Ergodic Theory and Topological Groups

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Throughout this talk $(G, \mathscr{B}, \mu)$ will denote a measure space. We call the space a probability space if $\mu(G) = 1$. We will also assume that G is a compact group.

1 Haar Measure

Theorem 1.1 (Haar Measure)**.** *Let G be a compact topological group. There exists a regular probability measure μ (called Haar measure) defined on Borel sets of G such that:*

$$\mu(xE) = \mu(E) \quad \forall x \in G, \ \forall E \in \mathscr{B}(G)$$

where $\mathscr{B}(G)$ denotes the Borel σ-algebra.

Note that with this definition Haar measure is unique. Also it follows that $\mu(Ex) = \mu(E), \ \forall x \in G, \ \forall E \in \mathscr{B}(G)$.

2 Measure Preserving Transformations

Definition 2.1. Let $(X_1, \mathscr{B}_1, \mu_1)$, $(X_2, \mathscr{B}_2, \mu_2)$ be probability spaces.

(i) A transformation $T : X_1 \to X_2$ is measurable if $T^{-1}(B_2) \in \mathscr{B}_1$ for any $B_2 \in \mathscr{B}_2$

(ii) A transformation T is measure preserving if it is measurable and $\mu_1(T^{-1}B_2) = \mu_2(B_2)$ for any $B_2 \in \mathscr{B}_2$

(iii) A transformation T is an invertible measure-preserving transformation if T is measure preserving, bijective and T^{-1} is also measure-preserving.

(iv) If $T : (X_1, \mathscr{B}_1, \mu_1) \to (X_1, \mathscr{B}_1, \mu_1)$ is measure-preserving then the measure μ_1 is said to be T-invariant and $(X_1, \mathscr{B}_1, \mu_1, T)$ is called a measure-preserving system.

In this talk we will be concerned with measure preserving-systems.

Theorem 2.2. *Let $(X_1, \mathscr{B}_1, \mu_1)$, $(X_2, \mathscr{B}_2, \mu_2)$ be probability spaces, $T : X_1 \to T_2$ be a transformation and $\mathscr{S}$ be a semi-algebra* [1] *which generates $\mathscr{B}_2$.*
If $A_2 \in \mathscr{S}_2 \Rightarrow T^{-1}(A_2) \in \mathscr{B}_1$ and $\mu_1(T^{-1}(A_2)) = \mu_2(A_2)$ then T is measure-preserving.

Proof. See [2, p. 20]. □

Examples

(i) If a is a fixed element of a compact group G then $T : G \to G$, $T(x) = ax$ preserves Haar measure and is called a rotation.

(ii) Circle Doubling Map:

Let $T_2 : \mathbb{T} \to \mathbb{T}$ be defined

$$T_2(t) = 2t \ (mod\ 1)$$

then T_2 preserves lebesgue measure, $\mu_{\mathbb{T}}$, on the circle.

By 2.2 it is enough to check this on intervals (since these generate the σ-algebra for $\mathbb{T}$).
Let $B = [a, b) \subseteq [0, 1)$

$$T_2^{-1}(B) = \left[\frac{a}{2}, \frac{b}{2}\right) \cup \left[a + \frac{1}{2}, b + \frac{1}{2}\right)$$

This is a disjoint union so:

$$\mu_{\mathbb{T}}(T_2^{-1}(B)) = \frac{1}{2}(b - a) + \frac{1}{2}(b - a) = b - a = \mu_{\mathbb{T}}(B)$$

as required.

Note: We have to study pre-images of these transformations. In the last example if I is a small interval then $T_2(I)$ is an interval with length $2(b - a)$.

Theorem 2.3. *Any continuous endomorphism of a compact group onto itself preserves Haar measure.*

Remark 2.4. A continuous endomorphism of a topological group is a group endomorphism which is continuous as a map between topological spaces.

[1] A set $\mathscr{S} \subseteq \mathbb{P}(G)$ is called a semi-algebra if

1. $\emptyset \in \mathscr{S}$
2. $A, B \in \mathscr{S}$ implies that $A \cap B \in \mathscr{S}$, and
3. if $A \in \mathscr{S}$ then the complement $G \backslash A$ is a finite union of pairwise disjoint elements in $\mathscr{S}$.

Note that this is a weaker condition than being an algebra

Proof. Let $T : G \to G$ be a continuous surjective endomorphism and let m be the Haar measure on G. Define a probability μ on Borel set of G by

$$\mu(E) = m(T^{-1}(E))$$

where E is a Borel set of G.
Note that μ is a regular measure since m is a regular measure.
Then for any $g \in G$, pick x with $T(x) = g$.
Then:

$$\mu(g \cdot E) = m(T^{-1}(g \cdot E)) = m(x \cdot T^{-1}(E)) = m(T^{-1}(E)) = \mu(E)$$

□

Theorem 2.5 (Poincare Recurrence)**.** *Let $T : G \to G$ be a measure-preserving transformation of a probability space $(G, \mathscr{B}, \mu)$ and $E \in \mathscr{B}$.*
Then almost every point $x \in E$ returns to E infinitely often,
i.e. there exists a measurable set $F \subseteq E$ with $\mu(F) = \mu(E)$ such that $\forall x \in F$, there exist naturals $0 < n_1 < n_2 < \ldots$ with

$$T^{n_i}(x) \in F \quad \forall i \geq 1$$

Proof. Let $E_n = \bigcup\limits_{n=N}^{\infty} T^{-n}(E)$ and consider $\bigcap\limits_{N=0}^{\infty} E_N$, which is the set of all points in G which enter E infinitely often under iteration by T.
So $F = E \cap \bigcap\limits_{N=0}^{\infty} E_N$ is the set of all points in E which enter E infinitely often under iteration by T.
So

$$x \in F \Rightarrow \quad \exists\, 0 < n_1 < n_2 < \ldots \quad \text{such that } T^{n_i}(x) \in E,\ \forall i \in \mathbb{N}$$

For each i we have $T^{n_i}(x) \in F$ since $T^{n_i - n_j}(T^{n_j}(x)) \in E_N$, for all j sufficiently large (this shows that $T^{n_i}(x) \in E_N,\ \forall N \in \mathbb{N}$).

Finally we have to show that $\mu(E) = \mu(F)$.
Since $T^{-1}E_N = E_{N+1}$ we get $\mu(E_N) = \mu(E_{N+1})$, $\forall N \in \mathbb{N}$.
Hence

$$\mu(E_0) = \mu(E_N), \quad \forall N \in \mathbb{N}$$

and

$$E_0 \supset E_1 \supset E_2 \supset \ldots \Rightarrow \mu\left(\bigcap_{N=0}^{\infty} E_N\right) = \mu(E_0)$$

So

$$\mu(F) = \mu(E \cap E_0) = \mu(E)$$

since $E_0 \subseteq E$. □

3 Ergodicity

We now move on to talking about the property of ergodicity, which can be thought of as indecomposability for measure-preserving transformations. So given a measure preserving system $(G, \mathscr{B}, \mu, T)$ ergodicity tells us we cannot split G into two subsets of positive measure, each of which are invariant under T.

Definition 3.1. A measure-preserving transformation T of a probability space $(G, \mathscr{B}, \mu)$ is ergodic if for any $B \in \mathscr{B}$ we have

$$T^{-1}B = B \Rightarrow \mu(B) = 0 \text{ or } \mu(B) = 1.$$

We call μ an ergodic measure for T.

Theorem 3.2. *The following are equivalent*

(i) T *is ergodic*

(ii) $\forall A \in \mathscr{B}$; $\mu(T^{-1}A \triangle A) = 0 \Rightarrow \mu(A) = 0$ *or* $\mu(A) = 1$

(iii) $\forall A \in \mathscr{B}$; $\mu(A) > 0 \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} T^{-n}A\right) = 1$

(iv) For $A, B \in \mathscr{B}$; $\mu(A), \mu(B) > 0 \Rightarrow \exists n > 0$ *s.t* $\mu(T^{-n}A \cap B) > 0$

Proof. See [1, p.23] or [2, p. 27]. □

Note that:

- Poincare recurrence implies that almost every orbit of G under T returns close to its starting point infinitely often.
- Ergodic implies that almost every orbit of G under T gets close to almost every point of G infinitely often

with the second remark following from (iii) in the above theorem (3.2).

4 Associated Operator

Now we move on to studying an isometry induced by a measure-preserving system. More details can be found in [1].

Definition 4.1. Given a measure-preserving map T define $U_T : L^2_\mu \to L^2_\mu$ as

$$U_T(f) = f \circ T$$

Recall that L^2_μ is a hilbert space and note that for all $f, g \in Ł^2_\mu$

$$\begin{aligned}\langle U_T f, U_T g\rangle =& \int f \circ T \cdot \overline{g \circ T}\ d\mu \\ =& \int f\overline{g}\ d\mu \qquad\qquad \text{(since } \mu \text{ is T-invariant)} \\ =& \langle f, g\rangle\end{aligned}$$

So U_T is an isometry whenever $(X, \mathscr{B}, \mu, T)$ is a measure-preserving transformation.

Furthermore if T is invertible then the associated operator U_T is a unitary operator[2], called the Koopman operator of T.

With this associated operator we have a new way to describe ergodicity.

Theorem 4.2. *Let $(G, \mathscr{B}, \mu, T)$ be a mesure-preserving system. The following are equivalent:*

1. *T is ergodic*
2. *Whenever f is measurable and $(f \circ T)(x) = f(x)\ \forall x \in G$, f is constant a.e.*
3. *Whenever f is measurable and $(f \circ T)(x) = f(x)$ a.e, f is constant a.e.*
4. *Whenever $f \in Ł^2_\mu$ and $(f \circ T)(x) = f(x)\ \forall x \in G$, f is constant a.e.*
5. *Whenever $f \in Ł^2_\mu$ and $(f \circ T)(x) = f(x)$ a.e, f is constant a.e.*

Proof. Clearly $(iii) \Rightarrow (ii) \Rightarrow (iv)$; $(iii) \Rightarrow (v) \Rightarrow (iv)$.
So if we can show $(i) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$ then we're done.

$(i) \Rightarrow (iii)$:
Let T be ergodic, f be a measurable function and assume $f \circ T = f$ a.e.
Assume that f is real valued, otherwise we can consider real and imaginary parts.
Define for $k, n \in \mathbb{Z}$, $n > 0$

$$X(k,n) = \left\{x : \frac{k}{2^n} \le f(x) < \frac{(k+1)}{2^n}\right\} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{(k+1)}{2^n}\right)\right)$$

Now

$$T^{-1}X(k,n) \triangle X(k,n) \subset \{x : (f \circ T)(x) \neq f(x)\}$$

[2] If $U : \mathscr{H} \to \mathscr{H}_2$ is a continuous linear operator between two Hilbert spaces then U is called unitary if U is invertible and

$$\langle Uh_1, Uh_2\rangle = \langle h_1, h_2\rangle$$

for all $h_1, h_2 \in \mathscr{H}_1$

and since by assumption $\mu(\{x : (f \circ T)(x) \neq f(x)\}) = 0$ this implies

$$\mu(T^{-1}X(k,n) \triangle X(k,n)) = 0.$$

So by (ii) of 3.2, $\mu(X(k,n)) = 0$ or 1.
For each $n \in \mathbb{N}$, $\bigcup_{k\in\mathbb{Z}} X(k,n) = G$ is a disjoint union so there exists a unique $k_n \in \mathbb{Z}$ with $\mu(X(k,n)) = 1$.
Let $Y = \bigcap_{n=1}^{\infty} X(k_n, n)$, then $\mu(Y) = 1$ (as $\{X(k_n,n)\}_{n=1}^{\infty}$ is a descending collection of sets).
Finally since f is constant on Y, f is constant a.e.

$(iv) \Rightarrow (i)$:
Suppose $T^{-1}E = E$ for some $E \in \mathscr{B}$ and let $\chi_E \in Ł^2_\mu$ be the charcteristic function on E. Then

$$(\chi_E \circ T)(x) = \chi_{T^{-1}E}(x) = \chi_E(x) \quad \forall x \in G$$

so by (iv) χ_E is constant a.e. Hence $\chi_E = 0$ a.e or $\chi_E = 1$ a.e.
This implies that $\mu(E) = \int \chi_E \, d\mu = 0$ or 1 as required. □

5 Theorems connecting Topological Groups with Ergodicity

We now consider three theorems which allow us to consider connections between the group properties of a topological group and ergodicity.

Theorem 5.1. *The rotation $T(z) = az$ of the unit circle S^1 is ergodic (relative to Haar measure) iff a is not a root of unity.*

Proof. Suppose a is a root of unity so $a^n = 1$ for some $n \in \mathbb{N}$.
Let $f(z) = z^n$, then $f \circ T = f$ and f is not constant a.e. so T is not ergodic by (ii) in 4.2.

Conversely suppose that a is not a root of unity and let $f \in Ł^2_\mu$ be such that $f \circ T = f$.
Let $f(z) = \sum_{-\infty}^{\infty} b_n z^n$ be its fourier series.
Then $(f \circ T)(z) = f(az) = \sum_{-\infty}^{\infty} b_n a^n z^n$ so $b_n(a^n - 1) = 0$ for all $n \in \mathbb{N}$.
So if $n \neq 0$ then $b_n = 0$, so f is constant a.e.
(v) from 4.2 implies that T is ergodic. □

$a \in S^1$ being a root of unity is equivalent to saying that $\{a^n\}_{-\infty}^{\infty}$ is dense in S^1. With this in mind we now want to generalise to a general compact group. Firstly we need the following lemma from character theory[3]:

[3] I'm hoping this was something covered in an earlier lecture

Lemma 5.2. If H is a closed subgroup of G and $H \neq G$ then there exists $\chi \in \widehat{G}$, $\chi \not\equiv 1$ such that $\chi(h) = 1\ \forall h \in H$.

Theorem 5.3. *Let $T(x) = ax$ be a rotation of G. Then T is ergodic iff $\{a^n\}_{-\infty}^{\infty}$ is dense in G.*

Proof. **Suppose T is ergodic.**
Let H denote the closure of the subgroup $\{a^n\}_{-\infty}^{\infty}$ of G.
Assume that $H \neq G$ then by the lemma there exists $\chi \in \widehat{G}$, $\chi \not\equiv 1$ such that

$$\chi(h) = 1\ \forall h \in H.$$

Then

$$\chi(T(x)) = \chi(ax) = \chi(a)\chi(x) = \chi(x)$$

with the last equality following from the fact that $a \in H$.
Since χ is not constant a.e this contradicts ergodicity.
Hence $H = G$.

Suppose $\{a^n\}_{-\infty}^{\infty}$ is dense in G.
Let $f \in L^2_\mu$ and $f \circ T = f$.
We can write f as a fourier series $f = \sum_i b_i\chi_i$, $\chi_i \in \widehat{G}$. Then

$$\sum_i b_i\chi_i(ax) = \sum_i b_i\chi_i(a)\chi_i(x) = \sum_i b_i\chi_i(x)$$

so if $b_i \neq 0$ then $\chi_i(a) = 1$ and since $\chi_i(a^n) = (\chi_i(a))^n = 1$, $\chi_i \equiv 1$ (since $\{a^n\}_{-\infty}^{\infty}$ is dense in G).
Hence only the constant term of the fourier series can be non-zero, so f is constant a.e.
So once again 4.2 tells us that T is ergodic. □

Theorem 5.4. *Let G be a compact abelian group equipped with Haar measure and $T : G \to G$ be a surjective continuous endomorphism of G. Then T is ergodic iff the trivial character $\chi_0 \equiv 1$ is the only $\chi \in \widehat{G}$ that satisfies $\chi \circ T^n = \chi$ for some $n > 0$.*

Proof. Suppose that whenever $\chi T^n = \chi$ for some $n \geq 1$ we have $\chi \equiv 1$.
Let $f L^2_\mu$ with $f \circ T = f$. Let $f(x)$ have the fourier series $\sum_{-\infty}^{\infty} a_n\chi_n$ for $\chi_n \in \widehat{G}$.
Then $\sum a_n\chi_n(T(x)) = \sum a_n\chi_n(x)$, so if $\chi_n,\ \chi_n \circ T,\ \chi_n \circ T^2, \ldots$ are all distinct then their coefficients are equal and therefore zero.
Hence if $a_n \neq 0$ then there exists $p > 0$ such that $\chi_n(T^p) = \chi_n$.
So $\chi_n \equiv 1$ by assumption and hence f is constant a.e.
4.2 tells us that T is ergodic.

Conversely let T be ergodic and $\chi T^n = \chi$ for some integer $n > 0$.
Choose n to be the least such number. Then

$$f = \chi + \chi T + \ldots + \chi T^{n-1}$$

is invariant under T and not constant a.e (it is the sum of orthogonal functions), which contradicts 4.2. □

References

[1] M.Einsiedler and T.Ward
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[2] Peter Walters
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