Cheat Sheet 2 for Tilings Lectures

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1 Complexes and collaring

The *Anderson-Putnam* complex Γ_{AP} of a tiling T is a CW complex obtained by taking one copy of each tile type that appears in T and gluing them as follows. If tiles A and B share an edge somewhere in T, identify the corresponding edges of A and B. A point in Γ_{AP} is essentially an instruction for placing a tile containing the origin. (Only "essentially" because of the ambiguities when the origin lies on the boundary of two or more tiles.)

Consider the tiles in a tiling T. Call two tiles *k-equivalent* if the patches containing these tiles *and neighbors up to k-th order* are translates of one another. An equivalence class is called a *k-times collared tile.*

Let $\Gamma_{AP}^{(k)}$ be the Anderson-Putnam complex built from k-collared tiles.

Theorem 1 (Gähler) *If T is any tiling with FLC, then $\Omega_T = \varprojlim(\Gamma^{(n)}, \rho_n)$ where ρ_n is the forgetful map. More precisely, any two tiles that are n-equivalent are $(n-1)$-equivalent, and the map ρ_n sends each n-equivalence class to the corresponding $(n-1)$-equivalence class.*

Proof: A point in $\varprojlim(\Gamma^{(n)}, \rho_n)$ is just a succession of consistent instructions for tiling bigger and bigger balls around the origin, and so is tantamount to instructions for tiling all of $\mathbb{R}^d$.

A *substitution* is a map $\sigma : \Omega_T \to \Omega_T$ that dilates each tiling by a factor λ around the origin, then subdivides each tile according to a fixed pattern. E.g., in one dimension, we can consider tiles a of length $\phi = (1+\sqrt{5})/2$ and b of length 1, and replace each a with ab and each b with a. A substitution also induces a map (also denoted σ) from $\Gamma^{(n)}$ to itself.

A cluster of tiles obtained by applying σ n times to a tile is called an n-supertile.

A substitution *forces the border* if there is an integer n such that every n-supertile of the same type has exactly the same patterns of tiles adjacent to the supertile.

Theorem 2 (Anderson-Putnam) *If a substitution forces the border, then $\Omega_T = \varprojlim(\Gamma^{(0)}, \sigma)$.*

A point in the inverse limit is a set of instructions for putting higher and higher order supertiles around the origin. Because of border-forcing, this determines the tiling on all of $\mathbb{R}^d$.

Theorem 3 (Anderson-Putnam) *If you rewrite a substitution tiling in terms of once-collared tiles, it always forces the border.*

Theorem 4 (Still A-P) *Every substitution tiling space can be written as an inverse limit where all the approximants are the same and all the maps are the same. Use the uncollared A-P complex if the substitution forces the border, and the collared A-P complex if it doesn't.*

2 Cohomology

Čech cohomology is a cohomology theory defined using open covers and refinements of open covers. It applies to any topological space, and so is more general than simplicial or singular (co)homology which only sees one path component at a time. The two key properties of Čech cohomolgy are:

1. If Γ is a CW complex, then the Čech cohomology $\check{H}^k(\Gamma)$ is isomorphic to every other kind of cohomology (e.g. simplicial, singular, or cellular).

2. If $\Omega = \varprojlim(\Gamma^n, \rho_n)$, then $\check{H}^k(\Omega) = \varinjlim(\check{H}^k(\Gamma^n), \rho_n^*)$.

Combining these two, we get that the Čech cohomology of an inverse limit of CW complexes is the direct limit of the (ordinary!) cohomology of the approximants. In practice, nobody every thinks about Čech cohomology from the definitions (which I'm not giving you!). They always think about limits.

Another cohomology theory is based on a single tiling T rather than on the tiling space Ω_T. T gives a partition of $\mathbb{R}^d$ into vertices, edges, faces, etc. A k-cochain is a function from the set of k-cells to a target group A, usually $\mathbb{Z}$ or $\mathbb{R}$.

A function $f : \mathbb{R}^d \to A$ is (strongly) pattern-equivariant (PE) if there is a radius R such that the value of $f(x)$ depends only on the pattern of T in an R-neighborhood of x. That is, if $T - x$ and $T - y$ agree on $B_R(0)$, then $f(x) = f(y)$.

We can similarly define PE differential forms and PE cochains. The exterior derivative of a PE form is PE. The coboundary of a PE cochain is PE.

Theorem 5 (Kellendonk-Putnam) *Consider the complex of PE differential forms. Let $H^k_{PE,dR}(T)$ be the closed PE k-forms mod d of the PE $(k-1)$-forms. (dR stands for de Rham) Then $H^k_{PE,dR}(T)$ is isomorphic to $\check{H}^k(\Omega_T, \mathbb{R})$.*

Theorem 6 (S) *Let A be any Abelian group. The cohomology of the complex of A-valued PE cochains on T is isomorphic to $\check{H}^k(\Omega_T, A)$.*

Proof: A cochain is PE if and only if it is the pullback of a cochain on a Gähler approximant. So working with PE cochains is equivalent to working on Γ^n and then taking a limit as $n \to \infty$. But that's the same as the Čech cohomology of the inverse limit space.

When $A = \mathbb{R}$, we can also speak of *weakly* PE functions, forms and cochains. These are uniform limits of strongly PE cochains. For a strongly PE function, knowing the R-neighborhood of x allows you to compute $f(x)$ exactly. For a weakly PE function, it allows you to approximate $f(x)$, with the error going to zero as $R \to \infty$.

3 Shape deformations

We return to general tilings (not necessarily substitution tilings).

A tiling T' is *locally derived* from T if there is an R such that the ball of radius 1 around every point x in T' is determined by the ball of radius R around x in T. This is very much like pattern equivariance. If $T-x$ and $T-y$ agree on $B_R(0)$, then $T'-x$ and $T'-y$ agree on $B_1(0)$. If T' is locally derived from T and T is locally derived from T', then T and T' are *mutually locally derivable*, or MLD.

The shape of a polyhedral tile is determined by the displacements along its edges. If two tiles share an edge, they must have the same displacement. So the shape is given by a vector valued 1-cochain on Γ_{AP}. This cochain is closed, and represents a class in $H^1(\Gamma, \mathbb{R}^d)$. Allowing collaring, and taking appropriate limits, gives the following theorem:

Theorem 7 (Clark-S) *Shape changes to a tiling, modulo MLD, are parametrized (at least infinitesimally) by* $\check{H}^1(\Omega_T, \mathbb{R}^d)$.

Changing the shapes of tiles always gives a homeomorphic tiling space. Sometimes it gives a space that is topologically conjugate. There is a subspace $H^1_{an}(\Omega_T, \mathbb{R}^)$ of *asymptotically negligible* classes such that

$$\frac{\text{Shape changes that are topological conjugacies (aka shape conjugacies)}}{\text{MLD}} = \check{H}^1(\Omega_T, \mathbb{R}^d).$$

If T is a substitution tiling with substitution σ, then σ maps Ω_T to itself, so σ^* maps $\check{H}^1(\Omega_T, \mathbb{R}^d)$ to itself.

Theorem 8 (Clark-S) *$\check{H}^1_{an}$ is the contracting subspace of $\check{H}^1$ under the action of σ^*, i.e. the span of all of the (generalized) eigenspaces with eigenvalue strictly smaller than 1.*

Corollary 9 *If σ is a Pisot substitution in 1 dimensional, then all changes to the length of the tiles yield topologically conjugate tiling spaces, up to an overall rescaling.*

Theorem 10 (Kellendonk) *In any tiling space (not just substitution), a closed strongly PE 1-cochain represents a class in H^1_{an} if and only if its integral is weakly PE. (Which is if and only if the integral is bounded)*

Partially open question: For tilings that come from cut-and-project or local matching rules, how can you compute H^1_{an}?

4 Cut-and-project tilings

In a cut-and-project tilings, we have

1. A *physical space* $\mathbb{R}^d$.
2. A locally compact Abelian group H. (For simplicity, imagine $H = \mathbb{R}^{N-d}$.)

3. Projections $\pi^{\|}$ and $\pi^{\perp}$ from $H \times \mathbb{R}^d$ to $\mathbb{R}^d$ and H, respectively.
4. A lattice $\Lambda \subset H \times \mathbb{R}^d$ such that $\pi^{\|} : \Lambda \to \mathbb{R}^d$ is injective and $\pi^{\perp}(\Lambda)$ is dense in Hi.
5. A *window* $W \subset H$ that is compact and the closure of its interior. When W is the projection of a unit lattice cell, we call the setup *canonical*.
6. A *strip* $S = W \times \mathbb{R}^d$.
7. A parameter $\xi \in H \times \mathbb{R}^d$. ξ is called *non-singular* if $\pi^{\perp}(\Lambda + \xi) \cap \partial W$ is empty. Since Λ is countable, if ∂W has measure zero then Lebesgue almost-every ξ is non-singular.

If ξ is non-singular, this data gives us a set

$$\pi^{\|}(S \cap (\xi + \Lambda))$$

in $\mathbb{R}^d$. There are several ways to convert from a point pattern to a tiling, either by using Voronoi cells or by "connecting the dots" so that the points become vertices of tiles.

Our next task is understanding spaces of cut-and-project tilings as cut tori, computing cohomology, and identifying the asymptotically negligible classes.