DEFINITIONS AND EXAMPLES FROM POINT SET TOPOLOGY

A **topological space** is a set X together with a subset τ of $\mathcal{P}(X)$ which satisfies

(i)
$$\emptyset, X \in \tau,$$

(ii)
$$\{A_i\}_{i\in I} \subseteq \tau \Rightarrow \bigcup_{i\in I} A_i \in \tau, \quad \text{and}$$

(iii)
$$A_1, \ldots, A_n \in \tau \Rightarrow \bigcap_{i=1}^{n} A_i \in \tau.$$

The collection τ is called the **topology** on X, the elements of τ are called **open sets**, and any subset of X which is the complement of an element of τ is called a **closed set**.

A subset $\mathcal{A} \subseteq \tau$ is called a **basis** for (X, τ) if every element of τ can be written as a union of elements of $\mathcal{A}$. In this case we also say that τ is the **topology generated by the basis** $\mathcal{A}$.

Examples:

(1) Every set X with more than one element has at least two topologies. The first is the **discrete topology**, in which we take $\tau = \mathcal{P}(X)$. The second is the **trivial topology**, in which we take $\tau = \{\emptyset, X\}$.

(2) If (X, d) is a metric space then the collection of open balls in X generates a topology called the **metric topology**. As a matter of definition, note that when we say that the collection of open balls generates a topology, we are saying that the collection of all sets which are unions of open balls satisfies the requirements for being a topology.

(3) If (X, τ) is a topological space and $S \subset X$ then the **subspace topology** on S is defined as
$$\{S \cap A : A \in \tau\}.$$
For example if we consider $\mathbb{Q}$ as a subset of $\mathbb{R}$, the latter being taken with the Euclidean metric topology, the collection of rational numbers lying in an open interval will be an open set. However a set consisting of a single rational point will not be open in $\mathbb{Q}$ with respect to this topology. By contrast if we are thinking of $\mathbb{Q}$ with respect to the discrete topology then every set is open. This illustrates the fact that in general there are many choices for the topology on a set X, and the natural choice of topology for one problem may not be the right choice for another.

If (X, τ_X) and (Y, τ_Y) are two topological spaces then we say that a function $f : X \to Y$ is **continuous** if $f^{-1}(A) \in \tau_X$ for all $A \in \tau_Y$. If there is a continuous bijective map $f : X \to Y$, for which f^{-1} is also continuous, then we say that (X, τ_X) and (Y, τ_Y) are **homeomorphic**.

Suppose that X is a set and $\{Y_i\}_{i \in I}$ is a collection of topological spaces, and for each $i \in I$ let f_i be a function from X to Y_i. The **initial topology** on X with respect to $\{f_i\}$ is the **coarsest** topology (i.e. the topology with fewest number of elements) with respect to which all of the functions f_i are continuous.

Examples:

(4) Suppose $\{(X_i, \tau_i)\}_{i \in I}$ is a collection of topological spaces, let X be the Cartesian product $\prod_{i \in I} X_i$ and for each $j \in I$ let $\pi_j : X \to X_j$ be the **projection map** (i.e. the map onto the jth coordinate). The **product topology** on X is defined to be the initial topology with respect to $\{\pi_j\}_{j \in I}$. A basis for this topology is

$$\left\{ \prod_{i \in I} A_i : A_i \in \tau_i,\ A_i = X_i \text{ for all but finitely many } i \right\}.$$

(5) With the same notation as in the previous example, the **box topology** on X is defined to be the topology generated by

$$\left\{ \prod_{i \in I} A_i : A_i \in \tau_i \right\}.$$

When I is finite this topology is the same as the product topology. *However in general the two topologies are not the same.* One strong argument in favor of using the product topology is that Tychonoff's Theorem (see below) is not true in general for the box topology.

(6) Let $\{(X_i, d_i)\}_{i=1}^n$ be metric spaces and consider the Cartesian product $X = \prod_{i=1}^n X_i$. This product is also a metric space with respect to the metric $d : X \times X \to [0, \infty)$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le n} \{d_i(x_i, y_i)\},$$

and it is not difficult to show that X with the induced metric topology is homeomorphic to X with the product topology (i.e. with each X_i taken with the metric topology).

It is also true that the product of countably many metric spaces $\{(X_i, d_i)\}_{i=1}^\infty$, taken with the product topology, is **metrizable** (i.e. there is a metric on the Cartesian product for which the induced metric topology is the product topology). One metric which realizes the product topology is given by

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}.$$

In general it is not always true that a product of metric spaces with respect to the product topology is metrizable.

(7) Let (X, τ) be a topological space and suppose that $X = \cup_{y \in Y} X_y$ is a partition of the set X. Let $\pi : X \to Y$ be the map which takes the constant value y on X_y, for each $y \in Y$. The **identification topology** on Y is defined to be the largest topology for which the map π is continuous. In this topology a set $A \subseteq Y$ is open if and only if $\pi^{-1}(A) \in \tau$. The topological space Y constructed in this way is called an **identification space**.

(8) Suppose G is a group which is also a topological space, and let H be a subgroup of G. Then there is a partition of G into distinct left cosets of H, which allows us to view the collection of cosets G/H as an identification space.

For example consider $\mathbb{R}$ under addition, taken with the usual metric topology. Then $\mathbb{Z}$ is a subgroup and the identification space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle in the complex plane with the subspace topology.

An **open cover** of a set K in a topological space (X, τ) is a collection of open sets whose union contains K. The set K is **compact** if every open cover of K can be replaced by a finite subcover. The set K is **locally compact** if every point in K has a compact neighborhood.

Examples:

(9) *Closed subsets of compact sets are compact.* To see this suppose that $C \subseteq K$ with K compact and C closed. If $\{A_i\}_{i \in I}$ is an open cover of C then, since $A = X \setminus C$ is open, we have that $\{A_i\}_{i \in I} \cup A$ is an open cover of K. By compactness this cover has a finite subcover $\{A'_i\}_{i=1}^n$. After removing A if necessary this gives a finite cover of C which is a subset of $\{A_i\}_{i \in I}$.

(10) *A continuous image of a compact set is compact.* Let $f : X \to Y$ be a continuous map between topological spaces and suppose that $K \subseteq X$ is compact. If $\{A_i\}_{i \in I}$ is an open cover of $f(K)$ then, by continuity of f we have that $\{f^{-1}(A_i)\}_{i \in I}$ is an open cover of K. By compactness this cover has a finite subcover $\{f^{-1}(A'_i)\}_{i=1}^n$, where each of the sets A'_i taken from the original cover, and then $\{A'_i\}_{i=1}^n$ gives a finite cover of $f(K)$.

Finally we mention an important theorem due to Tychonoff, which says that the product of any collection of compact spaces, taken with the product topology, is compact.

In the following lectures we will also assume that everyone has a firm grasp on the definitions of: Hausdorff space, connected, path connected, totally disconnected.