Cheat Sheet 3 for Tilings Lectures

Lorenzo Sadun

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1 Cut-and-project tilings

In a cut-and-project tilings, we have

1. A *physical space* $\mathbb{R}^d$.
2. A locally compact Abelian group H. (For simplicity, imagine $H = \mathbb{R}^{N-d}$.)
3. Projections $\pi^{\parallel}$ and $\pi^{\perp}$ from $H \times \mathbb{R}^d$ to $\mathbb{R}^d$ and H, respectively.
4. A lattice $\Lambda \subset H \times \mathbb{R}^d$ such that $\pi^{\parallel} : \Lambda \to \mathbb{R}^d$ is injective and $\pi^{\perp}(\Lambda)$ is dense in Hi.
5. A *window* $W \subset H$ that is compact and the closure of its interior. When W is the projection of a unit lattice cell, we call the setup *canonical*.
6. A *strip* $S = W \times \mathbb{R}^d$.
7. A parameter $\xi \in H \times \mathbb{R}^d$. ξ is called *non-singular* if $\pi^{\perp}(\Lambda + \xi) \cap \partial W$ is empty. Since Λ is countable, if ∂W has measure zero then Lebesgue almost-every ξ is non-singular.

If ξ is non-singular, this data gives us a set

$$Y_\xi = \pi^{\parallel}(S \cap (\xi + \Lambda))$$

in $\mathbb{R}^d$. There are several ways to convert from a point pattern to a tiling, either by using Voronoi cells or by "connecting the dots" so that the points become vertices of tiles.

Our next task is understanding spaces of cut-and-project tilings as cut tori, computing cohomology, and identifying the asymptotically negligible classes.

If parameters ξ_1 and ξ_2 differ by a lattice element, then $\Lambda + \xi_1 = \Lambda + \xi_2$, so $Y_{\xi_1} = Y_{\xi_2}$. In other words, we should think of ξ as living in the torus $\Omega_{max} = H \times \mathbb{R}^d/\Lambda$.

Theorem 1 *There is a factor map $\Omega \to \Omega_{max}$ that is 1:1 for all non-singular ξ and is finite-to-1 when ξ is singular.*

Corollary 2 *For all measure-theoretic purposes (e.g. computing spectrum, ergodicity, etc.), there is no difference between Ω and Ω_{max}. Our dynamical system is essentially an irrational rotation on a torus.*

Corollary 3 *To understand the topology of Ω, we just have to understand the singular ξ's.*

Example: If $H = \mathbb{R}$ and $n = 1$ and our window is an interval, the only question is whether the two ends of the window are related by an element of Λ. If so, then our space is the inverse limit of once-punctured tori, and $H^1 = \mathbb{Z}^2$. If not, then it is the inverse limit of twice-punctured tori, and $H^1 = \mathbb{Z}^3$.

2 Uses for PE cohomology

2.1 Deformations

Let Y be a point pattern. A *shape deformation* sends each point $y \in Y$ to $y + F(y)$. In order to preserve FLC, δF must be strongly PE (sPE). (Every left edge of a 17-times collared A tile needs to wind up with the same edge vector). But if F is itself sPE, then the shape deformation is a local derivation. So

$$\frac{\text{Shape deformations}}{\text{MLD}} = \frac{\text{closed sPE 1-cochains } \delta F}{\delta(\text{sPE 0-cochains})} = H^1_{PE}(Y, \mathbb{R}^d)$$

The deformations tF with t going from 0 to 1, induces a family of topological conjugacies if (and only if) F is weakly PE (wPE). In other words

Theorem 4 *A closed sPE 1-cochain α represents a class in H^1_{an} if and only if $\alpha = \delta F$ with F wPE.*

Lemma 5 *The integral of a closed 1-cochain is wPE if and only if it is bounded.*

Moral: Asymptotic negligibility, which is a dynamical and topological notion, is closely tied to bounded integrals, which (for cut-and-project tilings) is closely tied to Diophantine approximation properties.

2.2 Visualization

It's much easier to understand cohomology if you can produce PE generators for the different classes.

1. For the Fibonacci substitution tiling, $H^1 = \mathbb{Z}^2$, and the two generators are 1-cochains that count a tiles and count b tiles.

2. The same thing goes for any Sturmian sequence space. $H^1 = \mathbb{Z}^2$ and the generators count the two types of tiles.

3. In the period-doubling substitution, $a \to ab$, $b \to aa$, $H^1 = \mathbb{Z}[1/2] \oplus \mathbb{Z}$. The class $(2^{-n}, 0)$ just count n-supertiles. The class $(0, 1)$ counts one (or the other) species of tile.

3 Barge-Diamond collaring

We got the Gähler complex by looking at equivalence classes of tiles and gluing them together. We get Barge-Diamond (BD) collaring by considering equivalence classes of points.

Let T be a substitution tiling, and pick an arbitrary $\epsilon > 0$. (ϵ doesn't have to be small, but we usually imagine it is.) We have an equivalence relatoin on points in $\mathbb{R}^d$:

$$x \sim_\epsilon y \text{ if } B_\epsilon \cap (T - x) = B_\epsilon \cap (T - y).$$

That is *we identify points whose ϵ-neighborhoods look the same in the tiling T.* Each interior point of tile type A is identified with the corresponding point of every other copy of A, *as long as the point is farther than ϵ from the boundary of A). If a point is within ϵ of the boundary, it also "sees" what sort of tile is on the other side of the border.*

For 1D tilings, this gives a complex with two kinds of edges.

1. *Big edges corresponding to tiles, only shortened by ϵ at each end, and*

2. *"Vertex flaps", one for each possible transition from one tile to the next.*

Let Γ_ϵ denote this complex. Substitution sends Γ_ϵ to itself and

$$\Omega = \varprojlim(\Gamma_\epsilon, \sigma),$$

$$H^k(\Omega) = \varinjlim(H^k(\Gamma_\epsilon, \sigma^*)).$$

Unfortunately, σ does not send cells to cells. But it is homotopic to a map σ' that does! Since $\sigma^* = (\sigma')^*$, we do all of our cohomology computations using σ' instead of σ.

Theorem 6 (BD) *For 1D substitutions, if the sub-complex of vertex flaps is contractible, then H^1 is the direct limit of $\mathbb{Z}^k$ under the transpose of the substitution matrix, where k is the number of letters.*

BD collaring is more powerful, but harder to describe, for higher-dimensional tilings. When it comes to computations by hand of tiling cohomology, it (combined with "quotient cohomology", which I don't have time to explain) is the state of the art.

4 Conjugacy invariants

Suppose that Ω_T and $\Omega_{T'}$ are topologically conjugate tiling spaces. What does that tell us about the tilings T and T'. Understanding this requires understanding topological conjugacies. Fortunately, these boil down to shape conjugacies.

Theorem 7 (Kellendonk-S) *Every topological conjugacy between repetitive FLC tiling spaces can be written as a composition of two map, one being an MLD map and the other being a shape conjugacy.*

Since nothing interesting changes under MLD, understanding what is preserved by topological conjugacies boils down to understand what is preserved by shape conjugacies.

A point pattern Y is called Meyer *if the displacements between points is uniformly discrete, i.e. there is a minimum spacing between elements of $Y - Y$.*

Theorem 8 (Frank-S, Kellendonk-S) *The Meyer property is not necessarily preserved by topological conjugacies.*

Theorem 9 (Kellendonk-S) *If Y is a cut-and-project set with H being the product of $\mathbb{R}^{N-d}$ and a finite group, and with the window being a finite union of polyhedra, then*
(1) $H^1_{an}(Y, \mathbb{R}^d)$ has rank $d(N-d)$ and is generated by linear functions from H to $\mathbb{R}^d$.
(2) Every pattern topologically conjugate to Y is MLD to a reprojection of Y. That is, a set with the same window and strip, only with a different $\pi^{\|}$.