Crash Course in Point Set Topology

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A **topological space** is a set X together with a subset τ of $\mathcal{P}(X)$ which satisfies the following three conditions:

(i)

$$\emptyset, X \in \tau,$$

(ii)

$$\{A_i\}_{i\in I} \subseteq \tau \Rightarrow \bigcup_{i\in I} A_i \in \tau, \quad \text{and}$$

(iii)

$$A_1, \ldots, A_n \in \tau \Rightarrow \bigcap_{i=1}^{n} A_i \in \tau.$$

The collection τ is called the **topology** on X, the elements of τ are called **open sets**, and any subset of X which is the complement of an element of τ is called a **closed set**. It follows from (ii) and (iii), using de Morgan's laws from set theory, that an arbitrary intersection or finite union of closed sets, is itself a closed set.

A subset $\mathcal{A} \subseteq \tau$ is called a **base** for (X, τ) if every element of τ can be written as a union of elements of $\mathcal{A}$. In this case we also say that τ is the **topology generated by** $\mathcal{A}$. If there is a countable base $\mathcal{A}$ for τ then we say that (X, τ) is a **second-countable** space.

Examples:

(1) Every set X with more than one element has at least two topologies. The first is the **discrete topology**, in which we take $\tau = \mathcal{P}(X)$. The second is the **trivial topology**, in which we take $\tau = \{\emptyset, X\}$.

(2) If (X, d) is a metric space then the collection of open balls in X generates a topology called the **metric topology**. As a matter of definition, note that when we say that the collection of open balls generates a topology, we are saying that the collection of all sets which are unions of open balls satisfies the requirements for being a topology.

(3) If (X, τ) is a topological space and $S \subseteq X$ then the **subspace topology** on S is defined as

$$\{S \cap A : A \in \tau\}.$$

When thinking of a subset S of X as a topological space with the subspace topology, we may refer to S simply as a **subspace** of X. The subspace topology is also referred to as the **relative topology**.

To develop this example a little more, if we consider $\mathbb{Q}$ as a subset of $\mathbb{R}$, the latter being taken with the Euclidean metric topology, the collection of rational numbers lying in an open interval will be an open set. However a set consisting of a single rational point will not be open in $\mathbb{Q}$ with respect to this topology. By contrast if we are thinking of $\mathbb{Q}$ with respect to the discrete topology then every set is open. This illustrates the fact that in general there are many choices for the topology on a set X, and the natural choice for one problem may not be the right choice for another.

(4) Suppose that τ and τ' are two topologies on X. We say that τ' is **coarser** than τ, and that τ is **finer** than τ', if $\tau' \subseteq \tau$. In other words, τ' is coarser than τ if every open set in (X, τ') is also open in (X, τ).

Given a topological space (X, τ), and any set $A \subseteq X$, the **interior** of A is the union of all open sets contained in A. The **closure** of A is the intersection of all closed sets which contain A. Equivalently, the interior of A is the largest open set contained in A, and the closure of A is the smallest closed set which contains A. We will denote the interior of a set A by A° and its closure by $\overline{A}$. The **boundary** of a set A, denoted by ∂A, is its closure minus its interior, that is

$$\partial A = \overline{A} \setminus A^\circ.$$

For $A, B \subseteq X$, we say that A is **dense** in B if $\overline{A} = B$. The topological space X is called **separable** if there is a countable set A which is dense in X.

For $x \in X$, a set $A \subseteq X$ is called a **neighborhood of** x if there is an open set $U \subseteq A$ with $x \in U$. The topological space (X, τ) is called a **Hausdorff space** if, for any pair of distinct points $x, y \in X$, there is a neighborhood U of x and a neighborhood V of y with $U \cap V = \emptyset$. Let $\mathcal{U}_x$ denote the collection of all neighborhoods of x. We say that a collection of neighborhoods $\mathcal{V}_x \subseteq \mathcal{U}_x$ is a **neighborhood base** for x if, for every $U \in \mathcal{U}_x$, there is a neighborhood $V \in \mathcal{V}_x$ with $V \subseteq U$.

A point $x \in X$ is called a **limit point** of a set $A \subseteq X$ is, for every neighborhood U of x, we have that

$$(U \cap A) \setminus \{x\} \neq \emptyset.$$

It is not difficult to verify that the closure of a set A is equal to the union of A with the set of all points $x \in X$ which are limit points of A.

If (X, τ_X) and (Y, τ_Y) are two topological spaces then we say that a function $f : X \to Y$ is **continuous** if $f^{-1}(A) \in \tau_X$ for all $A \in \tau_Y$. If there is a continuous bijective map $f : X \to Y$, for which f^{-1} is also continuous, then we say that (X, τ_X) and (Y, τ_Y) are **homeomorphic**.

Examples:

(5) Suppose that X is a set and $\{Y_i\}_{i \in I}$ is a collection of topological spaces, and for each $i \in I$ let f_i be a function from X to Y_i. The **initial topology** on X with respect to $\{f_i\}$ is the coarsest topology on X with respect to which all of the functions f_i are continuous.

(6) A 'dual' notion to the topology defined in the previous example is the following. Suppose X is a set and $\{Y_i\}_{i \in I}$ is a collection of topological spaces, and for each $i \in I$ let f_i be a function from Y_i to X. Then the **final topology** on X with respect to $\{f_i\}$ is the finest topology on X with respect to which all of the functions f_i are continuous.

(7) Suppose $\{(X_i, \tau_i)\}_{i \in I}$ is a collection of topological spaces, let X be the Cartesian product $\prod_{i \in I} X_i$ and for each $j \in I$ let $\pi_j : X \to X_j$ be the **projection map** (i.e. the map onto the jth coordinate). The **product topology** on X is defined to be the initial topology with respect to $\{\pi_j\}_{j \in I}$. A base for this topology is

$$\left\{ \prod_{i \in I} A_i : A_i \in \tau_i,\ A_i = X_i \text{ for all but finitely many } i \right\}.$$

(8) With the same notation as in the previous example, the **box topology** on X is defined to be the topology generated by

$$\left\{ \prod_{i \in I} A_i : A_i \in \tau_i \right\}.$$

When I is finite this topology is the same as the product topology. *However in general the two topologies are not the same.* One strong argument in favor of using the product topology is that Tychonoff's Theorem (see below) is not true in general for the box topology.

(9) Let $\{(X_i, d_i)\}_{i=1}^n$ be metric spaces and consider the Cartesian product $X = \prod_{i=1}^n X_i$. This product is also a metric space with respect to the metric $d : X \times X \to [0, \infty)$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le n} \{d_i(x_i, y_i)\},$$

and it is not difficult to show that X with the induced metric topology is homeomorphic to X with the product topology (i.e. with each X_i taken with the metric topology).

It is also true that the product of countably many metric spaces $\{(X_i, d_i)\}_{i=1}^{\infty}$, taken with the product topology, is **metrizable** (i.e. there is a metric on the Cartesian product for which the induced metric topology is the product topology). One metric which realizes the product topology is given by

$$d(\mathbf{x},\ \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}.$$

In general it is not always true that a product of metric spaces with respect to the product topology is metrizable.

(10) Let (X, τ) be a topological space and suppose that $X = \cup_{y \in Y} X_y$ is a partition of the set X. Let $\pi : X \to Y$ be the map which takes the constant value y on X_y, for each $y \in Y$. The **identification topology** on Y is defined to be the finest topology for which the map π is continuous. In this topology a set $A \subseteq Y$ is open if and only if $\pi^{-1}(A) \in \tau$. The topological space Y constructed in this way is called an **identification space**.

(11) Suppose G is a group which is also a topological space, and let H be a subgroup of G. Then there is a partition of G into distinct left cosets of H, which allows us to view the collection of cosets G/H as an identification space.

For example consider $\mathbb{R}$ under addition, taken with the usual metric topology. Then $\mathbb{Z}$ is a subgroup and the identification space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle in the complex plane with the subspace topology.

An **open cover** of a set K in a topological space (X, τ) is a collection of open sets whose union contains K. The set K is **compact** if every open cover of K can be replaced by a finite subcover. The set K is **locally compact** if every point in K has a compact neighborhood.

Examples:

(12) *Closed subsets of compact sets are compact.* To see this suppose that $C \subseteq K$ with K compact and C closed. If $\{A_i\}_{i \in I}$ is an open cover of C then, since $A = X \setminus C$ is open, we have that $\{A_i\}_{i \in I} \cup A$ is an open cover of K. By compactness this cover has a finite subcover $\{A_i'\}_{i=1}^{n}$. After removing A if necessary this gives a finite cover of C which is a subset of $\{A_i\}_{i \in I}$.

(13) *A continuous image of a compact set is compact.* Let $f : X \to Y$ be a continuous map between topological spaces and suppose that $K \subseteq X$ is compact. If $\{A_i\}_{i\in I}$ is an open cover of $f(K)$ then, by continuity of f we have that $\{f^{-1}(A_i)\}_{i\in I}$ is an open cover of K. By compactness this cover has a finite subcover $\{f^{-1}(A'_i)\}_{i=1}^n$, where each of the sets A'_i taken from the original cover, and then $\{A'_i\}_{i=1}^n$ gives a finite cover of $f(K)$.

An important theorem due to Tychonoff says that *the product of any collection of compact spaces, taken with the product topology, is compact.*

A topological space (X, τ) is **disconnected** if it can be written as a disjoint union of two nonempty open sets. In other words X is disconnected if there exist nonempty $U, V \in \tau$ with $U \cap V = \emptyset$ and $X = U \cup V$. If X is not disconnected then we say that it is **connected**. A subset S of X is connected (resp. disconnected) if it is connected (resp. disconnected) as a topological space with the subspace topology.

Examples:

(14) Suppose $\{S_i\}_{i\in I}$ is a nonempty collection of connected subsets of (X, τ) with the property that, for any $i, j \in I$, the intersection $S_i \cap S_j$ is non-empty. Then the union $S = \cup_{i\in I} S_i$ is also connected.

(15) If S is a connected subset of X then the closure $\overline{S}$ is also connected.

(16) For any point $x \in X$, the set $\{x\}$ is trivially seen to be connected. By (14) from above, the union of all connected subsets of X which contain x is also a connected set, called the **connected component** of x. It is clear from (15) that the connected component of any point is a closed subset of X.

(17) For any $x, y \in X$, the connected components of x and y are either the same, or they are disjoint. Therefore the collection of connected components in a topological space is a partition of the space into disjoint, closed, connected subsets. A space X in which the connected components are all singleton sets (i.e. sets consisting of only one element) is called **totally disconnected**.