Fourier analysis, measures, and distributions

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§1 Mathematics of diffraction

Physical diffraction

- As a physical phenomenon, diffraction refers to interference of waves passing through some medium or aperture.
- 'Pure point' diffraction (i.e. sharp peaks in the diffraction pattern) is an indication of order in the atomic or molecular structure of a material.

Mathematical descriptions of diffraction

- Under one commonly used assumption, called the Fraunhofer far field limit, the intensity of the diffraction pattern is given by the modulus squared of the Fourier transform of the indicator function of the aperture.

- In physical applications it is sometimes desirable to describe the ‘aperture’ as a measure. This point of view is flexible enough to accommodate both situations where we have a continuous distribution in space, and where we have an array of point particles.

Defining the diffraction of a measure

At the outset we face several challenges with this measure theoretic approach to diffraction:

(i) The commonly used definition of the Fourier transform of a finite measure, is not well defined for infinite measures.

(ii) Even with the right definition, the Fourier transform of an infinite measure may not be a measure.

(iii) We have to find a reasonable way to interpret what is meant by the ‘modulus squared’ of a measure.

§2 Review of Fourier analysis

Important function spaces

- $C_c(\mathbb{R}^d)$ denotes the vector space of complex valued continuous functions on $\mathbb{R}^d$ with compact support.
- $\mathcal{S}(\mathbb{R}^d)$ denote the **Schwartz space** on $\mathbb{R}^d$, which is the vector space of complex-valued C^∞ functions on $\mathbb{R}^d$ whose higher order multiple derivatives all tend to zero as $|x| \to \infty$ faster than $|x|^{-r}$, for any $r \geq 1$.
- Both of these vector spaces can be made into topological spaces in a natural way.

Definition of the Fourier transform and its inverse

- The **Fourier transform** of a function $\phi \in L^1(\mathbb{R}^d)$ is defined by
$$(\mathcal{F}\phi)(t) = \widehat{\phi}(t) = \int_{\mathbb{R}^d} \phi(x) e(-x \cdot t)\, dx,$$
where $e(x) = \exp(2\pi i x)$.

- The **inverse Fourier transform** of a function $\psi \in L^1(\mathbb{R}^d)$ is defined by
$$(\mathcal{F}^{-1}\psi)(x) = \check{\psi}(x) = \int_{\mathbb{R}^d} \psi(t) e(t \cdot x)\, dt.$$

Fourier Inversion Formula

- **Fourier Inversion Formula**: If ϕ is a continuous function in $L^1(\mathbb{R}^d)$ and if $\mathcal{F}(\phi) \in L^1(\mathbb{R}^d)$, then

$$\mathcal{F}^{-1}(\mathcal{F}\phi) = \phi.$$

- The Fourier transform is a linear map, which provides a bijection from $\mathcal{S}(\mathbb{R}^d)$ to itself, with $\mathcal{F}^{-1}$ being the inverse map.

Fourier series and the Poisson Summation Formula

- Functions $\phi \in \mathcal{S}(\mathbb{R}^d)$ which are periodic modulo $\mathbb{Z}^d$ (i.e. so that $\phi(x+n) = \phi(x)$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$) also have a **Fourier series expansion**

$$\phi(x) = \sum_{m \in \mathbb{Z}^k} c_\phi(m) e(m \cdot x),$$

where

$$c_\phi(m) = \int_{[0,1)^d} \phi(x) e(-m \cdot x)\, dx.$$

- **Poisson Summation Formula**: For any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have that

$$\sum_{n \in \mathbb{Z}^d} \phi(n) = \sum_{n \in \mathbb{Z}^d} \widehat{\phi}(n).$$

Convolution of functions

- If ψ and ϕ are in $L^1(\mathbb{R}^d)$ then their **convolution** is the function $\phi * \psi \in L^1(\mathbb{R}^d)$ defined by

$$(\phi * \psi)(x) = \int_{\mathbb{R}^d} \phi(t)\psi(x - t)\, dt.$$

- It is straightforward to show that

$$\phi * \psi = \psi * \phi$$

and that

$$\widehat{\phi * \psi} = \widehat{\phi}\widehat{\psi}.$$

Exercises from lecture notes

(5.2.1) For $\sigma > 0$, let $\aleph_\sigma \in \mathcal{S}(\mathbb{R}^d)$ be the d-dimensional Gaussian density defined by

$$\aleph_\sigma(x) = \frac{1}{(2\pi)^{d/2}\sigma^d} \cdot \exp\left(\frac{-|x|^2}{2\sigma^{2d}}\right).$$

Prove that if $f \in L^1(\mathbb{R}^d)$ is continuous at $x = 0$ then

$$f(0) = \lim_{\sigma \to 0^+} \int_{\mathbb{R}^d} f(x)\aleph_\sigma(x)\, dx.$$

(5.2.2) Prove that for every $x \in \mathbb{R}^d$, the sequence $\{\widehat{\aleph}_{1/n}(x)\}_{n\in\mathbb{N}}$ is increasing and converges to 1.

§3 The space of complex regular measures on $\mathbb{R}^d$

Positive regular measures as linear functionals

- A positive regular Borel measure μ on $\mathbb{R}^d$ defines a linear functional on $C_c(\mathbb{R}^d)$ by the rule that, for $g \in C_c(\mathbb{R}^d)$,

$$\mu(g) = \int_{\mathbb{R}^d} g(x)\, d\mu(x).$$

- **Riesz-Markov-Kakutani Representation Theorem**: If F is a positive (real valued) linear functional on $C_c(\mathbb{R}^d)$ then F is determined, in the manner mentioned above, by a positive regular Borel measure.

Complex regular measures as linear functionals

- Now consider the collection of all complex valued linear functionals F on $C_c(\mathbb{R}^d)$ satisfying the condition that for every compact K there exists a c_K such that, for all $g \in C_c(\mathbb{R}^d)$ with support in K,

$$|F(g)| \leq c_K \|g\|_\infty.$$

- By an extended form of the Riesz-Markov-Kakutani Representation Theorem, each such functional is determined, in the way above, by a linear combination of the form

$$\mu^+ - \mu^- + i(\nu^+ - \nu^-),$$

  where μ^+, μ^-, ν^+, and ν^- are positive regular Borel measures. Such a linear combination is called a **complex measure.**

Definitions and terminology, part 1

- If μ is a measure on $\mathbb{R}^d$ then the **conjugate** of μ is the measure $\overline{\mu}$ defined, for $g \in C_c(\mathbb{R}^d)$, by

$$\overline{\mu}(g) = \overline{\mu(\overline{g})}.$$

- μ is **real** if $\overline{\mu} = \mu$ and it is **positive** if it is real and if $\mu(g) \geq 0$ whenever $g \geq 0$.

Definitions and terminology, part 2

- The **total variation measure** $|\mu|$ of μ is defined to be the smallest positive measure such that, for all $g \geq 0$,

$$|\mu|(g) \geq |\mu(g)|.$$

- μ is **translation bounded** if

$$\sup_{x \in \mathbb{R}^d} |\mu|(x + K) < \infty,$$

for all compact $K \subseteq \mathbb{R}^d$, and it is **finite** if $|\mu|(\mathbb{R}^d) < \infty$.

Complex measures as a topological space

- The collection of all complex regular Borel measures on $\mathbb{R}^d$, which we denote by $\mathcal{M}(\mathbb{R}^d)$, becomes a topological space with the weak-$*$ topology which it inherits from $C_c(\mathbb{R}^d)$.
- Explicitly, a sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$ converges as $n \to \infty$ to μ in the weak-$*$ topology if and only if

$$\lim_{n\to\infty} \mu_n(g) = \mu(g)$$

for every $g \in C_c(\mathbb{R}^d)$.

Fourier transform of a finite measure

- The Fourier transform of a finite measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is defined to be the measure which is absolutely continuous with respect to Lebesgue measure, whose Radon-Nikodym derivative is given by

$$\widehat{\mu}(t) = \int_{\mathbb{R}^d} e(-t \cdot x)\, d\mu(x).$$

- As mentioned, this definition does not generalize well to infinite measures.

§4 Tempered distributions

Space of tempered distributions on $\mathbb{R}^d$

- The space of **tempered distributions** is the space of complex valued linear functionals on $\mathcal{S}(\mathbb{R}^d)$, which we denote by $\mathcal{S}'(\mathbb{R}^d)$.

- Similar to the space of measures, we take $\mathcal{S}'(\mathbb{R}^d)$ to be equipped with its weak-$*$ topology. To be clear, a sequence $\{T_n\}_{n\in\mathbb{N}}$ of tempered distributions converges to $T \in \mathcal{S}'(\mathbb{R}^d)$ if and only if

$$\lim_{n\to\infty} T_n(\phi) = T(\phi)$$

for every $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Regular distributions

- An important subspace of $\mathcal{S}'(\mathbb{R}^d)$ is the space of **regular distributions**, which are defined, for each continuous function g with at most polynomial growth, by

$$T_g(\phi) = \int_{\mathbb{R}^k} \phi(x)g(x)\, dx.$$

- The space of regular distributions is dense in $\mathcal{S}'(\mathbb{R}^d)$, but not all tempered distributions are regular distributions. For example, if $x \in \mathbb{R}^d$ then the **Dirac delta** distribution $\delta_x \in \mathcal{S}'(\mathbb{R}^d)$, defined by

$$\delta_x(\phi) = \phi(x),$$

is not a regular distribution.

Fourier transforms of tempered distributions

- If T is a regular distribution then the **Fourier transform** of T is the tempered distribution $\widehat{T}$ defined by

$$\widehat{T}(\phi) = T(\hat{\phi}).$$

- The Fourier transform thus defined extends to a unique continuous function on all of $\mathcal{S}'(\mathbb{R}^d)$.

First example

- Exercise: Prove, using continuity of the Fourier transform, that for $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$\widehat{\delta_x}(\phi) = \int_{\mathbb{R}^d} e(-x \cdot t)\phi(t)\, dt.$$

In other words, $\widehat{\delta_x}$ is the regular distribution defined by the function $e(-x \cdot t)$.

- It follows, for example, that

$$\widehat{\delta_0} = \lambda,$$

where λ denotes Lebesgue measure on $\mathbb{R}^d$, viewed as a tempered distribution.

Second example

- Suppose that $Y \subseteq \mathbb{R}^d$ is such a point set, and that $w : Y \to \mathbb{C}$ is a complex valued function defined on Y, with the property that $|w(y)|$ grows at most polynomially in $|y|$. Then the **weighted Dirac comb** ω defined by w is the tempered distribution given by

$$\omega = \sum_{y \in Y} w(y)\delta_y.$$

  The growth condition on w guarantees that this is in fact an element of $\mathcal{S}'(\mathbb{R}^d)$.

- For example suppose that $Y = \mathbb{Z}^d$ and $w(y) = 1$ for all $y \in Y$. In this case, what is $\widehat{\omega}$?

Moving back and forth between distributions and measures

- It is not the case that every measure is a tempered distribution, nor is it the case that every tempered distribution is a measure. A measure that is also a tempered distribution is called a **tempered measure.**
- If μ is a tempered measure which is also a **positive definite tempered distribution** (to be defined on the next slide), then $\widehat{\mu}$ is a positive, translation bounded measure.

Positive definite measures and distributions

- First of all, for any function g on $\mathbb{R}^d$, let $\tilde{g}$ be the function defined by
$$\tilde{g}(x) = \overline{g(-x)}.$$
- A measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is a **positive definite measure** if, for any $g \in C_c(\mathbb{R}^d)$,
$$\mu(g * \tilde{g}) \geq 0.$$
- Similarly, a tempered distribution is a **positive definite tempered distribution** if the above equation holds for all $g \in \mathcal{S}(\mathbb{R}^d)$.

Exercise

(5.3.2) Prove that the collection of positive definite tempered distributions is a closed subspace of $\mathcal{S}'(\mathbb{R}^d)$.

§5 Decomposition of measures

Absolute continuity and the Radon-Nikodym Theorem

- For $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$, we say that μ is **absolutely continuous** with respect to ν if there is a continuous function f with the property that
$$\int_K |f(x)| \, d\nu(x) < \infty$$
for all compact measurable sets K, and such that
$$\mu(g) = \nu(fg) \text{ for all } g \in C_c(\mathbb{R}^d).$$
- In this case the function f is called the **Radon-Nikodym derivative** of μ with respect to ν.
- **Radon-Nikodym Theorem**: μ is absolutely continuous with respect to ν if and only if $\mu(A) = 0$ whenever $\nu(A) = 0$ for a measurable set A.

Singular measures and a decomposition theorem

- At the extreme opposite from absolute continuity, we say that μ is **singular** with respect to ν if there is a measurable set A for which $\mu(A) = \nu(\mathbb{R}^d \setminus A) = 0$.
- We can write any regular Borel measure μ as a sum

$$\mu = \mu_{\text{ac}} + \mu_{\text{sing}},$$

where μ_{ac} is absolutely continuous with respect to Lebesgue measure, and μ_{sing} is singular with respect to Lebesgue measure.

Pure point part of a measure

- We can decompose μ further by defining its **pure points** to be

$$P_\mu = \{x \in \mathbb{R}^d : \mu(\{x\}) > 0\}.$$

- Define the (singular) measure μ_{pp} by

$$\mu_{\text{pp}}(A) = \sum_{x \in A \cap P_\mu} \mu(\{x\}).$$

- Writing

$$\mu_{\text{sc}} = \mu_{\text{sing}} - \mu_{\text{pp}}$$

  for the **singular continuous** part of μ, we have

$$\mu = \mu_{\text{ac}} + \mu_{\text{sc}} + \mu_{\text{pp}}. \tag{1}$$

- If $\mu = \mu_{\text{pp}}$ then μ is a **pure point measure.**