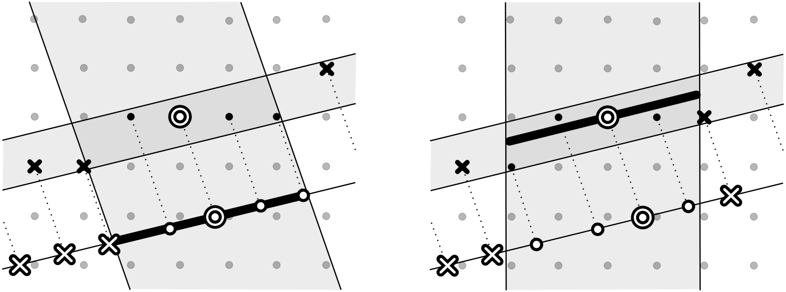
Patches in cut and project sets



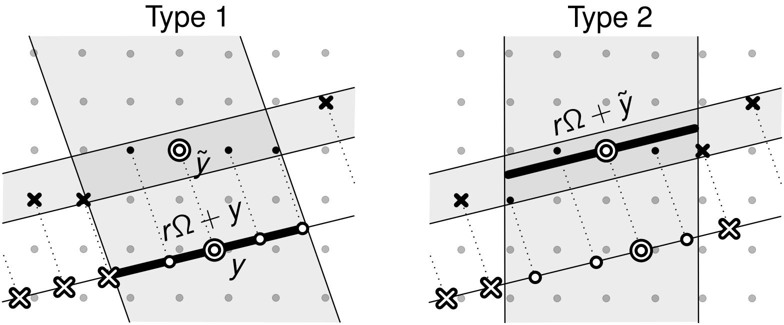
Alan Haynes

§1 Patches, complexity, and repetitivity

Two types of patches

- Let $\Omega \subseteq E$ be a bounded convex set with 0 in its interior.
- For each $y \in Y$ and $r > 0$, the **type 1 patch of size r at y** is
$$P_1(y, r) = Y \cap (y + r\Omega).$$
- Writing $\tilde{y}$ for the point in $\mathcal{S} \cap (\mathbb{Z}^k + s)$ with $\pi(\tilde{y}) = y$, we define the **type 2 patch of size r at y** of size r at y by
$$P_2(y, r) := \{y' \in Y : \rho(\tilde{y}' - \tilde{y}) \in r\Omega\}.$$

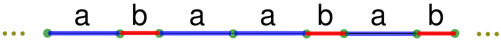
Jamie Walton's picture

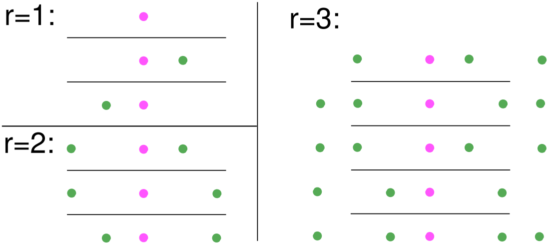


Example: Type 1 patches for the Fibonacci tiling

$$a \mapsto a\,b$$
$$b \mapsto a$$

$\Omega =$ 





Differences between type 1 and type 2 patches

- Since $\mathcal{W}$ is bounded, the type 1 and type 2 patches of size r at y differ at most within a (fixed) constant neighborhood of the boundary of $y + r\Omega$.
- Type 1 patches are more geometric, but type 2 patches are substantially easier to work with. So the strategy is to prove results about type 2 patches, and transfer them to results about type 1 patches.

Equivalence classes and complexity

- For $i = 1, 2$, $r > 0$, and $y_1, y_2 \in Y$, we say that $P_1(y_1, r)$ and $P_1(y_2, r)$ are equivalent if

$$P_i(y_1, r) = P_i(y_2, r) + y_1 - y_2.$$

  Denote the equivalence class of $P_i(y, r)$ by $\mathcal{P}_i(y, r)$.

- For $i = 1, 2$, the **complexity function** $p_i : [0, \infty) \to \mathbb{N}$ is defined by taking $p_i(r)$ to be the number of equivalence classes of patches of type i of size r.

Observations about the complexity function

A couple of things to note:

- The number of equivalence classes of size r is always finite (Y has **finite local complexity**).
- If E acts minimally on $\mathbb{T}^k$ then $p_i(r)$ does not depend on the (nonsingular) choice of s.

Repetitivity

- Cut and project sets satisfying our basic conditions are **repetitive**: for all $y \in Y, r \geq 0$, there is an $R > 0$ such that every ball $B_R(x) \subseteq E$ contains a point $y' \in Y$ with
$$P_i(y', r) \in \mathcal{P}_i(y, r).$$
- For $i = 1, 2$ define the **repetitivity function** $R_i : [0, \infty) \to \mathbb{R}$: $R_i(r)$ is the smallest real number with the property that every ball of radius $R_i(r)$ in E contains the distinguished point of a patch from every equivalence class of type i patches of size r.
- We say that Y is **linearly repetitive** (LR) if there exists a constant $C > 0$ such that, for all $r \geq 1$,
$$R_i(r) \leq Cr.$$

LR cubical cut and project sets

Theorem (H., Koivusalo, Walton): A k to d cubical cut and project set defined by linear forms $\{L_i\}_{i=1}^{k-d}$ is LR if and only if

(LR1) The sum of the ranks of the kernels of the maps $\mathcal{L}_i : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ defined by

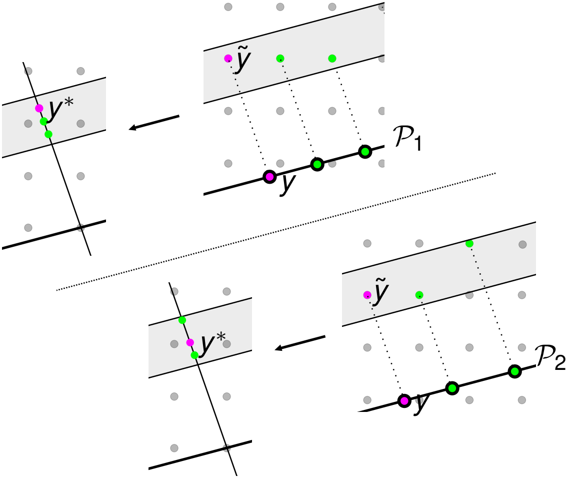
$$\mathcal{L}_i(n) = L_i(n) \mod 1$$

is equal to $d(k-d-1)$, and

(LR2) Each L_i is **relatively badly approximable.**

§2 Dynamical coding of patches

Patterns, lifted patterns, and their projections to F



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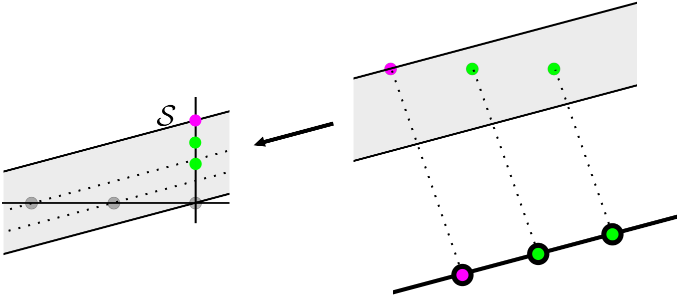

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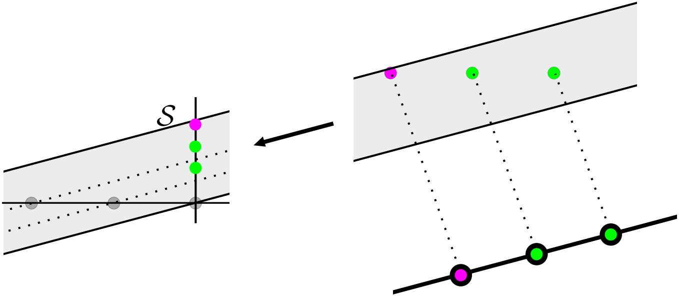

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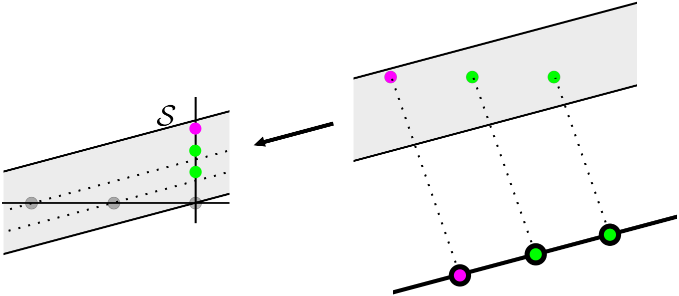

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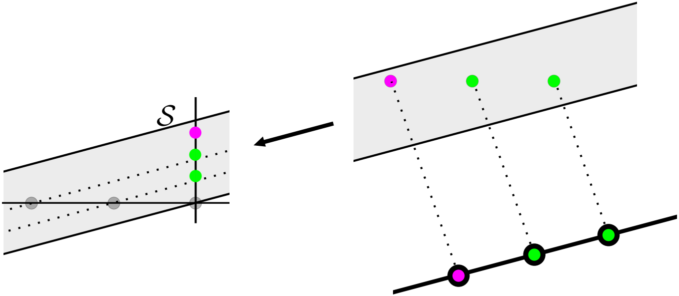

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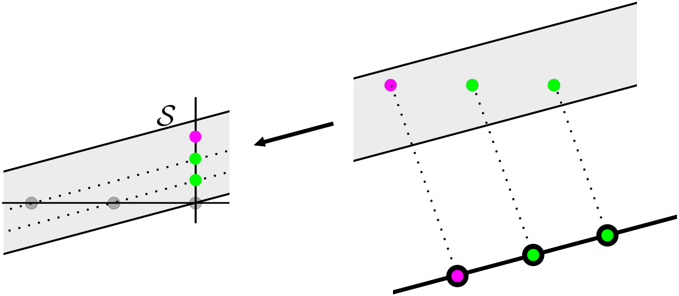

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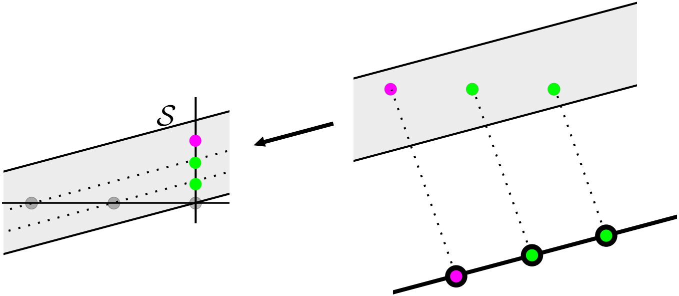

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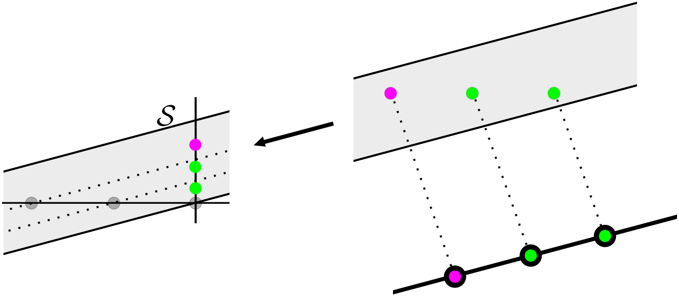

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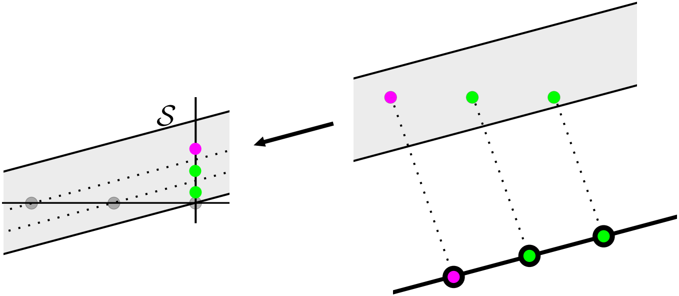

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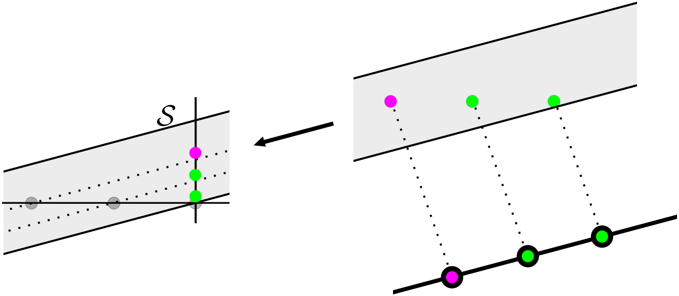

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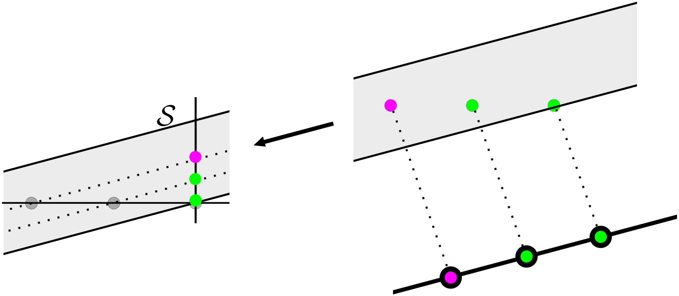

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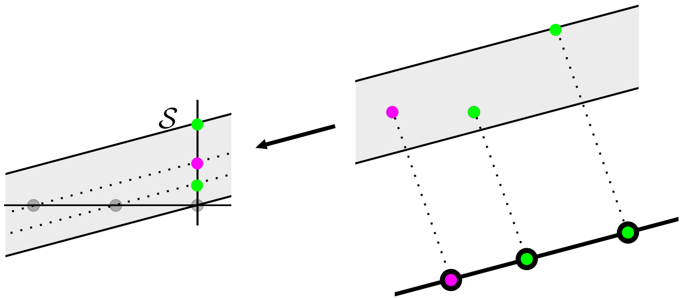

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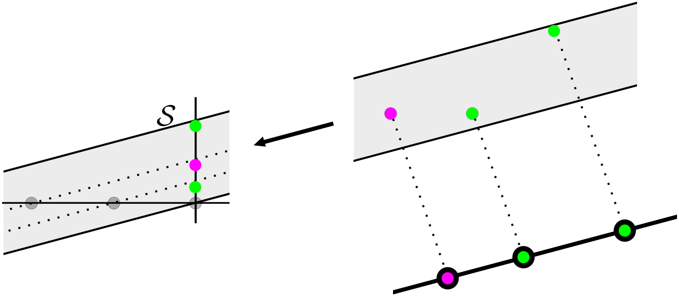

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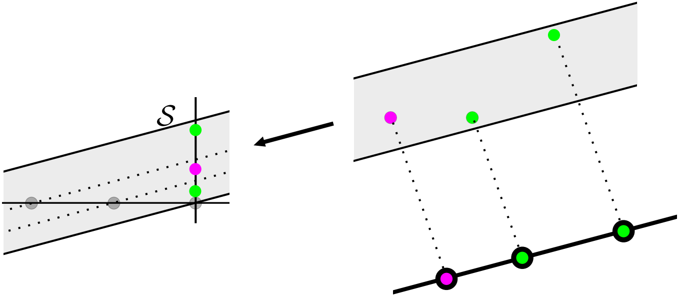

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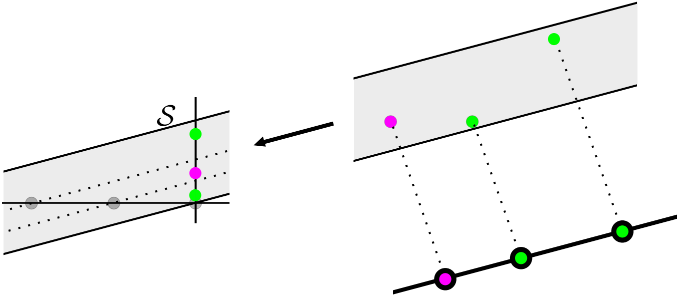

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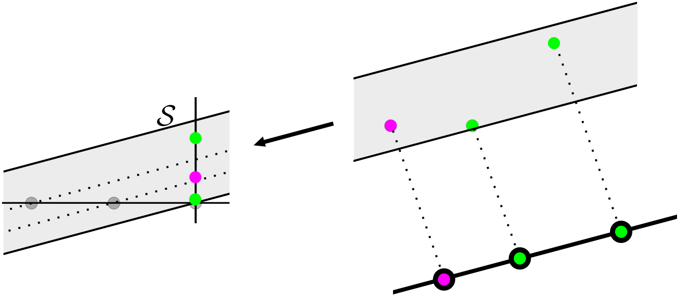

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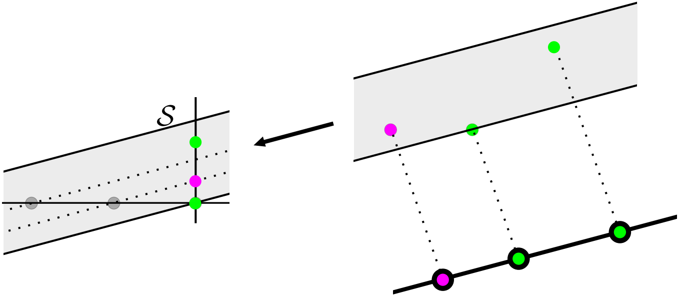

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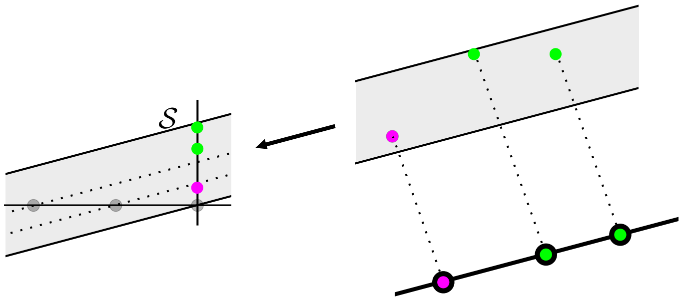

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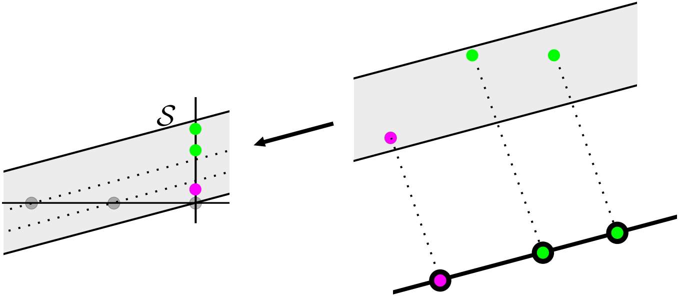

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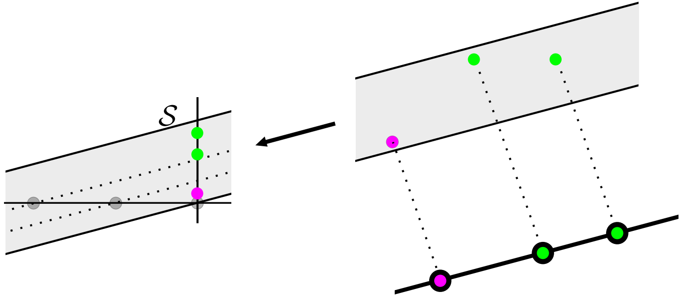

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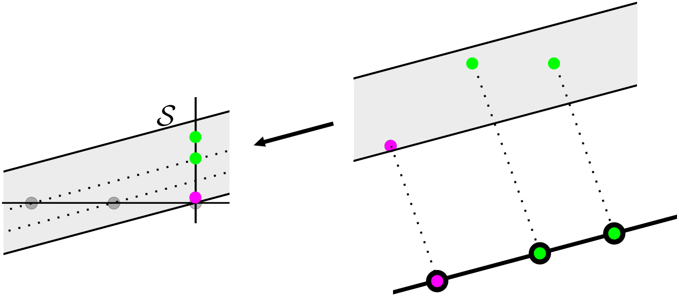

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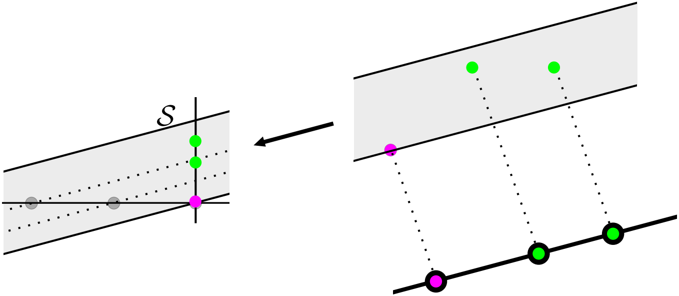


Singular and regular points

There is a natural action of $\mathbb{Z}^k$ on F_ρ, given by

$$n.w = \rho^*(n) + w = w + (0, n_2 - L(n_1)),$$

for $n = (n_1, n_2) \in \mathbb{Z}^k = \mathbb{Z}^d \times \mathbb{Z}^{k-d}$ and $w \in F_\rho$. For each $r \geq 0$ we define the *r-singular points of type* 1 by

$$\mathrm{sing}_1(r) := \mathcal{W} \cap \left((-\pi^{-1}(r\Omega) \cap \mathbb{Z}^k).\partial\mathcal{W} \right),$$

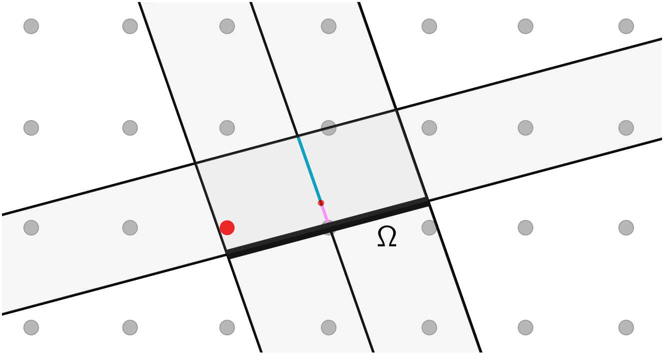
and, similarly, the **r-singular points of type** 2 by

$$\mathrm{sing}_2(r) := \mathcal{W} \cap \left((-\rho^{-1}(r\Omega) \cap \mathbb{Z}^k).\partial\mathcal{W} \right).$$

For $i = 1$ or 2 we define the **r-nonsingular points of type i** by

$$\mathrm{nsing}_i(r) := \mathcal{W} \setminus \mathrm{sing}_i(r).$$

Singular and regular points



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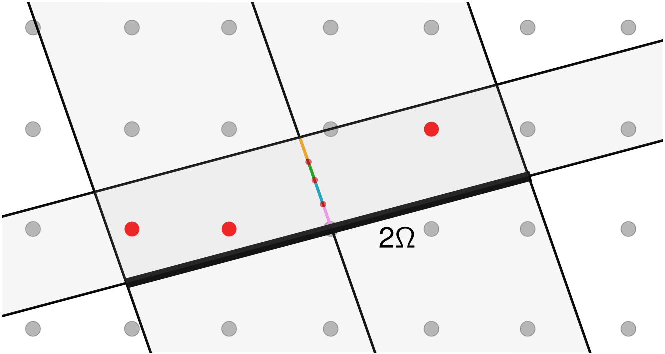

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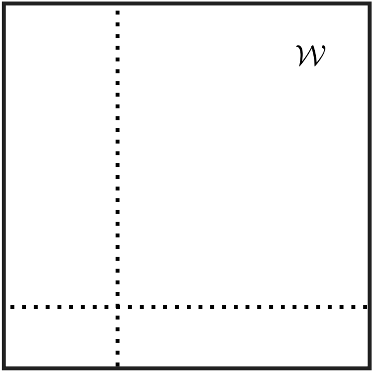

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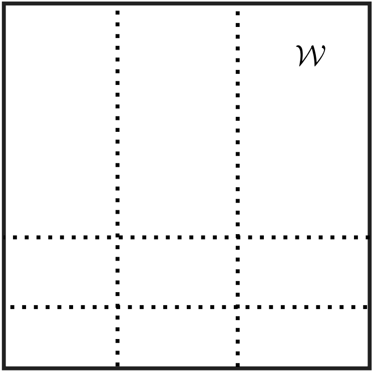

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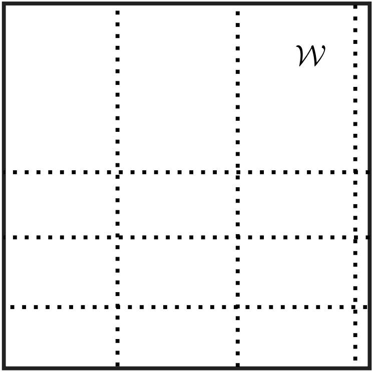

Correspondence with patterns, part 1

Lemma: Let $i = 1$ or 2, suppose that E acts minimally on $\mathbb{T}^k$, and suppose that $Y = Y_s$ is nonsingular. Suppose that U is any connected component of $\mathrm{nsing}_i(r)$. Then for any points $y, y' \in Y$,

$$\text{if} \quad \rho^*(\tilde{y}), \rho^*(\tilde{y'}) \in U \quad \text{then} \quad \mathcal{P}_i(y, r) = \mathcal{P}_i(y', r).$$

Remark: The reverse implication is not true, in general.

Proof of lemma

For each $y \in Y$, let $y^* = \rho^*(\tilde{y}) \in \mathcal{W}$. The point y^* determines the pattern around y, as follows. Each point $y' \in Y$ lifts to a point $\tilde{y}' = \tilde{y} + n$. But such a point is in $\mathcal{S}$ if and only if $\pi^*(\tilde{y}') = n.y^*$ lies in $\mathcal{W}$. As we vary y^*, the pattern around y can only change when some $n.y^*$ passes through $\partial\mathcal{W}$, that is when y^* passes from one connected component of $\mathrm{nsing}_i(r)$ to another. The only difference between $i = 1$ and $i = 2$ is the set of n's being considered. In both cases, each connected component of $\mathrm{nsing}_i(r)$ corresponds to a single equivalence class of patches.

Correspondence with patterns, part 2

Lemma: Suppose that E acts minimally on $\mathbb{T}^k$, that Y is non-singular, and that $\mathcal{W}$ is a parallelotope generated by integer vectors. Then for every equivalence class $\mathcal{P}_2 = \mathcal{P}_2(y, r)$ of type 2 patches, there is a unique connected component U of $\mathrm{nsing}_2(r)$ with the property that, for any $y' \in Y$,

$$\mathcal{P}_2(y', r) = \mathcal{P}_2(y, r) \text{ if and only if } \rho^*(\tilde{y'}) \in U.$$

Remark: The conclusion of the Lemma is not true, in general, for type 1 patches.

Proof of lemma

Suppose that y_1 and $y_2 \in Y$, and that $P_2(y_1, r)$ is equivalent to $P_2(y_2, r)$. Imagine varying y^* in a straight line from y_1^* to y_2^*. In moving y^* from one connected component to another, the patch $P_2(y, r)$ gains and/or loses points whenever y^* crosses from one component to another. We will show that none of the points of $P_2(y_1, r)$ may be removed in going from y_1^* to y_2^*, and that no points may be added without removing other points. Combining these observations, no points can be added or removed, so y_1^* and y_2^* must lie in the same component.

Proof of lemma

No points may be removed: $\mathcal{W}$ is convex. Thus, for each n for which $\pi(\tilde{y}_i + n)$ is in $P_2(y_i, r)$, the set of points y^* satisfying $n.y^* \in \mathcal{W}$ is convex. Since $n.y_1^*$ and $n.y_2^*$ are in $\mathcal{W}$, all points on the line segment connecting them must also be in $\mathcal{W}$. Thus all points y^* on the line segment correspond to patches that contain a translate of $P_2(y_i, r)$.

No points may be added: $\mathcal{W}$ is a parallelotope generated by integer vectors, after possibly modifying a subset of its boundary it is a fundamental domain for a sublattice of $\mathbb{Z}^k \cap F_\rho$, of some index I. This implies that for each $n_1 \in \mathbb{Z}^d$, and each $\tilde{y}$, there are exactly I points $n_2 \in \mathbb{Z}^{k-d}$ such that $\tilde{y} + (n_1, n_2) \in \mathcal{S}$. In other words, as we cross a boundary between connected components, a point is removed from $P_2(y, r)$ for each point added. But, we have already shown that this can't happen.

Correspondence with patterns, part 3

Lemma: With notation as above, suppose that $\mathcal{W}$ is cubical and that $(\alpha_{ij})_{j=1}^{d} \in \mathcal{B}_{d,1}$ for each $1 \leq i \leq k - d$. Then there exist constants $c_1, c_2 > 0$ such that, for all $r > 0$, every element connected component of $\mathrm{nsing}_1(r)$ is a union of at most c_1 connected components of $\mathrm{nsing}_2(r + c_2)$.

Correspondence with patterns, part 4

Open problem: Give an example of a cut and project set satisfying the hypotheses of the previous lemma, but without the badly approximable hypothesis, for which there is no uniform bound on the number of connected components corresponding to each equivalence class of type 1 patches.