms are linearly dependent, we can understand the orbit of a point in $F_\rho/\mathbb{Z}^3$ under the $\mathbb{Z}^2$-action by considering only the values of $\mathcal{L}_1$ and $\mathcal{L}_3$. In other words, we can consider the projection of the problem onto the e_1e_3-plane. Consider the intersection of a connected component of $\text{nsing}(r)$ with the subspace $x + y + z = 0$. This is a two dimensional region, bounded by the intersections of the subspace with translates (by the $\mathbb{Z}^5$ action) of the hyperplanes forming the boundary of the canonical window. Computing the vertices of the region is an operation which takes place in $\mathbb{Q}(\sqrt{5})$. As in the previous example, this leads to the conclusion that the intersection of any connected component of $\mathrm{nsing}(r)$ with the subspace $x+y+z=0$, when projected to the e_1e_3-plane, contains a square of side length $\gg r^{-1}$. The remainder of the proof follows exactly as before, allowing us to conclude that Y is LR.

Chapter 5

Diffraction

In this chapter we give a brief account of part of the mathematical theory of diffraction. The goal of the chapter is to prove that Dirac combs supported on cut and project sets (satisfying our usual assumptions) have pure point diffraction, thus reinforcing their potential usefulness as models for physical crystals and quasicrystals. The contents of this chapter are included mostly for completeness, and are not intended to be a complete treatise on the theory of diffraction, either from a physical or a mathematical point of view. Most of what we will cover, together with many more examples and insights, can be found in the introductory book by Baake and Grimm on aperiodic order [5, Chapters 8,9]. Having been influenced very much by those authors and their collaborators, we follow closely their notation and adopt much of their point of view in our exposition.

5.1 Physical diffraction

In physics the term diffraction refers to the pattern created by wave interference. This is an extremely well studied phenomenon in optics and, given a function which describes the shape of an aperture through which waves or particles pass, the wave equation (a differential equation which describes the propagation of waves) can be used, at least theoretically, to predict the exact pattern and intensities which will be measured on the other side of the aperture. Without some simplifying assumptions it is not always possible to deduce an exact analytic solution for the wave equation. However there are several approximations which are accurate for different practical purposes. One of these is Fraunhofer's far field limit, under which the diffraction image is predicted to be the Fourier transform of the indicator function of the aperture. In this model, the intensities of peaks in the diffraction spectrum are computed as the modulus squared of the Fourier transform. As curious as it may seem at first, the validity of this solution is easily substantiated by experiments which are simple enough to perform at home (assuming access to a strong concentrated light source).

One of the uses of diffraction is to try to form an image which tells us about the arrangement of atoms in physical materials. If the molecular structure is disordered then x-ray diffraction will appear as noise. However if there is a strong tendency towards order then in the diffraction we expect this to show up as regularly arranged sharp peaks. This is referred to, in physics, as pure point diffraction.

Before the 1980's it was widely assumed that the presence of pure point diffraction implied that the molecular arrangement of a material was completely periodic. If this were the case then, by the Crystallographic Restriction Theorem in the form of Theorem 3.5.1, we would never see diffraction with rotational symmetries of orders other than $1, 2, 3, 4$, or 6. This view was challenged in the early 1980's by Dan Shechtman, who created a metal alloy with diffraction showing 10-fold symmetry. Although these results were opposed for many years by well known scientists, in the end they were proved to be correct and, in 2011, Shechtman received the Nobel Prize for this discovery.

Of course, the existence of quasicrystals does not contradict the Crystallographic Restriction Theorem. The 'forbidden symmetries' are possible because the arrangement in the molecular structure of those materials is a patch which, if extended to infinity, would form an aperiodic tiling of Euclidean space. From this point of view, and from what we have learned so far about rotational symmetries, cut and project sets are natural candidates to model the molecular arrangements found in quasicrystals. Furthermore, as we will show in this chapter, such arrangements do in fact produce pure point diffraction patterns.

Now we return to our discussion of the mathematics of diffraction. In order to deal with obstacles which are given as continuous distributions in space, it is natural from a mathematical point of view to replace the indicator function of the aperture in our description above by a measure in Euclidean space. This point of view is also attractive for the reason that it is also flexible enough to model the situation where our obstacle is given as an array of particles, by using a measure which is a Dirac comb (this will be defined below) supported on a Delone set.

At the outset, we immediately encounter two difficulties in formulating a measure theoretic approach to diffraction. The first is that, while the Fourier transform of a measure can easily be defined for finite measures, the usual definition does not always make sense for infinite measures, which are some of the most natural examples we will want to study. Secondly, in order to connect our analysis in a physically meaningful way with the above description we must try to find a reasonable way of understanding how we should interpret the 'modulus squared' of a measure.

In order to overcome the first of these difficulties, we work simultaneously with the space of measures and with the space of tempered distributions (the latter being the natural setting in which to define the Fourier transform). This is a necessary technical tool, and care must be taken in passing back and forth between the two spaces. To overcome the second difficulty mentioned above, we introduce the notion of an autocorrelation measure and work with the Fourier transform of the autocorrelation. The fact that the intensities of the diffraction are accurately described by the resulting object is an analogue of the well known result that the Fourier transform of a convolution of two functions is the product of the Fourier transforms.

5.2 Background from Fourier analysis

In what follows we let $C_c(\mathbb{R}^d)$ denote the linear space of complex valued continuous functions on $\mathbb{R}^d$ with compact support, endowed with the metric topology inherited from the sup-norm $\|\cdot\|_\infty$. We also let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space on $\mathbb{R}^d$, which is the collection of complex-valued C^∞ functions on $\mathbb{R}^d$ whose higher order multiple derivatives all tend to zero as $|x| \to \infty$ faster than $|x|^{-r}$, for any $r \geq 1$. We use the usual topology on $\mathcal{S}(\mathbb{R}^d)$, with which it is a complete normed linear space. We do not define this topology fully, except to say that a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^d)$ converges, as $n \to \infty$, to f if and only if

$$\lim_{n\to\infty} \|f_n - f\|_{\alpha,\beta} = 0,$$

for all multi-indices α and β, where the semi-norms $\|\cdot\|_{\alpha,\beta}$ are defined, for $g \in \mathcal{S}(\mathbb{R}^d)$, by

$$\|g\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta g(x)|.$$

The *Fourier transform* of a function $\phi \in L^1(\mathbb{R}^d)$ is defined by

$$(\mathcal{F}\phi)(t) = \int_{\mathbb{R}^d} \phi(x)e(-x\cdot t)\ dx,$$

where $e(x) = \exp(2\pi i x)$. Similarly, the inverse Fourier transform of a function $\psi \in L^1(\mathbb{R}^d)$ is defined by

$$(\mathcal{F}^{-1}\psi)(x) = \int_{\mathbb{R}^d} \psi(t)e(t\cdot x)\ dt.$$

The Fourier inversion formula says that if ϕ is a continuous function in $L^1(\mathbb{R}^d)$ and if $\mathcal{F}(\phi) \in L^1(\mathbb{R}^d)$, then

$$\mathcal{F}^{-1}(\mathcal{F}\phi) = \phi.$$

We note that the Fourier transform is a linear map, which provides a bijection from $\mathcal{S}(\mathbb{R}^d)$ to itself, with $\mathcal{F}^{-1}$ being the inverse map. From here on we use the abbreviation $\widehat{\phi}$ for $\mathcal{F}\phi$, and $\check{\phi}$ for $\mathcal{F}^{-1}\phi$.

For functions $\phi \in \mathcal{S}(\mathbb{R}^d)$ which are periodic modulo $\mathbb{Z}^d$ (i.e. so that $\phi(x+n) = \phi(x)$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$) we also have the *Fourier series* expansion

$$\phi(x) = \sum_{m\in\mathbb{Z}^k} c_\phi(m)e(m\cdot x),$$

where

$$c_\phi(m) = \int_{[0,1)^d} \phi(x)e(-m\cdot x)\ dx.$$

This leads immediately to the following well known result.

Theorem 5.2.1 (Poisson Summation Formula)**.** *For any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have that*

$$\sum_{n\in\mathbb{Z}^d} \phi(n) = \sum_{n\in\mathbb{Z}^d} \widehat{\phi}(n).$$

Proof. Since $\phi \in \mathcal{S}(\mathbb{R}^d)$ it follows that the sums on both sides of the above equation converge absolutely, and that the function $\Phi : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\Phi(x) = \sum_{n \in \mathbb{Z}^d} \phi(n + x)$$

is an element of $\mathcal{S}(\mathbb{R}^d)$. By construction, Φ is periodic modulo $\mathbb{Z}^d$, so it has the Fourier series expansion

$$\Phi(x) = \sum_{m \in \mathbb{Z}^d} c_\Phi(m) e(m \cdot x),$$

with

$$\begin{aligned} c_\Phi(m) &= \int_{[0,1)^d} \Phi(x) e(-m \cdot x)\ dx \\ &= \sum_{n \in \mathbb{Z}^k} \int_{[0,1)^d} \phi(n + x) e(-m \cdot x)\ dx \\ &= \int_{\mathbb{R}^d} \phi(x) e(-m \cdot x)\ dx \\ &= \widehat{\phi}(m). \end{aligned}$$

It is clear that all sums and integrals in this calculation are absolutely convergent. Finally, we have that

$$\sum_{n \in \mathbb{Z}^d} \phi(n) = \Phi(0) = \sum_{n \in \mathbb{Z}^d} \widehat{\phi}(n),$$

as required. □

Finally, if ψ and ϕ are two elements of $L^1(\mathbb{R}^d)$ then their *convolution* is the function $\phi * \psi \in L^1(\mathbb{R}^d)$ defined by

$$(\phi * \psi)(x) = \int_{\mathbb{R}^d} \phi(t) \psi(x - t)\ dt.$$

It is not difficult to check that

$$\phi * \psi = \psi * \phi$$

and that

$$\widehat{\phi * \psi} = \widehat{\phi} \widehat{\psi}. \tag{5.2.2}$$

EXERCISES

Exercise 5.2.1. *For $\sigma > 0$, let $\aleph_\sigma \in \mathcal{S}(\mathbb{R}^d)$ be the d-dimensional Gaussian density defined by*

$$\aleph_\sigma(x) = \frac{1}{(2\pi)^{d/2}\sigma^d} \cdot \exp\left(\frac{-|x|^2}{2\sigma^{2d}}\right). \tag{5.2.3}$$

Prove that if $f \in L^1(\mathbb{R}^d)$ is continuous at $x = 0$ then

$$f(0) = \lim_{\sigma \to 0^+} \int_{\mathbb{R}^d} f(x)\aleph_\sigma(x)\ dx.$$

Exercise 5.2.2. *With $\aleph_\sigma$ as above, prove that for every $x \in \mathbb{R}^d$, the sequence $\{\widehat{\aleph}_{1/n}(x)\}_{n\in\mathbb{N}}$ is increasing and converges to 1.*

Exercise 5.2.3. *Let Λ be a lattice in $\mathbb{R}^d$, let and let Λ^* denote the dual lattice to Λ, which is defined by*

$$\Lambda^* = \{\lambda^* \in \mathbb{R}^d : (\lambda \cdot \lambda^*) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

Prove that, for any $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\sum_{\lambda\in\Lambda} \phi(\lambda) = |\mathrm{covol}(\Lambda)|^{-1} \sum_{\lambda^*\in\Lambda^*} \widehat{\phi}(\lambda^*),$$

where $\mathrm{covol}(\Lambda)$*, called the covolume of Λ, is the volume of any measurable fundamental domain for $\mathbb{R}^d/\Lambda$.*

5.3 Measures and distributions

We assume that the reader has some experience working with positive regular Borel measures on $\mathbb{R}^d$, and take this as the starting point for our discussion. A positive regular Borel measure μ on $\mathbb{R}^d$ defines a linear functional on $C_c(\mathbb{R}^d)$, the space of continuous functions on $\mathbb{R}^d$ with compact support, by the rule that

$$\mu(g) = \int_{\mathbb{R}^d} g(x)\ d\mu(x),$$

for $g \in C_c(\mathbb{R}^d)$.

If μ and ν are two such measures then we say that μ is *absolutely continuous* with respect to ν if there is a continuous function f with the property that

$$\int_K |f(x)|\, d\nu(x) < \infty$$

for all compact measurable sets K, and such that

$$\mu(g) = \nu(fg) \text{ for all } g \in C_c(\mathbb{R}^d).$$

In this case the function f is called the *Radon-Nikodym derivative* of μ with respect to ν. By the well known Radon-Nikodym Theorem, μ is absolutely continuous with respect to ν if and only if $\mu(A) = 0$ whenever $\nu(A) = 0$ for a measurable set A. At the extreme opposite from absolute continuity, we say that μ is *singular* with respect to ν if there is a measurable set A for which $\mu(A) = \nu(\mathbb{R}^d \setminus A) = 0$.

We can write any regular Borel measure μ as a sum

$$\mu = \mu_{\text{ac}} + \mu_{\text{sing}},$$

where μ_{ac} is absolutely continuous with respect to Lebesgue measure, and μ_{sing} is singular with respect to Lebesgue measure. We can decompose the singular part of μ further by defining the collection of *pure points* of μ to be

$$P_\mu = \{x \in \mathbb{R}^d : \mu(\{x\}) > 0\}.$$

Then we define a measure μ_{pp} by the rule that, for any measurable set A,

$$\mu_{\text{pp}}(A) = \sum_{x \in A \cap P_\mu} \mu(\{x\}).$$

This measure is clearly singular with respect to Lebesgue measure, so writing

$$\mu_{\text{sc}} = \mu_{\text{sing}} - \mu_{\text{pp}}$$

for the *singular continuous* part of μ, we arrive at the decomposition

$$\mu = \mu_{\text{ac}} + \mu_{\text{sc}} + \mu_{\text{pp}}. \tag{5.3.1}$$

If μ_{pp} is the only non-zero part of this decomposition then we say that μ is a *pure point* measure.

In order to begin speaking about Fourier transforms of measures, we must now introduce complex measures. At the beginning of this section we remarked that every positive regular Borel measure on $\mathbb{R}^d$ defines a linear functional on $C_c(\mathbb{R}^d)$. The Riesz-Markov-Kakutani Representation Theorem tells us that a partial converse also holds, in the sense that if F is a positive (real valued) linear functional on $C_c(\mathbb{R}^d)$ which satisfies the condition that, for every compact set $K \in \mathbb{R}^d$, there exists a constant $c_K > 0$ such that, for all $g \in C_c(\mathbb{R}^d)$ with support contained in K,

$$|F(g)| \leq c_K \|g\|_\infty,$$

then F is determined, in the manner mentioned above, by a positive regular Borel measure. Now we simply broaden the scope and consider the collection of all complex valued linear functionals F on $C_c(\mathbb{R}^d)$ satisfying the condition that for every compact K there exists a c_K such that, for all $g \in C_c(\mathbb{R}^d)$ with support in K,

$$|F(g)| \leq c_K \|g\|_\infty.$$

By an extended form of the Riesz-Markov-Kakutani Representation Theorem, each such functional is determined, in the way above, by a linear combination of the form

$$\mu^+ - \mu^- + i(\nu^+ - \nu^-),$$

where μ^+, μ^-, ν^+, and ν^- are positive regular Borel measures. Such a linear combination is called a *complex measure*. Thus in what follows we will think of complex measures and their corresponding linear functionals as being the same.

All of the notions that we developed above concerning absolute continuity, singularity, and pure point measures, as well as the existence of a decomposition as in (5.3.1) with respect to Lebesgue measure, generalize in the obvious ways to complex measures.

If μ is a measure then the *conjugate* of μ is the measure $\overline{\mu}$ defined by

$$\overline{\mu}(g) = \overline{\mu(\overline{g})},$$

for $g \in C_c(\mathbb{R}^d)$. A measure μ is *real* if $\overline{\mu} = \mu$ and it is *positive* if it is real and if $\mu(g) \geq 0$ whenever $g \geq 0$ (this is clearly consistent with our previous notion of positivity of measures). For any measure μ, the *total variation measure* $|\mu|$ of μ is defined to be the smallest positive measure with the property that

$$|\mu|(g) \geq |\mu(g)|$$

for all $g \geq 0$. A measure μ is *translation bounded* if

$$\sup_{x \in \mathbb{R}^d} |\mu|(x + K) < \infty,$$

for all compact sets K, and it is *finite* if $|\mu|(\mathbb{R}^d) < \infty$.

The collection of all complex regular Borel measures on $\mathbb{R}^d$, which we denote by $\mathcal{M}(\mathbb{R}^d)$, becomes a topological space with the weak-$*$ topology which it inherits from $C_c(\mathbb{R}^d)$. Explicitly, a sequence of measures $\{\mu_n\}_{n \in \mathbb{N}}$ converges as $n \to \infty$ to μ if and only if

$$\lim_{n \to \infty} \mu_n(g) = \mu(g)$$

for every $g \in C_c(\mathbb{R}^d)$.

The Fourier transform of a finite measure μ is defined to be the measure which is absolutely continuous with respect to Lebesgue measure, whose Radon-Nikodym derivative is given by

$$\widehat{\mu}(t) = \int_{\mathbb{R}^d} e(-t \cdot x)\ d\mu(x).$$

As was mentioned, this definition does not generalize well to infinite measures. In order to move in that direction, we need to take a slightly different approach.

With a view towards extending the definition of the Fourier transform, we define the space of *tempered distributions* to be the space of complex valued linear functionals on $\mathcal{S}(\mathbb{R}^d)$. We denote this space by $\mathcal{S}'(\mathbb{R}^d)$ and, similar to the space of measures, we take it to be equipped with its weak-$*$ topology. To be clear, a sequence $\{T_n\}_{n \in \mathbb{N}}$ of tempered distributions converges to $T \in \mathcal{S}'(\mathbb{R}^d)$ if and only if

$$\lim_{n \to \infty} T_n(\phi) = T(\phi)$$

for every $\phi \in \mathcal{S}(\mathbb{R}^d)$.

An important subspace of $\mathcal{S}'(\mathbb{R}^d)$ is the space of *regular distributions*, which are defined, for each continuous function g with at most polynomial growth, by

$$T_g(\phi) = \int_{\mathbb{R}^k} \phi(x)g(x)\ dx.$$

The space of regular distributions is dense in $\mathcal{S}'(\mathbb{R}^d)$, but not all tempered distributions are regular distributions. For example, if $x \in \mathbb{R}^d$ then the *Dirac delta* distribution $\delta_x \in \mathcal{S}'(\mathbb{R}^d)$, defined by

$$\delta_x(\phi) = \phi(x),$$

is not a regular distribution. If T is a regular distribution then the *Fourier transform* of T is the tempered distribution $\widehat{T}$ defined by

$$\widehat{T}(\phi) = T(\hat{\phi}).$$

The Fourier transform thus defined extends to a unique continuous function on all of $\mathcal{S}'(\mathbb{R}^d)$.

Now we consider a couple of examples. For the first, let $x \in \mathbb{R}^d$ and define $\delta_x \in \mathcal{S}'(\mathbb{R}^d)$, the *Dirac delta* distribution at x, by the rule that

$$\delta_x(\phi) = \phi(x)$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. This is clearly a tempered distribution, but it is not a regular distribution. This means that, technically, to compute $\widehat{\delta_x}$ we need to realize δ_x as a limit of regular distributions and then use the continuity of the Fourier transform. Therefore, for $\sigma > 0$, let $\aleph_\sigma \in \mathcal{S}(\mathbb{R}^d)$ be the Gaussian density defined in (5.2.3). By the result of Exercise 5.2.1 we have that, for any $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\delta_x(\phi) = \lim_{\sigma \to 0^+} \int_{\mathbb{R}^d} \phi(x+t)\aleph_\sigma(t)\ dt = \lim_{\sigma \to 0^+} \int_{\mathbb{R}^d} \phi(t)\aleph_\sigma(t-x)\ dt.$$

This shows that δ_x is the weak-$*$ limit as $\sigma \to 0^+$ of the sequence of regular distributions given by the functions $\aleph_{\sigma,x}$ defined by

$$\aleph_{\sigma,x}(t) = \aleph_\sigma(t-x).$$

Taking Fourier transforms, and using similar steps as above, we have for $\phi \in \mathcal{S}(\mathbb{R}^d)$ that

$$\widehat{\delta}_x(\phi) = \lim_{\sigma \to 0^+} \widehat{T}_{\aleph_{\sigma,x}}(\phi) = \lim_{\sigma \to 0^+} T_{\aleph_{\sigma,x}}(\widehat{\phi}) = \widehat{\phi}(x).$$

Since

$$\widehat{\phi}(x) = \int_{\mathbb{R}^d} e(-x \cdot t)\phi(t)\ dt,$$

this proves that $\widehat{\delta}_x$ is the regular distribution defined by the function $e(-x \cdot t) \in \mathcal{S}(\mathbb{R}^d)$. Although we could have guessed this directly from the definition of the Fourier transform for regular distributions, we have now verified it rigorously. As a special case of this result, we observe that $\widehat{\delta}_0$ corresponds to the distribution which integrates a function over $\mathbb{R}^d$. In other words, $\widehat{\delta}_0$ is Lebesgue measure, viewed as a tempered distribution.

In our applications below we will be considering distributions which are supported on point sets in $\mathbb{R}^d$. Therefore, suppose that $Y \subseteq \mathbb{R}^d$ is such a point set, and that $w : Y \to \mathbb{C}$ is a complex valued function defined on Y, with the property that $|w(y)|$ grows at most polynomially in $|y|$. Then the *weighted Dirac comb* ω defined by w is the tempered distribution given by

$$\omega = \sum_{y \in Y} w(y)\delta_y.$$

The growth condition on w guarantees that this is in fact an element of $\mathcal{S}'(\mathbb{R}^d)$. Of particular importance is the special case when $Y = \mathbb{Z}^d$ and $w(y) = 1$ for all $y \in Y$. We leave it as an exercise (see below) to compute the Fourier transform of the resulting Dirac comb.

Now we return to the problem of defining the Fourier transform of a measure. It is not the case that every measure is a tempered distribution, nor is it the case that every tempered distribution is a measure. Therefore, although these two collections of objects do have a nontrivial intersection, it will take some justification to pass back and forth between the two. In our upcoming application to diffraction, the basic strategy that we would like to employ is to argue that we are working with a measure that is also a tempered distribution (such a measure is called a *tempered measure*), then compute the Fourier transform, then argue that the resulting object is not only a tempered distribution but also a measure.

Fortunately, in our application things are not too complicated. First of all, we will be starting with translation bounded measures which, it is easy to see, also define linear functionals on $\mathcal{S}(\mathbb{R}^d)$. Secondly, our measures will also be *positive definite measures*, which means that, for any $g \in C_c(\mathbb{R}^d)$,

$$\mu(g * \tilde{g}) \geq 0, \tag{5.3.2}$$

where $\tilde{g} \in C_c(\mathbb{R}^d)$ is defined, here and in what follows, by

$$\tilde{g}(x) = \overline{g(-x)}. \tag{5.3.3}$$

In this situation we can appeal to a well known result, the Bochner-Schwartz Theorem, which tells us that if μ is any tempered measure which is a *positive definite tempered distribution* (which means that (5.3.2) holds for all $g \in \mathcal{S}(\mathbb{R}^d)$) then $\widehat{\mu}$ is a positive, translation bounded measure.

EXERCISES

Exercise 5.3.1. *Let $\omega \in \mathcal{S}'(\mathbb{R}^d)$ be the Dirac comb defined by*

$$\omega = \sum_{n \in \mathbb{Z}^d} \delta_n.$$

Prove that

$$\widehat{\omega} = \omega.$$

Exercise 5.3.2. *Prove that the collection of positive definite tempered distributions is a closed subspace of $\mathcal{S}'(\mathbb{R}^d)$.*

5.4 Autocorrelation and diffraction

Finally we are in a position to introduce diffraction. Recall that we are looking for a mathematical definition of diffraction, for measures, which matches the physical observation that the intensity of the diffraction pattern is the modulus squared of the Fourier transform. For measures, one way to arrive at such a definition is to begin by forming an autocorrelation measure, which is a limit of convolutions of finite measures, and then pass to the Fourier transform.

First of all, the convolution of two finite measures μ and ν is the finite measure $\mu * \nu$ defined by

$$(\mu * \nu)(g) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x + y)\ d\mu(x)\ d\nu(y).$$

Functionally, convolutions of finite measures satisfy similar properties as convolutions of functions. For example, the Fourier transform $\widehat{\mu * \nu}$ is easily seen to be the absolutely continuous measure whose Radon-Nikodym derivative is given by

$$\widehat{\mu}\widehat{\nu}.$$

As will become clearer below, this is one justification for pursuing this line of thought, in the scope of our discussion of diffraction, in attempting to make sense of the 'modulus squared of the Fourier transform'.

Now suppose that $\mu \in \mathcal{M}(\mathbb{R}^d)$ is an arbitrary (possibly infinite) measure and for each $R > 0$ let $\mu_R \in \mathcal{M}(\mathbb{R}^d)$ be defined by

$$\mu_R(g) = \int_{B_R(0)} g(x)\ d\mu(x).$$

Then, writing $\tilde{\mu}$ for the measure defined by

$$\tilde{\mu}(g) = \overline{\mu(\tilde{g})},$$

with $\tilde{g}$ given by (5.3.3), define $\gamma_\mu^{(R)} \in \mathcal{M}(\mathbb{R}^d)$ by

$$\gamma_\mu^{(R)} = \frac{\mu_R * \tilde{\mu}_R}{|B_R(0)|},$$

where $|B_R(0)|$ denotes the volume of the ball of radius R centered at 0 in $\mathbb{R}^d$. If μ is a translation bounded measure then it follows that the collection of measures $\{\gamma_\mu^{(R)}\}_{R>0}$ is bounded in $\mathcal{M}(\mathbb{R}^d)$, in the weak-$*$ topology, and that any accumulation point of this sequence is also translation bounded (see [5, Proposition 9.1]). If there is a unique accumulation point of this sequence, then we call it the *autocorrelation measure* of μ and we denote it by γ_μ. It is an important fact that, if μ is translation bounded and if γ_μ exists, then it is a positive definite tempered distribution. This is often taken for granted but, as a test of understanding up to this point, the reader is encouraged to try to prove it in Exercise 5.4.1 below.

Now we arrive at the definition which has motivated most of our analysis in this chapter. Suppose that μ is a translation bounded measure and that its autocorrelation γ_μ exists. Then γ_μ is a translation bounded measure, hence a tempered distribution, and it is also positive definite as a tempered distribution. Therefore, by the comments at the end of the previous section, the Fourier transform $\widehat{\gamma}_\mu$ is a positive, translation bounded measure, called the *diffraction measure* of μ.

EXERCISES

Exercise 5.4.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a translation bounded measure and suppose that the autocorrelation γ_μ exists. Prove that γ_μ is a positive definite tempered distribution.*

Exercise 5.4.2. *Let $Y \subseteq \mathbb{R}^d$ be a uniformly discrete set and let $\omega \in \mathcal{M}(R^d)$ be the Dirac comb defined by*

$$\omega = \sum_{y \in Y} \delta_y.$$

Prove that, for $R > 0$,

$$\omega_R * \tilde{\omega}_R = \sum_{z \in Y - Y} c(z, R)\delta_z,$$

where

$$c(z, R) = \#\{(y, y') : y, y' \in Y \cap B_R(0),\ y - y' = z\}.$$

5.5 Diffraction from cut and project sets

Now we turn to the problem of computing the diffraction of the Dirac comb of a cut and project set. Much of what is in this chapter was originally formulated in a rigorous way by Hof [18], although our presentation more closely follows that in [4, Chapter 9].

Recall the notation of Section 3.2. Suppose that E is a d-dimensional subspace of $\mathbb{R}^k$ which acts minimally on $\mathbb{T}^k$. Let F_π be a complementary subspace satisfying our usual hypothesis that $\pi|_{\mathbb{Z}^k}$ is injective, and suppose that the window $\mathcal{W}_\pi \subseteq F_\pi$ is bounded and has non-empty interior. The choice of s in the definition of our cut and project set makes no difference in the resulting diffraction measure, therefore let $s = 0$ and let $Y = Y_s$ be defined as in (3.2.1). In what follows set $L = \pi(\mathbb{Z}^k)$ and $L^* = \pi^*(\mathbb{Z}^k)$. Then for any $z \in L$ we use $\tilde{z}$ to denote the unique point in $\mathbb{Z}^k$ with the property that $\pi(\tilde{z}) = z$, and we set $z^* = \pi^*(\tilde{z})$.

Let $\omega \in \mathcal{M}(\mathbb{R}^d)$ be the Dirac comb defined by

$$\omega = \sum_{y \in Y} \delta_y.$$

It is clear that ω is translation bounded, so all of our comments from the end of the previous section will apply, as soon as we show that the autocorrelation γ_ω exists.

By Exercise 5.4.2 we have that

$$\gamma_\omega^{(R)} = \sum_{z \in Y - Y} \eta(z, R)\delta_z,$$

with

$$\eta(z, R) = \frac{\#\{(y, y') : y, y' \in Y \cap B_R(0),\ y - y' = z\}}{|B_R(0)|_E}.$$

We have used a subscript E here to indicate that $|B_R(0)|_E$ is the d-dimensional volume of a ball of radius R centered at 0 in E. Now if $y, y' \in Y$ then we have that

$$y - y' = \pi(\tilde{y} - \tilde{y'}) \in L,$$

which shows that the measures $\gamma_\omega^{(R)}$ are supported on L.

Now for any $z \in L$ we have that

$$\begin{aligned}
|B_R(0)|_E \cdot \eta(z, R) &= \#\{y, y' \in Y \cap B_R(0) : y - y' = z\} \\
&= \#\{y, y' \in L \cap B_R(0) : y^*, y'^* \in \mathcal{W}_\pi,\ y - y' = z\} \\
&= \#\{y \in (L \cap B_R(0) \cap B_R(z)) : y^*, y^* - z^* \in \mathcal{W}_\pi\} \\
&= \#\{y \in (L \cap B_R(0) \cap B_R(z)) : y^* \in (\mathcal{W}_\pi \cap (\mathcal{W}_\pi + z^*))\}.
\end{aligned}$$

Now since E acts minimally on $\mathbb{T}^k$, the action is uniquely ergodic and we have that

$$\begin{aligned}
&\#\{y \in (L \cap B_R(0) \cap B_R(z)) : y^* \in (\mathcal{W}_\pi \cap (\mathcal{W}_\pi + z^*))\} \\
&\qquad \sim |B_R(0)|_E \cdot |\mathcal{W}_\pi \cap (\mathcal{W}_\pi + z^*)|_{F_\pi},\ \text{as } R \to \infty,
\end{aligned}$$

where $|\cdot|_{F_\pi}$ denotes $(k-d)$-dimensional volume in F_π. This shows that, for any $z \in L$,

$$\eta(z) := \lim_{R \to \infty} \eta(z, R) = |\mathcal{W}_\pi \cap (\mathcal{W}_\pi + z^*)|_{F_\pi}. \tag{5.5.1}$$

It follows from this formula that the support of the function $\eta : L \to [0, \infty)$ just defined is also a cut and project set, defined using the same data as Y, but with the window $\mathcal{W}_\pi - \mathcal{W}_\pi$. This implies, in particular, that the support of η is a Delone set in E (cf. Exercise 3.2.5).

Finally, it is now straightforward to see that, for any $g \in C_c(\mathbb{R}^d)$,

$$\lim_{R\to\infty} \gamma_\omega^{(R)}(g) = \sum_{z\in L} \eta(z)g(z),$$

and so we have that the limiting autocorrelation of ω exists and is given by

$$\gamma_\omega = \sum_{z\in L} \eta(z)\delta_z.$$

Furthermore, by what we have said previously, γ_ω is a translation bounded measure (hence a tempered distribution), and positive definite tempered distribution. Therefore the diffraction measure $\widehat{\gamma}_\mu$ is a positive, translation bounded measure.

The basic strategy for computing the diffraction measure is to lift $\gamma_\omega \in \mathcal{S}'(E)$ to a tempered distribution in $\mathcal{S}'(\mathbb{R}^k)$ supported on $\mathbb{Z}^k$, apply the Poisson Summation Formula in the form of Exercise 5.3.1, and then marginalize the resulting measure in the E component to obtain a measure on $\mathcal{M}(E)$. In what follows, if $g : E \to \mathbb{C}$ and $h : F_\pi \to \mathbb{C}$ then we write $g \otimes h$ to denote the function from $\mathbb{R}^k$ to $\mathbb{C}$ defined by

$$(g \otimes h)(x) = g(\pi(x))h(\pi^*(x)).$$

First of all, let $\nu \in \mathcal{S}'(\mathbb{R}^k)$ be defined by

$$\nu = \sum_{n\in\mathbb{Z}^k} \eta(\pi(n))\delta_n.$$

For $\sigma > 0$ write $\aleph_\sigma \in \mathcal{S}(F_\pi)$ for Gaussian density defined in Exercise 5.2.1. Then we have, for $g \in \mathcal{S}(E)$, that

$$\nu(\widehat{g} \otimes \widehat{\aleph}_\sigma) \;\overset{\sigma\to 0^+}{\longrightarrow}\; \gamma_\omega(\widehat{g}) \;=\; \widehat{\gamma}_\omega(g), \tag{5.5.2}$$

where, for the limit, we have used the result of Exercise 5.2.2. Now we would like to go further in expanding out the left hand side of this equation but, before we do so, let us return to the definition of $\eta(z)$ from (5.5.1) and notice that it can be rewritten as

$$\eta(z) = \int_{F_\pi} \chi_{\mathcal{W}_\pi}(t)\chi_{\mathcal{W}_\pi}(t-z^*) = (\chi_{\mathcal{W}_\pi} * \widetilde{\chi}_{\mathcal{W}_\pi})(z^*),$$

where $\chi_{\mathcal{W}_\pi}$ denotes the indicator function of $\mathcal{W}$. Therefore let $\xi \in L^1(F_\pi)$ be defined by

$$\xi(x) = (\chi_{\mathcal{W}_\pi} * \widetilde{\chi}_{\mathcal{W}_\pi})(x),$$

so that $\eta(z) = \xi(z^*)$ and, by (5.2.2), we have that

$$\widehat{\xi}(t) = \widehat{\chi}_{\mathcal{W}_\pi}(t) \cdot \widehat{\widetilde{\chi}}_{\mathcal{W}_\pi}(t) = |\widehat{\chi}_{\mathcal{W}_\pi}(t)|^2.$$

Returning to left hand side of equation (5.5.2), we have that

$$\nu(\widehat{g} \otimes \widehat{\aleph}_\sigma) = \sum_{n\in\mathbb{Z}^k} \xi(\pi^*(n))\ \widehat{g}(\pi(n))\ \widehat{\aleph}_\sigma(\pi^*(n)).$$

Now notice that

$$\xi(\pi^*(n))\ \widehat{g}(\pi(n))\ \widehat{\aleph}_\sigma(\pi^*(n)) = (g \otimes (\xi \cdot \widehat{\aleph}_\sigma))(n),$$

and apply the Possion Summation Formula to obtain

$$\begin{aligned}\nu(\widehat{g} \otimes \widehat{\aleph}_\sigma) &= \sum_{n\in\mathbb{Z}^k} g(\pi(n)) \cdot \widehat{\xi \cdot \widehat{\aleph}_\sigma}(\pi^*(n)) \\ &= \sum_{n\in\mathbb{Z}^k} g(\pi(n)) \cdot (\widehat{\xi} * \aleph_\sigma)(-\pi^*(n)).\end{aligned}$$

Here we have also used the fact that

$$\begin{aligned}\widehat{\xi \cdot \widehat{\aleph}_\sigma}(\pi(n)) &= \widehat{\widehat{\check{\xi} * \aleph_\sigma}}(\pi^*(n)) \\ &= (\check{\xi} * \aleph_\sigma)(-\pi^*(n)) \\ &= (\widehat{\xi} * \aleph_\sigma)(-\pi^*(n)).\end{aligned}$$

Continuing our calculation, we have

$$\nu(\widehat{g} \otimes \widehat{\aleph}_\sigma) = \sum_{z\in L} \delta_z(g) \int_{F_\pi} \widehat{\xi}(t+z^*)\aleph_\sigma(t)\ dt.$$

Finally taking the limit as $\sigma \to 0^+$ and using (5.5.2) and Exercise 5.2.1, we have that

$$\widehat{\gamma}_\omega = \sum_{z \in L} \widehat{\xi}(z^*) \cdot \delta_z = \sum_{z \in L} |\widehat{\chi}_{\mathcal{W}_\pi}(z^*)|^2 \cdot \delta_z.$$

This completes our calculation of the diffraction measure of Y, showing that it is a pure point measure supported on the Delone set $Y - Y$.

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