Topology of tiling spaces 2/4

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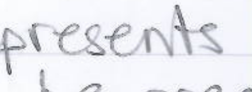
As we saw last time, tiling spaces are generically totally disconnected spaces, Therefore the fundamental group is not interesting, and we need to use different algebraic data to study them. It turns out that Cech cohomology is a good invariant for us to study, especially because it behaves well with respect to inverse limits.

Recall that repetitive FLC tilings have an inverse limit structure:

$$\Gamma^0 \xleftarrow{\rho_0} \Gamma^1 \xleftarrow{\rho_1} \Gamma^2 \longleftarrow \cdots$$

Our running example is the Fibonacci tiling, given by the substitution $a \mapsto ab$, $b \mapsto a$. We start with the fixed point abaababaabaab... and consider the collection of all tilings in tiles a and b consisting only of patterns which occur in this fixed point. As we saw last time the 0^{th} order approximant is the bouquet of two circles

Γ_0: 

,

which represents the fact that an "a" tile can be preceded or followed by a "b", and the "b" tile can be preceded or followed by an "a" tile. This is called an <u>Anderson-Putnam complex</u>.

To obtain the higher order approximants, as last time, we can use collared tiles.

Thm (Gähler): $\Omega_T = \varprojlim \Gamma_{AP}^{(n)}$.

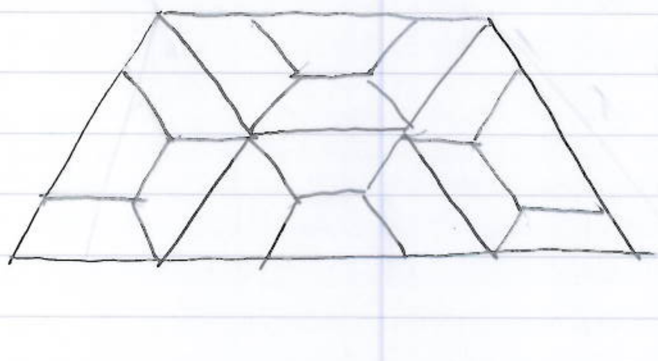
(the And.-Put. complex made with n-times collared tiles)

Although this is a wonderful theorem, in general it is different to compute with, because the approximants change at each step. For substitution tilings there is an even nicer inverse limit construction.

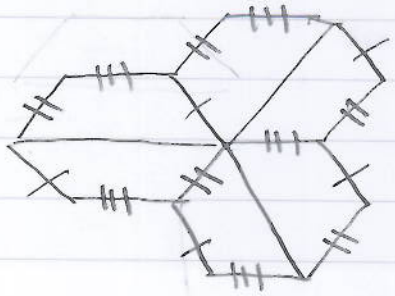
Suppose we have a substitution rule σ on a collection of tiles $a_1, \dots, a_m$. The collection of tiles $\sigma^i(a_j)$, $1 \leq j \leq m$, are called <u>supertiles of level i</u>. We can form inverse limits w.r.t. supertiles.

As an example consider the <u>half-hex tiling</u>, given by the rule

The next iteration would give



The Γ° AP-complex is



.

Then we build an inverse lim. using the approximants

Γ_0 = instrucs. for placing a tile at 0,

Γ_1 = place supertile of level 1 @ 0,

⋮

Γ_n = place supertile of level n @ 0,

with the maps ρ_n taken to be the "forgetful maps".

Thm 1 (AP): If σ forces the border then

$$\Omega_T = \varprojlim \Gamma^n_{AP} .$$

The precise def. of "forces the border" will be given on a separate sheet. It basically means that successive substitutions of tilings always fill up the whole space in a unique way.

Ok, here is the def: σ forces the border if $\exists\, n$ s.t. any two n-supertiles of the same type have the same neighbors.

This property guarantees that a point in the inverse limit uniquely determines a full tiling of $\mathbb{R}^d$.

Ex: Fibonacci tiling, nth order approximant is Γ_n: 

, and the map

$p_n : \Gamma_{n+1} \to \Gamma_n$ is given by stretching each loop and wrapping the stretched "a" around an "a" and a "b" in Γ_n, and wrapping the stretched "b" around an "a" in Γ_n. One problem though is that the Fib. subst. doesn't force the border. However there is a way around this problem:

Thm 2(AP): In any substitution, if you collar once, you force the border.

Exercise: Verify this for collared tiles in the Fib. tiling: $A_1 = (b)a(a)$, $A_2 = (a)a(b)$, $A_3 = (b)a(b)$, $B = (a)b(a)$.

An example where collaring once makes a difference in the top. of the inv. lim. in the corresp. AP complex:
Thue Morse subst: $a \mapsto ab$, $b \mapsto ba$. (exercise)

<u>Cohomology:</u>

To tell the difference between two manifolds, e.g. S^2 and $\mathbb{T}^2$, we could use topological invariants, e.g. π_1, H_1, or H^1.

First think about simplicial complexes.
A 0-cochain is a function on vertices, 1-cochain a function on edges, etc.
There is a boundary map on each cell of a simplicial complex, and this gives a map on cochains: $(\delta\alpha)(e) = \alpha(\partial e)$.

Tiling spaces have infinitely many path components, and π_1 and H_1 are useless for studying them.
The right invariant turns out to be Čech cohomology. (notation: $\check{H}^* =$ Čech cohom.)

Facts:

1) If Γ is a CW complex then
$\check{H}^k(\Gamma) = H^k(\Gamma)$

2) $\check{H}^k(\varprojlim(\Gamma^n, \rho_n)) = \varinjlim(\check{H}^k(\Gamma^n), (\rho_n)^*)$

$= \varinjlim(H^k(\Gamma^n), (\rho_n)^*)$

Ex: Fib. tiling

$H^0 \cong \mathbb{Z}$, $H^1 \cong \mathbb{Z}^2$.

The maps ρ_n are all the same, and are represented by the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
(on the generators of H^1).

This matrix defines an automorphism of $\mathbb{Z}^2$, so by fact 2 above,

$$\check{H}(\Omega_T) \cong \varinjlim\left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^t\right) \cong \mathbb{Z}^2.$$

Exercise: Prove that for the Thue-Morse tiling, $\check{H}^1(\Omega_T) \cong \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$.

Deformations of Tilings

Assume we have a tiling by collection of finitely many tiles which meet full edge to full edge. How do we define the shape of a tile? Answer: By vector valued 1 cochains on Γ^n:

Since the sum of the vectors is 0, we should consider closed vector valued 1 cochains, i.e. elements of the cohom. group.

Thm (Clark, Sadun): Shape changes / local equivalence

$\approx \check{H}^1(\Omega, \mathbb{R}^d)$

(= mutual local derivability (MLD))

Shape changes always result in new tilings which are homeomorphic, but not necessarily topologically conjugate (means there is a homeomorphism which commutes w/ the translation map). However there is a subset of $\check{H}^1$, called asymptotically negligable and denoted $\check{H}^1_{an}$, for which

$H^1_{an} \cong$ shape conjugacies / MLD

For substitutions it turns out that H^1_{an} is the contracting subspace of $\check{H}^1(\Omega, \mathbb{R}^d) \to \check{H}^1(\Omega, \mathbb{R}^d)$ under the natural linear action induced by the substitution σ.

Pattern equivariant cohomology

We say that a function f or cochain is pattern-equivariant (PE) if $\exists R$ s.t. $f(x)$ only depends on the pattern to radius R around x.