Cheat Sheet for Tilings Lectures

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1 Some Definitions

- A *tile* is a topological disk in Euclidean space that is the closure of its interior, possibly with an additional label. For our purposes, tiles will always be closed intervals in 1 dimension, closed polygons in 2 dimensions, or closed polyhedra in 3 or more dimensions. We do *not* assume that our tiles are convex.

- Tiles can be moved around. Translating a tile does not change its label. Two tiles are *equivalent* if they carry the same label and are translates of one another.

- A *tiling* is a covering of the plane by tiles that intersect only on their boundaries. Usually we assume that all the tiles in a tiling are translates of a finite set of *prototiles*.

- A *patch* is a finite collection of tiles in a tiling. As with tilings, we consider patches P_1 and P_2 equivalent if one is the translate of the other. (And vice-versa, of course.)

- A tiling has *Finite Local Complexity*, or FLC, if for each radius R there are only finitely many patches of diameter at most R, up to translation.

  Exercise: Show that a tiling T has FLC if and only if there are finitely many connected 2-tile patches, up to translation.

- A tiling is *repetitive* if, for each patch P, there exists a radius $R(P)$ such that every ball of radius $R(P)$ contains at least one copy of P. In other words, all patches appear infinitely often, and with bounded gaps.

- A tiling T is *non-periodic* if it has no translational symmetries. That is, if $T - x = T$ implies $x = 0$.

- The *tiling metric*, and associated topology, are defined as follows. Two tilings T, T' are considered ϵ-close if there exist $x, x' \in \mathbb{R}^d$ of size ϵ or less such that $T - x$ and $T' - x'$ agree exactly on a ball of radius $1/\epsilon$ around the origin. There is nothing magical about the function $1/\epsilon$. The main idea is that two tilings are close if they agree on a large ball around the origin, up to small translations.

- A *tiling space* is a non-empty, closed, and translation-invariant set of tilings.

- The *continuous hull* Ω_T of a tiling T is the closure of its translational orbit. This is the smallest tiling space that contains T.

Exercise: Show that a tiling T' is in Ω_T if and only if every patch of T' is found somewhere in T.

Theorem 1 *Ω_T is compact if and only if T has FLC. The action of $\mathbb{R}^d$ on Ω_T is minimal if and only if T is repetitive.*

We almost always assume that our tilings have FLC, are repetitive, and are non-periodic.

2 Inverse limits and collaring

Let $\Gamma_0, \Gamma^1, \ldots$ be a sequence of topological spaces and let $\rho_n : \Gamma^{n+1} \to \Gamma^n$ be continuous maps. Then the *inverse limit* of the *approximants* Γ^n is

$$\varprojlim(\Gamma^n, \rho^n) = \{(x_0, x_1, \ldots) \in \prod \Gamma^n | \text{each } x_j = \rho_{j+1}(x_{j+1})\}.$$

Standard example: The *dyadic solenoid* is the inverse limit where each Γ^n is a circle and each ρ_n wraps an approximant twice around the previous approximant. This can be viewed in two ways:

1. All Γ^n are the same circle $\mathbb{R}/\mathbb{Z}$, and each ρ_n is multiplication by 2, or
2. $\Gamma^n = \mathbb{R}/(2^n\mathbb{Z})$, and each ρ_n is just a quotient map where we identify x and $x + 2^{n-1}$.

Note that convergence in Γ^n implies convergence in all Γ^m with $m < n$, and that convergence in the inverse limit is the same as convergence in Γ^n for all n. In practice, almost all calculations with inverse limits proceed along the lines of "pick n sufficiently large, do some estimates on Γ^n, and then go to the pub".

The *Anderson-Putnam* complex Γ_{AP} of a tiling T is a CW complex obtained by taking one copy of each tile type that appears in T and gluing them as follows. If tiles A and B share an edge somewhere in T, identify the corresponding edges of A and B. A point in Γ_{AP} is essentially an instruction for placing a tile containing the origin. (Only "essentially" because of the ambiguities when the origin lies on the boundary of two or more tiles.)

Consider the tiles in a tiling T. Call two tilings *1-equivalent* if the patches containing these tiles *and their immediate neighbors* are translates of one another. An equivalence class is called a *collared tile.*

Likewise, we can consider twice-collared tiles, 3-times collared tiles, etc.