Pontryagin Duality

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We follow [1] closely.

Let G be an abelian topological group. The *Pontryagin dual* of G is the group of continuous homomorphisms $G \to S^1 \subset \mathbb{C}$, and we now describe its (compact-open) topology. A neighbourhood base of the trivial character is given by the set of all

$$W(K, V) = \{\chi \in \hat{G} : \chi(K) \subseteq V\},$$

for compact $K \subseteq G$ and $V \subseteq S^1$ a neighbourhood of 1. The set of all $W(K, V)$ and their translates therefore forms a base for the topology on $\hat{G}$.[1]

Proposition 1. *Let* $N = \{e^{2\pi i\theta} : -\frac{1}{3} < \theta < \frac{1}{3}\}$. *Then*

(i) A group homomorphism $\chi : G \to S^1$ *is continuous if and only if* $\chi^{-1}(N)$ *is a neighbourhood of the identity in* G.

(ii) The set of $W(K, N)$, *for compact* $K \subseteq G$, *is a neighbourhood base for the trivial character.*

(iii) If G *is discrete then* $\hat{G}$ *is compact.*

(iv) If G *is compact then* $\hat{G}$ *is discrete.*

(v) If G *is locally compact then* $\hat{G}$ *is locally compact.*[2]

Proof. (i) We don't need arbitarily small neighbourhoods V, for we can just take sufficiently large compact $K \subseteq G$. The technical details are in [1].

(ii) Likewise.

(iii) Assume that G is discrete. Then $\hat{G}$ is the set of group homomorphisms $G \to S^1$. The *topology of pointwise convergence* on $\hat{G}$ is inherited from

[1] If we allow V to be any open subset of S^1 then the $W(K, V)$ form a base for the topology. Hence this is the compact-open topology (in general these will only form a subbase).

[2] Recall that a topological group is *locally compact* if it is locally compact and Hausdorff as a topological space.

the product topology on $(S^1)^G = \{f : G \to S^1\}$. The projection maps are, for $g \in G$,

$$\begin{aligned} p_g : (S^1)^G &\to S^1 \\ f &\mapsto f(g). \end{aligned}$$

A base for the product topology is given by finite intersections of $p_g^{-1}(U)$, with $g \in G$ and U open in S^1. As the compact subsets of G are precisely the finite ones (since G is compact), the compact-open topology on $\hat{G}$ matches the topology of pointwise convergence.

By Tychonoff's theorem, $(S^1)^G$ is compact, so it remains to show that $\hat{G}$ is a closed subset of $(S^1)^G$. Let $f : G \to S^1$ be a limit point of $\hat{G}$. Suppose for the sake of contradiction that f is not a homomorphism. Then there exist $g, h \in G$ such that $|f(gh) - f(g)f(h)| = 3\varepsilon$ for some $\varepsilon > 0$. Every neighbourhood of f contains a homomorphism, and in particular there exists a homomorphism $F : G \to S^1$ such that

$$|f(g) - F(g)| < \varepsilon, \tag{1}$$
$$|f(h) - F(h)| < \varepsilon, \text{and} \tag{2}$$
$$|f(gh) - F(gh)| < \varepsilon. \tag{3}$$
$$\tag{4}$$

As $F(gh) = F(g)F(h)$ and $|F(g)| = |f(h)| = 1$, we now have

$$\begin{aligned} 3\varepsilon &= |f(gh) - f(g)f(h)| && (5) \\ &\leq |f(gh) - F(gh)| + |F(g)F(h) - F(g)f(h)| + |F(g)f(h) - f(g)f(h)| && (6) \\ &< 3\varepsilon, && (7) \end{aligned}$$

contradiction. We conclude that $\hat{G}$ is closed in $(S^1)^G$, and therefore compact.

(iv) Assume that G is compact. Then the subset $W(G, N) = \{\chi_0\}$ is open in $\hat{G}$, where $\chi_0 : G \to S^1$ is the trivial character $g \mapsto 1$. Its translates are therefore also open, so $\hat{G}$ is discrete.

(v) See [1].

□

Theorem 2. *Let G be a locally compact abelian group (LCA). Then the evaluation map*

$$\begin{aligned}\alpha : G &\to \hat{\hat{G}} \\ g &\mapsto \alpha(g) : \hat{G} \to S^1 \\ &\qquad\qquad \chi \mapsto \chi(g)\end{aligned}$$

is an isomorphism of topological groups.

Our focus will be the proof of this theorem. We can put Haar measure on G since it's LCA. We'll take the theory of positive definite functions as a black box. Fourier inversion holds pointwise for continuous, L^1, positive definite functions to $G \to \mathbb{C}$. Let $V^1(G)$ denote the set of such functions.

For $f \in L^1(G)$, define

$$\begin{aligned}L_z f : G &\to S^1 \\ t &\mapsto f(z^{-1}t).\end{aligned}$$

Lemma 3. *The map α is injective.*

Proof. Let $z \in G \backslash \{1\}$. Suppose, for the sake of contradiction, that $\alpha(z) = 1$. Then $\chi(z) = 1$ for all $\chi \in \hat{G}$. For $f \in L^1(G)$ and $\chi \in \hat{G}$, using the definition of Haar measure,

$$\widehat{L_z f}(\chi) = \int_G f(z^{-1}y)\overline{\chi}(y)dy = \int_G f(y)\overline{\chi(y)}dy = \hat{f}(\chi).$$

Hence $\hat{f} = \widehat{L_z f}$ for all $f \in L^1(G)$. By Fourier inversion, $L_z f = f$ for all $f \in V^1(G)$.

As G is Hausdorff,[3] there exists an open neighbourhood U of the identity such that $U \cap (z^{-1}U) = \emptyset$, and we may choose U small enough to lie within a compact neighbourhood of the identity. Using Urysohn's lemma, we can show that there exists a continuous, positive definite function $f \neq 0$ with support in U. Then $f \in L^1(G)$, being compactly supported, and so $f \in V^1(G)$. Now the supports of f and $L_z f$ are disjoint, contradicting $L_z f = f$. □

[3]Recall that this is part of the definition of a locally compact topological group.

For a compact neighbourhood $\hat{K}$ of the identity character in $\hat{G}$ and an open neighbourhood V of the identity in S^1, let

$$W(\hat{K}, V) = \{\psi \in \hat{\hat{G}} : \psi(\hat{K}) \subseteq V\}.$$

These subsets and their translates form a base for the topology of $\hat{\hat{G}}$.

We use these to construct a base for the topology of G. Put

$$W_G(\hat{K}, V) = W(\hat{K}, V) \cap \alpha(G),$$

and regard these as subsets of G (since α is injective).

Proposition 4. *The subsets $W_G(\hat{K}, V)$ and their translates form a base for the topology of G.*

Proof. (sketch) Let U be an open neighbourhood of the identity $e \in G$. Use Urysohn's lemma to construct a continuous, positive definite function $g : G \to \mathbb{C}$ with support contained in U such that $g(e) = 1$. Use the Fourier transform and some measure theory to show that $g \simeq 1$ on $W_G(K, V)$ (for large enough $\hat{K}$ and small enough V), thereby establishing that

$$W_G(K, V) \subseteq \text{supp}(g) \subseteq U. \tag{8}$$

□

Corollary 5. *The map α is bicontinuous (open and continuous), so α is a homeomorphism onto its image.*

Proof. By construction,[4]

$$\alpha(W_G(\hat{K}, V)) = W(\hat{K}, V) \cap \alpha(G). \tag{9}$$

This shows that α is bicontinuous at the identity, and the result follows by translation. □

Corollary 6. *The image of α is closed in $\hat{\hat{G}}$.*

[4]Recall that the $W_G(\hat{K}, V)$ are considered as subsets of G.

Proof. Equivalently, we show that $\alpha(G)$ is closed in its closure. Since every open subgroup of a topological group is closed,[5] it suffices to show that $\alpha(G)$ is open in its closure. This follows because $\alpha(G)$ is locally compact and dense in its closure.[6] □

It remains to show that $\alpha(G)$ is dense in $\hat{\hat{G}}$. Refer to the book if you're interested.

[5]Efthymios covered this: write the whole group as a disjoint union of cosets. Efthymios also showed us that the closure of a subgroup is a subgroup.

[6]Locally compact and dense in a Hausdorff space implies open.

References

[1] D. Ramakrishnan and R. J. Valenza, *Fourier analysis on number fields* (Graduate Texts in Mathematics, vol. 186, Springer, 1999), chapter 3.