Practice problems

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Problem 1

A subspace E of $\mathbb{R}^k$ acts minimally on $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ if

$$\overline{E + \mathbb{Z}^k} = \mathbb{R}^k.$$

Give an example of an integer $k \in \mathbb{N}$ and a subspace E of $\mathbb{R}^k$ with the properties that:

(i) E acts minimally on $\mathbb{R}^k$, and

(ii) E contains an element of $\mathbb{Q}^k \setminus \{0\}$.

Problem 2

Give an example of an integer $k \in \mathbb{N}$ and a subspace E of $\mathbb{R}^k$ with the properties that:

(i) E does not act minimally on $\mathbb{R}^k$, and

(ii) E does not contain an element of $\mathbb{Q}^k \setminus \{0\}$.

Problem 3

Let $\varrho : S^d \to \mathbb{R}^d$ be the stereographic projection. For point sets $Y_1, Y_2 \subseteq \mathbb{R}^d$, we define

$$d(Y_1, Y_2) = d_H\left(\overline{\varrho^{-1}(Y_1)}\,,\ \overline{\varrho^{-1}(Y_2)}\right),$$

where d_H is the Hausdorff metric on S^d.

Take $d = 1$ and let

$$Y = -2\mathbb{N} \cup \{0\} \cup \mathbb{N} = \{\ldots - 6, -4, -2, 0, 1, 2, 3, \ldots\}.$$

Describe the topology of the space

$$\overline{\{Y + x : x \in \mathbb{R}\}},$$

where the closure is taken with respect to the metric d above.

Problem 4

A point set $Y \in \mathbb{R}^k$ has *n*-fold symmetry if there is an element $A \in \mathrm{SO}_k(\mathbb{R})$ of order n which stabilizes Y (i.e. with the property that $AY = Y$).

Give an example of a lattice in $\mathbb{R}^6$ which possesses 15-fold symmetry.

Problem 5

A crystallographic point set $\Gamma \subseteq \mathbb{R}^k$ is a set which can be written as

$$\Gamma = \Lambda + F,$$

where Λ is a lattice in $\mathbb{R}^k$ and $F \subseteq \mathbb{R}^k$ is a finite set.

Prove that every crystallographic point set can be obtained as a cut and project set.

Problem 6

Prove that a crystallographic point set Y in $\mathbb{R}^2$ or $\mathbb{R}^3$ can have only $1, 2, 3, 4,$ or 6-fold rotational symmetry.

Hint: Prove that if A is a rotation of $\mathbb{R}^k$ which maps Y into itself, then it must stabilize the group of periods of Y.

Problem 7

Suppose that Y is a cut and project set formed from a physical space which acts minimally, using a bounded window with non-empty interior, and a projection map π with the property that $\pi|_{\mathbb{Z}^k}$ is injective.

Prove that $Y - Y$ is also a cut and project set, and explain why it is a Delone set.

Problem 8

Let Λ be a lattice in $\mathbb{R}^d$, let and let Λ^* denote the dual lattice to Λ, which is defined by

$$\Lambda^* = \{\lambda^* \in \mathbb{R}^d : (\lambda \cdot \lambda^*) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

Prove that, for any $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\sum_{\lambda \in \Lambda} \phi(\lambda) = |\mathrm{covol}(\Lambda)|^{-1} \sum_{\lambda^* \in \Lambda^*} \widehat{\phi}(\lambda^*),$$

where $\mathrm{covol}(\Lambda)$, called the covolume of Λ, is the volume of any measurable fundamental domain for $\mathbb{R}^d/\Lambda$.

Problem 9

Let $\mu \in \mathcal{M}(\mathbb{R})$ be the restriction of Lebesgue measure to the unit interval $[0, 1)$, and let $\omega \in \mathcal{M}(\mathbb{R})$ be the tempered measure defined by

$$\omega = \sum_{n \in \mathbb{Z}} \delta_n.$$

Compute the Fourier transform

$$\widehat{\mu * \omega}.$$