TILING SPACES AND CUT AND PROJECT SETS

Instructions: Divide into groups of 2 or 3 people each, and choose one of the following two topics to work through. The exercises under Topic I are designed to help gain an understanding of the topology of tiling spaces, with a goal of making precise the statement that 'tiling spaces are Cantor set fiber bundles.' The exercises under Topic II introduce basic properties of cut and project sets, with an emphasis on studying patterns and the dual problem of understanding orbits of points of a related dynamical system on the internal space.

Topic I: Topology of tiling spaces

In this topic we will study metric spaces composed of tilings of Euclidean space. Good references for the material in this section are [5, 6].

For all tilings under consideration we will assume throughout that:

(i) All of the tiles are poyhedra and, in any given tiling, only a finite number of tile types are allowed, up to translation.

(ii) All tiles meet full face to full face. In other words, a face of one tile can not partially overlap with a face of another.

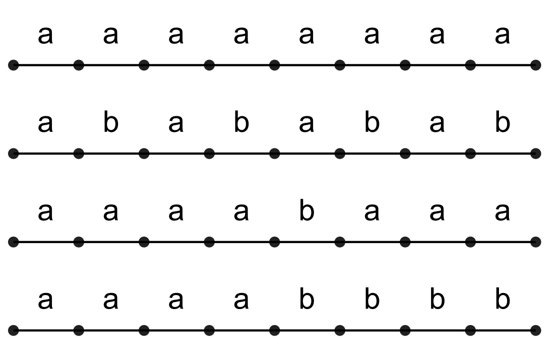
For any two tilings T and T' of $\mathbb{R}^d$, let $\partial(T, T')$ be the infimum of the set of $\epsilon > 0$ with the property that there are vectors $x, x' \in \mathbb{R}^d$ with $|x|, |x'| < \epsilon/2$, and such that the tilings $T+x$ and $T'+x'$ agree on a ball of radius $1/\epsilon$ centered at the origin. Then the function d defined by

$$d(T, T') = \min\{1, \partial(T, T')\}$$

defines a metric, called the **tiling metric**, on the collection of all tilings of $\mathbb{R}^d$. A **tiling space** is a collection of tilings of $\mathbb{R}^d$ which is closed under translation by elements of $\mathbb{R}^d$, and complete in the tiling metric. For any fixed tiling T of $\mathbb{R}^d$, the **hull** of T, denoted Ω_T, is defined to be the smallest tiling space containing T. It consists of the closure with respect to d of the collection of all translates of T.

(P1) Prove that the hull of any tiling is compact.

(P2) Describe the hulls of each the following tilings of $\mathbb{R}$:



Next, suppose that $X_1, X_2, \ldots$ are topological spaces and that, for each $i \geq 1$, we are given a continuous map $f_i : X_{i+1} \to X_i$. The **inverse limit** of the spaces

X_i with respect to the maps f_i is defined by

$$X_\infty = \varprojlim X_i = \left\{(x_1, x_2, \ldots) \in \prod_{i\in\mathbb{N}} X_i : f_i(x_{i+1}) = x_i \text{ for all i}\right\}.$$

The inverse limit is assumed to have the subspace topology inherited from the product topology on $\prod X_i$.

(P3) For each $i \geq 1$ let X_i be a finite, discrete space with 2^i elements. Find a collection of maps $\{f_i\}$ as above, with respect to which the inverse limit X_∞ is homeomorphic to the 'middle third' Cantor set.

(P4) For each i let $X_i = \mathbb{R}/\mathbb{Z}$ and let $f_i(x) = 2x$ mod 1. Prove that X_∞ is locally homeomorphic to the product of an interval and a Cantor set.

(P5) Let T be a tiling of $\mathbb{R}$ with two tiles a and b, determined by the substitution rule $a \mapsto ab$ and $b \mapsto aab$. Write Ω_T as an inverse limit of 'bouquets of two circles,' with connecting maps f_i determined by the substitution.

(P6) Explain why the hull of any tiling is locally homeomorphic to a 'Euclidean ball with Cantor set fibers'.

(P7) Let T be a tiling of $\mathbb{R}^2$. Prove that there is a homeomorphism of $\mathbb{R}^2$ which maps T to a tiling T', with the exact same pattern of tiles as T, and with the property that all of the vertices of T' are elements of $\mathbb{Q}^2$.

(P8) Show that the homeomorphism in (P7) can be chosen so that it extends to a homeomorphism from Ω_T to $\Omega_{T'}$.

(P9) Using the result from (P8), prove that a tiling space constructed from a tiling of $\mathbb{R}^2$ is a fiber bundle over a torus, with totally disconnected fibers.

The result from (P9) was proved in [6, Theorem 1], for tiling spaces of Euclidean space of any dimension.

Topic II: Introduction to cut and project sets

In this topic we will study cut and project sets. Some references for the material found here are [2, 3].

Let E (the **physical space**) be a d-dimensional subspace of $\mathbb{R}^k$, and let F_π (the **internal space**) be a subspace of $\mathbb{R}^k$ complementary to E, so that $E \cap F_\pi = \{0\}$ and $\mathbb{R}^k = E + F_\pi$. Write π for the projection onto E with respect to this decomposition. Choose a set $\mathcal{W}_\pi \subseteq F_\pi$, and define $\mathcal{S} = \mathcal{W}_\pi + E$. The set $\mathcal{W}_\pi$ is referred to as the **window**, and $\mathcal{S}$ as the **strip**. For the purposes of this worksheet, we will define the **cut and project set** $Y \subseteq E$ by

$$Y = \pi(\mathcal{S} \cap \mathbb{Z}^k).$$

In this situation we refer to Y as a **k to d cut and project set**. *We also adopt the conventional assumption that $\pi|_{\mathbb{Z}^k}$ is injective.*

First we introduce some basic terminology related to cut and project sets. We say that a subspace E of $\mathbb{R}^k$ is **totally irrational** if $E + \mathbb{Z}^k$ is dense in $\mathbb{R}^k$, and in this case we also refer to Y as totally irrational. We say that Y is **aperiodic** if $Y + x \neq Y$, for any nonzero $x \in E$.

As a point of reference, we make use of the fixed subspace $F_\rho = \{0\} \times \mathbb{R}^{k-d} \subseteq \mathbb{R}^k$, and *we assume throughout this worksheet, with little loss of generality, that F_ρ is complementary to E.* Define $\rho : \mathbb{R}^k \to E$ and $\rho^* : \mathbb{R}^k \to F_\rho$ to be the projections onto E and F_ρ with respect to the decomposition $\mathbb{R}^k = E+F_\rho$. We write $\mathcal{W} = \mathcal{S}\cap F_\rho$, and for convenience we also refer to this set as the *window* defining Y.

With the above assumptions, we can write E as the graph of a linear function with respect to the standard basis vectors in F_ρ. In other words,

$$E = \{(x, L(x)) : x \in \mathbb{R}^d\},$$

where $L : \mathbb{R}^d \to \mathbb{R}^{k-d}$ is a linear function. For each $1 \leq i \leq k-d$, we define the linear form $L_i : \mathbb{R}^d \to \mathbb{R}$ by

$$L_i(x) = L(x)_i = \sum_{j=1}^{d} \alpha_{ij} x_j,$$

and we use the points $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$ to parametrize the choice of E.

(P1) Prove that E is totally irrational if and only if the collection of points

$$\{L(n) : n \in \mathbb{Z}^d\} + \mathbb{Z}^{k-d}$$

is dense in $F_\rho/\mathbb{Z}^{k-d}$.

(P2) Suppose that $\mathcal{W}$ is the (half open) unit cube in F_ρ. Prove that Y is aperiodic if and only if the map $\mathcal{L} : \mathbb{Z}^d \to \mathbb{R}^{k-d}/\mathbb{Z}^{k-d}$ defined by

$$\mathcal{L}(n) = L(n) + \mathbb{Z}^{k-d}$$

has trivial kernel.

(P3) Give an example of a cut and project set which is aperiodic but not totally irrational, and an example which is totally irrational but not aperiodic.

For $y \in Y$, write $\tilde{y}$ for the point in $\mathbb{Z}^k$ which satisfies $\pi(\tilde{y}) = y$. Assume that $\Omega \subseteq E$ is a bounded convex set which contains a neighborhood of 0 in E. Then, for each $r \geq 0$, define the **patch of size r at y**, by

$$P(y, r) := \{y' \in Y : \rho(\tilde{y'} - \tilde{y}) \in r\Omega\}.$$

In other words, $P(y, r)$ consists of the projections (under π) to Y of all points of $\mathcal{S}$ whose first d coordinates are in a certain neighborhood of the first d coordinates of $\tilde{y}$. For $y_1, y_2 \in Y$, we say that $P(y_1, r)$ and $P(y_2, r)$ are equivalent if

$$P(y_1, r) = P(y_2, r) + y_1 - y_2.$$

This defines an equivalence relation on the collection of patches of size r, and we denote the equivalence class of the patch of size r at y by $\mathcal{P}(y, r)$.

(P4) Prove that $P_1 = P(y_1, r)$ and $P_2 = P(y_2, r)$ are equivalent if and only if the sets $\pi^{-1}(P_1) \cap \mathbb{Z}^k$ and $\pi^{-1}(P_2) \cap \mathbb{Z}^k$ are translates of each other.

(P5) Conclude from (P4) that P_1 and P_2 are equivalent if and only if the sets

$$\rho^*\left(\mathcal{S} \cap \left(\tilde{y}_i + (\rho^{-1}(r\Omega) \cap \mathbb{Z}^k)\right)\right), \quad i = 1, 2,$$

are translates by an element of F_ρ.

Motivated by (P5), we consider the natural action of $\mathbb{Z}^k$ on F_ρ, given by

$$n.w = \rho^*(n) + w = w + (0, n_2 - L(n_1)),$$

for $n = (n_1, n_2) \in \mathbb{Z}^k = \mathbb{Z}^d \times \mathbb{Z}^{k-d}$ and $w \in F_\rho$. For each $r \geq 0$ we define the **r-singular points** of $\mathcal{W}$ by

$$\text{sing}(r) = \mathcal{W} \cap \left((-\rho^{-1}(r\Omega) \cap \mathbb{Z}^k).\partial\mathcal{W}\right),$$

and the **r-regular points** by

$$\mathrm{reg}(r) = \mathcal{W} \setminus \mathrm{sing}(r).$$

(P6) Suppose that $\mathcal{W}$ is the unit cube in F_ρ. Prove that, for every $y \in Y$ and $r > 0$, there is a unique connected component U of $\mathrm{reg}(r)$ with the property that, for any $y' \in Y$,

$$\mathcal{P}(y', r) = \mathcal{P}(y, r) \text{ if and only if } \rho^*(\tilde{y'}) \in U.$$

Hint: Use the result from (P4), and the fact that $\mathcal{W}$ is a fundamental domain for $F_\rho/\mathbb{Z}^{k-d}$.

(P7) Let $\mathcal{W}$ be the unit cube in F_ρ. Prove that, for almost every choice of $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$, with respect to Lebesgue measure and with reference to the linear forms described above, the number of equivalence classes of patches of size r in Y is bounded above and below by constants times $r^{d(k-d)}$. Hint: Consider how the number of connected components of $\mathrm{reg}(r)$ depends on the ranks of the kernels of the maps $\mathcal{L}_i : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}$ defined by

$$\mathcal{L}_i(n) = L_i(n) + \mathbb{Z}, \quad 1 \le i \le k - d.$$

(P8) For each integer τ with $2 \le \tau \le 4$, give an explicit example of a totally irrational, aperiodic, 4 to 2 cut and project set where the number of patches of size r is bounded above and below by constants times r^τ.

Finally, we introduce an important notion which allows us to group together cut and project sets which have essentially the same combinatorics of patches. Given cut and project sets Y and Y' in the same subspace E, we will say that Y is **derivable** from Y' if there exists a constant $c > 0$ such that all balls of radius c in E intersect Y', and such that, for all sufficiently large $r > 0$, any patch of size r at a point $y \in Y$ uniquely determines the patch of size $r - c$ at any point $y' \in Y'$ with $|y' - y| \le c$. If Y and Y' are both derivable from each other then we say that they are **mutually locally derivable (MLD)**.

(P9) Suppose that $\mathcal{W}$ is the unit cube in F_ρ and that $\mathcal{W}'$ is the image under ρ^* of the unit cube in $\mathbb{R}^k$ (the window $\mathcal{W}'$ is often called the **canonical window**). Let Y be a totally irrational cut and project set constructed from the window $\mathcal{W}$, and let Y' be a cut and project set constructed with the same data, but with the window $\mathcal{W}'$. Prove that Y and Y' are MLD.

(P10) Let $\zeta = \exp(2\pi i/5)$ and let Y be the cut and project set defined using the canonical window and the two dimensional subspace E of $\mathbb{R}^5$ generated by the vectors

$$(1, \mathrm{Re}(\zeta), \mathrm{Re}(\zeta^2), \mathrm{Re}(\zeta^3), \mathrm{Re}(\zeta^4))$$

and

$$(0, \mathrm{Im}(\zeta), \mathrm{Im}(\zeta^2), \mathrm{Im}(\zeta^3), \mathrm{Im}(\zeta^4)).$$

Parameterize E in terms of linear forms as above, and prove that Y is aperiodic but not totally irrational.

(P11) Let Y be the cut and project set defined in (P10). Use the result of problem (P9) to prove that the number of patches of size r is bounded above and below by constants times r^2.

Well known results of de Bruijn [1] and Robinson [4] show that the cut and project set Y from problem (P10) is the image under a linear transformation of the collection of vertices of a Penrose tiling, and in fact that all Penrose tilings can be obtained in a similar way from cut and project sets.

References

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