Topology of Tiling Spaces, 1/4

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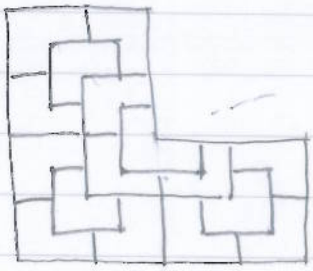
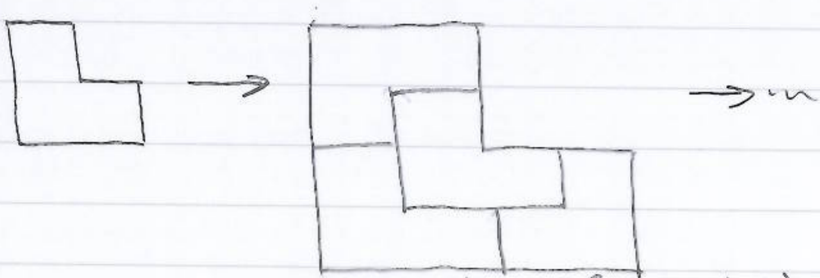
Tilings: We want to understand tilings with some long range order, but with enough complexity to make them interesting.

Ex 1: Chair tiling (ex. of a substitution tiling)



Think about this as a tiling generated by a substitution rule, which expands out at each iteration:



(supertile of level 1)

By starting with a supertile of appropriate level and choosing the origin appropriately, we can ensure that this tiling eventually fills the plane.

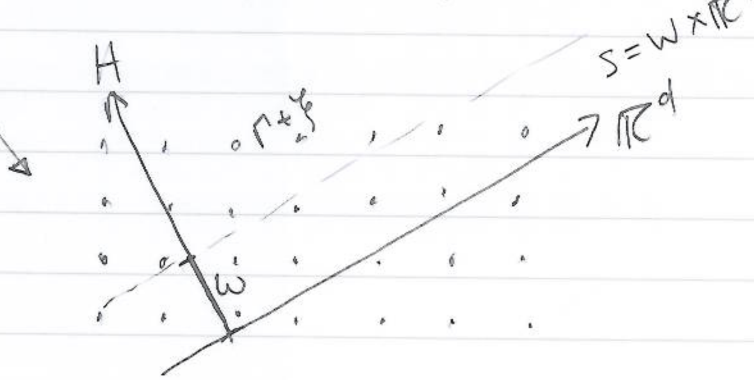
We want to consider the collection of all possible "admissible" tiles with chair tiles. Admissible means that every pattern which occurs in our tiling must occur somewhere in the original tiling which we described above.

This is an example of a collection of aperiodic tilings (exercise) with large scale order.

Ex 2: Cut-and-project tilings:
Start with an Abelian group H, a positive integer d, and let Γ be a lattice (a discrete co-compact subgroup) of $H \times \mathbb{R}^d$.

Choose a subset $W \subseteq H$ (a <u>window</u>) and an element $\xi \in H \times \mathbb{R}^d$, and let $S = W \times \mathbb{R}^d$ (the <u>strip</u>). If we consider the set $(\Gamma + \xi) \cap S$ and project this set to $\mathbb{R}^d$, we obtain a collection of points in $\mathbb{R}^d$ called a <u>cut-and-project set</u> or a <u>model set</u>. These are closely related to tilings, which can be obtained by either connecting points or looking at Voronoi cells.

Terminology: The space $\mathbb{R}^d$ in this construction is called the <u>physical space</u> and the space E is called the <u>internal space</u>.



Not every subst. tiling comes from cut-and-project, and vice-versa. However there is a non-trivial intersection between the two collections of tilings.

Ex. 3: Tilings coming from local matching rules, i.e. a finite collection of rules which can be enforced by looking in a fixed size neighborhood of any tile.
e.g.: Penrose tiling, and many other exs.

Tiling spaces: Two tilings T and T' (of $\mathbb{R}^d$) are close if they agree on a big ball around 0 up to a small "wiggle"

Precisely: close $\leftrightarrow$ within ε
big ball $\leftrightarrow$ ball of radius $1/\varepsilon$
small "wiggle" $\leftrightarrow$ translation, or rigid motion, or shear moving every pt. by at most ε.

About "wiggle" – there are several different notions of what a wiggle is, depending on the context. We will say that a small wiggle is a rigid translation of at most ε.

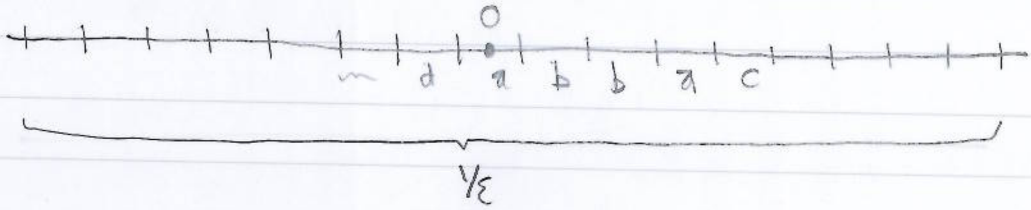
A tiling space is a set of tilings that is:

a) closed w.r.t. the metric topology defined as above, and
b) $\mathbb{R}^d$ translation invariant.

Write $\Omega_T = \overline{\text{orbit of } T}$ = the closure of the set of all $\mathbb{R}^d$ translates of T.

This is also called the hull of T.

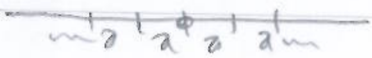
How to build a neighborhood of T:



Start with a $1/\varepsilon$ neighborhood of 0, wiggle it by ε. Then consider the collection of all possible tiles that could preceede or follow a patch with this pattern, anywhere in T. Then consider all possible tiles that could preceede or follow the extended patch, and continue. This construction shows that Ω_T locally is $\mathbb{R}^d \times C$, for some Cantor set C. (this is an ex. of a matchbox manifold)

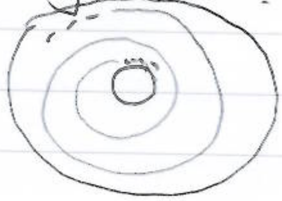
Ex: T:

All translates of T are in Ω_T, but the tilings



and

are also in Ω_T. The topology of the all "a" and all "b" tilings are each homeomorphic to circles. So the topology of Ω_T looks like two circles with a copy of $\mathbb{R}$ that asymptotically approaches the circles:



(a "slinky").

Def: A tiling has finite local complexity (FLC) if for each radius R, $\exists$ finitely many patches of radius R up to translation.

Exercise 1:

$$T \text{ has FLC} \iff \Omega_T \text{ is compact.}$$

Exercise 2:

$$T' \in \Omega_T \iff \text{every pattern in } T' \text{ is found in } T$$

Recall that a dynamical system is minimal if every orbit is dense. It follows from exercise 2 above that Ω_T (w.r.t. translation) is minimal if and only if all tilings have the same patches. Furthermore, Ω_T is minimal iff it is repetitive, i.e. every patch of T appears infinitely often with bounded gaps.

For the most part, we want to look at repetitive tilings with FLC.

Inverse limits

A good ex. of a matchbox manifold is the 2-adic solenoid:

$\forall n \in \mathbb{N}$, there is a natural map ρ_n from $\mathbb{R}/2^n\mathbb{Z}$ to $\mathbb{R}/2^{n-1}\mathbb{Z}$, given by wrapping around two times. A point in the 2-adic solenoid is a point $(x_0, x_1, \ldots) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\mathbb{Z} \times \ldots$ with the property that $\rho_n(x_n) = x_{n-1}$ for all n.

This is formalized by the concept of <u>inverse limits</u> of top. spaces: Given a collection of top. spaces $\Gamma^0, \Gamma^1, \ldots$, and a collection of maps $\rho_n : \Gamma^n \to \Gamma^{n-1}$, we define the top. space

$$\varprojlim(\Gamma^n, \rho_n) = \{(x_0, x_1, \ldots) \in \prod \Gamma^n \mid x_n = \rho_{n+1}(x_{n+1}), \forall n$$

taken with the subspace top. of the product top. on $\prod \Gamma^n$.

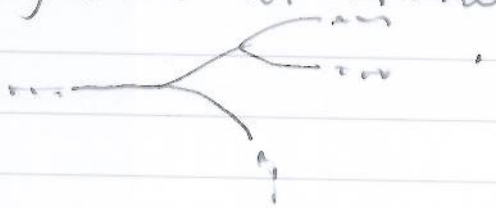
We call Γ^n the <u>nth approximant</u> to $\varprojlim(\Gamma^n, \rho_n)$. Two points in the inverse limit are close if and only if they are close in Γ^n, for large enough n.

Thm: Let T be a (repetitive) FLC tiling. Then $\Omega_T = \varprojlim(\Gamma^n, \rho_n)$, where each Γ^n

is a connected <u>branched manifold</u>.

A branched manifold is a collection of manifolds "glued" together at branch points.

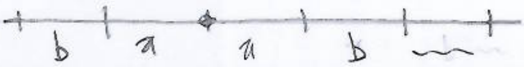
e.g.:



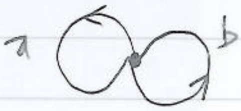
$$\Gamma^n = \left\{ \begin{array}{c} \text{Information needed to describe} \\ \text{a patch of radius } r_n \end{array} \right\}$$

Then Γ^n consists of sets of instructions for building larger and larger patches of tiles at the origin. The map ρ_n is the "forgetful map", which erases part of a patch of radius r_n to obtain a patch of radius r_{n-1}.

Ex:



There is an interval of ways to place an "a" tile at the origin, and an interval of ways to place a "b" at the origin. If the origin is at an endpt. then there are multiple possibilities, depending on the tile just to the left and the one just to the right. To remedy this we identify endpts. wherever tiles meet to obtain a CW-complex which describes all ways that a tile in Ω_T can sit at the origin. For the tiling pictured above we have the complex



. This is our Γ^0.

Suppose we are interested in a particular tiling where the only patterns of "supertiles of level 1" (i.e. tiles together with information about which tiles can precede or follow them) are:

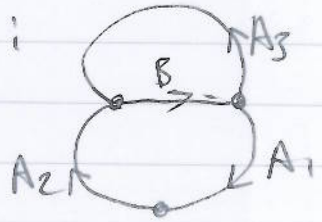
$B = (a)\,b\,(a)$

$A_1 = (b)\,a\,(a)$

$A_2 = (a)\,a\,(b)$

$A_3 = (b)\,a\,(b)$

Then we can encode our tiling in the letters B, A_1, A_2, and A_3 to obtain a new tiling. The information about supertiles of level 1 at the origin can be encoded in the approximant Γ^1:



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The tiles B, A_1, A_2, A_3 are called "collared tiles". If we collar again we obtain Γ^2, then Γ^3, etc. This is a way of capturing the topology of Ω_T using an inverse limit, but in practice it is very difficult to compute with, because the spaces of collared tiles and maps between them change at each level.