UNIQUE EQUILIBRIUM STATES FOR THE ROBUSTLY TRANSITIVE DIFFEOMORPHISMS OF MAÑÉ AND BONATTI–VIANA

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Abstract. We show that the families of robustly transitive diffeomorphisms of Mañé and Bonatti–Viana have unique equilibrium states for natural classes of potentials. In particular, for any Hölder continuous potential on the phase space of one of these families, we construct a C^1-open neighborhood of a diffeomorphism in that family for which the potential has a unique equilibrium state. We also characterize the SRB measures for these diffeomorphisms as unique equilibrium states for a suitable geometric potential. These results are an application of general machinery developed by the first and last named authors, and are among the first results on uniqueness of equilibrium states in the setting of diffeomorphisms with partial hyperbolicity or dominated splittings.

1. Introduction

An *equilibrium state* for a diffeomorphism $f\colon M \to M$ and a potential $\varphi\colon M \to \mathbb{R}$ is an invariant Borel probability measure that maximizes the quantity $h_\mu(f) + \int \varphi\, d\mu$. Results on existence and uniqueness of equilibrium states have a long history [11, 31, 34, 14, 54, 23, 30, 40, 44, 45], and are one of the main goals in thermodynamic formalism. Results of this type are a powerful tool to understand the orbit structure and global statistical properties of dynamical systems, and often lead to further applications, including large deviations principles, central limit theorems, and knowledge of dynamical zeta functions [43, 59].

The benchmark result of this type is that there is a unique equilibrium state μ when (M, f) is uniformly hyperbolic, mixing, and φ is Hölder continuous. Moreover, when φ is the *geometric potential* $\varphi(x) = -\log|\det Df|_{E^u(x)}|$, this unique equilibrium state is the SRB

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measure [11, 53]. Extending this type of result beyond uniform hyperbolicity is a major challenge in the field. The first and third authors have developed techniques to establish existence and uniqueness of equilibrium states in the presence of non-uniform versions of specification and expansivity [26], generalizing the classic work of Bowen [10]. We apply these results to higher dimensional smooth systems with weak forms of hyperbolicity, where alternative approaches based on symbolic dynamics or transfer operators appear to meet with fundamental difficulties.

As test cases for our techniques, we focus on two classes of diffeomorphisms originally introduced by Mañé [39] and Bonatti–Viana [6]. These are well-studied classes of smooth systems which exhibit interesting dynamical phenomena that do not arise in the Anosov setting. Both families are C^0 perturbations of hyperbolic toral automorphisms with positive irrational simple eigenvalues; the Mañé family is constructed from a d-dimensional automorphism f_A with a 1-dimensional unstable bundle, while the Bonatti–Viana family is constructed from a 4-dimensional automorphism f_B with a 2-dimensional unstable bundle. The Mañé examples are partially hyperbolic, while the Bonatti–Viana examples admit a dominated splitting, but are not partially hyperbolic.

Our main results are Theorem 4.1 for the Mañé family, and Theorem 6.1 for the Bonatti-Viana family. These results give a quantitative criterion for existence and uniqueness of the equilibrium state involving the topological pressure, the norm and variation of the potential, the tail entropy of the system, and the C^0 size of the perturbation from the original Anosov map. All quantities under consideration vary continuously under a C^1 perturbation of the map. The following statements follow from the more technical statements of Theorems 4.1 and 6.1.

Theorem A. *Let f_A be as above and let $\varphi\colon \mathbb{T}^d \to \mathbb{R}$ be Hölder continuous. Then in any C^0-neighborhood of f_A there exists a C^1-open set $\mathcal{U} \subset \mathrm{Diff}(\mathbb{T}^d)$ which contains diffeomorphisms from the Mañé family of examples, such that for every $g \in \mathcal{U}$ we have:*

- *g is partially hyperbolic and not Anosov;*
- *the system $(\mathbb{T}^d, g, \varphi)$ has a unique equilibrium state.*

Theorem B. *Let f_B be as above and let $\varphi\colon \mathbb{T}^4 \to \mathbb{R}$ be Hölder continuous. Then in any C^0 neighborhood of f_B there is a C^1-open set $\mathcal{V} \subset \mathrm{Diff}(\mathbb{T}^4)$ which contains diffeomorphisms from the Bonatti-Viana family of examples, such that for every $g \in \mathcal{V}$ we have:*

- *g has a dominated splitting and is not partially hyperbolic;*
- *the system $(\mathbb{T}^4, g, \varphi)$ has a unique equilibrium state.*

We can also fix a diffeomorphism g and obtain unique equilibrium states for all Hölder continuous φ subject to a bounded range hypothesis $\sup\varphi - \inf\varphi < D$ for some $D > 0$. This result applies exactly when our criterion guarantees the existence of a unique measure of maximal entropy for g. Precise statements of this 'bounded range' theorem are given as Theorem 4.2 and Theorem 6.2.

We can characterize the SRB measures for these families as equilibrium states for a suitable geometric potential φ^{geo}. For the Mañé family, $\varphi^{\text{geo}} = -\log J^u(x)$, and for the Bonatti–Viana family $\varphi^{\text{geo}} = -\log J^{cu}(x)$, where $J^u(x)$ and $J^{cu}(x)$ are the Jacobian determinants in the unstable and center-unstable direction respectively, see §8 for details. Our results on SRB meaures are given in Theorems 8.2 and 8.4, and follow from a quantitative criterion on the diffeomorphism and the potential φ^{geo}. An immediate consequence of these results is the following statement.

Theorem C. *In any C^0-neighborhood of f_A (resp. f_B), there exists a C^1-open set $\mathcal{U}$ (resp. $\mathcal{V}$) of C^2 diffeomorphisms which is a neighborhood of a Mañé (resp. Bonatti-Viana) example, such that for every $g \in \mathcal{U}\cup\mathcal{V}$ the following are true.*

- *$t = 1$ is the unique root of the function $t \mapsto P(t\varphi^{geo})$.*
- *There is an $\varepsilon > 0$ such that $t\varphi^{geo}$ has a unique equilibrium state μ_t for each $t \in (-\varepsilon, 1+\varepsilon)$.*
- *μ_1 is the unique SRB measure for g.*

In §9, we derive consequences of this result for the multifractal analysis of the largest Lyapunov exponent, and obtain a large deviations result for many of the equilibrium states produced by our main theorems. Our large deviations result is as follows.

Theorem D. *Let μ be a unique equilibrium state provided by the conclusion of Theorem A, or by Theorem B with $\varphi = 0$, or by Theorem C applied with $g \in \mathcal{U}$. Then μ satisfies the upper inequality of the level-2 large deviations principle.*

The upper level-2 large deviations principle includes the following estimate on the rate of decay of the measure of points whose Cesàro sums experience a 'large deviation' from the expected value:

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \mu \left\{ x : \left| \frac{1}{n} S_n\varphi(x) - \int \varphi \, d\mu \right| > \varepsilon \right\} \leq -q(\varepsilon) < 0, \tag{1.1}$$

where $\varepsilon > 0$, φ is any potential, and $q(\varepsilon)$ is a *rate function*, whose precise value is given in (9.2). Our result is a consequence of a general large deviations result of Pfister and Sullivan [48], and a weak upper

Gibbs property which is satisfied by our equilibrium states. See §9 for more details. It is still an open question to establish this upper large deviations bound for equilibrium states of the Bonatti–Viana examples beyond the MME case, or to establish lower large deviations bounds for either class of examples.

Techniques. Our results are proved constructively using general machinery developed by the first and last named authors [26]. The main idea is to *decompose* the collection of orbit segments to obtain a 'large' collection of 'good' orbit segments on which the map has uniform expansion and contraction properties, leading to a non-uniform version of the specification property and the Bowen property.

The diffeomorphisms we consider are not expansive. In particular, the Bonatti-Viana examples may not even be asymptotically h-expansive, and thus may have positive tail entropy [17]. We handle this by showing that any measure with large enough free energy is *almost expansive* (Definition 2.3), so the failure of expansivity does not affect equilibrium states.

The analysis of the Bonatti-Viana family involves some serious difficulties which are not present in the Mañé family. The key difference is that the Mañé family, and its C^1 perturbations, have uniform expansion on the unstable distribution, and zero tail entropy; this considerably simplifies our analysis in that setting. These fundamental differences motivate why we choose these two families of examples to demonstrate our techniques.

Future directions. The techniques introduced in this paper are expected to apply beyond the Mañé and Bonatti–Viana familes. For instance, the Shub class of robustly transitive diffeomorphisms [33] should follow from modifications of the arguments we present. Our approach is based on exploiting the uniform expansion/contraction of the system away from a finite collection of neighborhoods, and as such is likely to be suitable in other settings beyond uniform hyperbolicity.

We anticipate that further results on statistical properties of the equilibrium measures obtained here will be possible (see [27, 22] for such results in the simpler symbolic setting). This is not pursued here except for the upper large deviations statement of Theorem D.

Context of the results. In the partially hyperbolic setting, there are some results on uniqueness of MME, i.e. equilibrium states for $\varphi \equiv 0$. For ergodic toral automorphisms, the Haar measure was shown to be the unique MME by Berg [4] using convolutions. Ures showed uniqueness of the MME for partially hyperbolic diffeomorphisms of the 3-torus homotopic to a hyperbolic automorphism [57].

For the Mañé and Bonatti-Viana examples, the existence of a unique MME was obtained in [16, 17]. It was shown that a semi-conjugacy π between these examples and the unperturbed toral automorphism f_L induces a measure theoretic isomorphism between 'large entropy' invariant measures for g and f_L. The result then follows from the uniqueness of the MME for the automorphism f_L. For this approach to generalize to equilibrium states, we would have to restrict to potentials φ where $\varphi \circ \pi$ is a well-defined function with good regularity properties; this would be a very strong assumption since we would need to know that φ is constant on fibers of the semiconjugacy, and we do not expect π to preserve Hölder continuity.

Equilibrium states for $\varphi \neq 0$ have been largely unexplored in the partially hyperbolic setting. Existence of equilibrium states for partially hyperbolic horseshoes was studied by Leplaideur, Oliveira, and Rios [38], but they do not deal with uniqueness. Other recent references which apply in higher dimensional settings include [18, 45, 19]. In particular, Pesin, Senti and Zhang [45] have used tower techniques to develop thermodynamic formalism for the Katok map, which is a non-uniformly hyperbolic DA map of the 2-torus.

The theory of SRB measures has received more attention. The fact that there is a unique SRB measure for the examples we study follows from [6, 1, 55], and the statistical properties of these measures is an active area of research [2, 45]. The characterization of the SRB measure as a unique equilibrium state is completely novel in this setting; immediate consequences include the upper large deviations principle of Theorem D and the multifractal results of §9.2.

Structure of the paper. In §2, we give background material on thermodynamic formalism, specification and expansivity, and state the general results from [26] which give the existence of a unique equilibrium state (Theorems 2.8 and 2.9). In §3, we collect results on Anosov systems and their C^0-perturbations, including a pressure estimate (Theorem 3.3) that will allow us to verify the conditions of Theorems 2.8 and 2.9 in the settings of Theorems A–C. Details and results concerning Mañé's examples are given in §4, and proved in §5. Details and results concerning the Bonatti–Viana examples are given in §6, and proved in §7. Theorem C on SRB measures is proved in §8. The large deviations and multifractal results are proved in §9. In §10, we prove the pressure estimates from §3, and in §11 we prove various lemmas used earlier in the paper.

2. Background

In this section, we state definitions and results that we will need throughout the paper. We begin with a review of facts from thermodynamic formalism, and then state the general results we will use for the existence and uniqueness of equilibrium states.

2.1. **Pressure.** Let X be a compact metric space and $f\colon X \to X$ be a continuous map. Henceforth, we will identify $X \times \mathbb{N}$ with the space of finite orbit segments for a map f via the correspondence

$$(x,n) \quad \longleftrightarrow \quad (x, f(x), \dots, f^{n-1}(x)). \tag{2.1}$$

Fix a continuous potential function $\varphi\colon X \to \mathbb{R}$. We write

$$S_n\varphi(x) = S_n^f\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k x)$$

for the ergodic sum along an orbit segment, and given $\eta > 0$, we write

$$\mathrm{Var}(\varphi,\eta) = \sup\{|\varphi(x) - \varphi(y)| \,:\, x,y \in X, d(x,y) < \eta\}.$$

Given $n \in \mathbb{N}$ and $x,y \in X$, we write

$$d_n(x,y) = \max\{d(f^k x, f^k y) \,:\, 0 \le k < n\}.$$

Given $x \in X$, $\varepsilon > 0$, and $n \in \mathbb{N}$, the *Bowen ball of order n with center x and radius ε* is

$$B_n(x,\varepsilon) = \{y \in X \,:\, d_n(x,y) < \varepsilon\}.$$

We say that $E \subset X$ is (n,ε)-separated if $d_n(x,y) \ge \varepsilon$ for all $x,y \in E$.

We will need to consider the *pressure of a collection of orbit segments*. More precisely, we interpret $\mathcal{D} \subset X \times \mathbb{N}$ as a collection of finite orbit segments, and write $\mathcal{D}_n = \{x \in X \,:\, (x,n) \in \mathcal{D}\}$ for the set of initial points of orbits of length n in $\mathcal{D}$. Then we consider the partition sum

$$\Lambda_n^{\mathrm{sep}}(\mathcal{D},\varphi,\varepsilon;f) = \sup\left\{\sum_{x\in E} e^{S_n\varphi(x)} \,:\, E \subset \mathcal{D}_n \text{ is } (n,\varepsilon)\text{-separated}\right\}.$$

When there is no confusion in the map we will sometimes omit the dependence on f in the above notation and the notation below. We will also sometimes require a partition sum $\Lambda_n^{\mathrm{span}}$ defined with (n,ε)-spanning sets. Given $Y \subset X$, $n \in \mathbb{N}$, and $\delta > 0$, we say that $E \subset Y$ is an (n,δ)-*spanning set for* Y if $\bigcup_{x\in E} \overline{B_n(x,\delta)} \supset Y$. Write

$$\Lambda_n^{\mathrm{span}}(\mathcal{D},\varphi,\delta;f) = \inf\left\{\sum_{x\in E} e^{S_n\varphi(x)} \,:\, E \subset \mathcal{D}_n \text{ is } (n,\delta)\text{-spanning}\right\}.$$

We will use the following basic result relating Λ_n^{sep} and $\Lambda_n^{\mathrm{span}}$, which is proved in §11.

Lemma 2.1. *For any $\mathcal{D} \subset X \times \mathbb{N}$, $\varphi \colon X \to \mathbb{R}$, and $\delta > 0$, we have*

$$\Lambda_n^{\mathrm{span}}(\mathcal{D}, \varphi, \delta) \leq \Lambda_n^{\mathrm{sep}}(\mathcal{D}, \varphi, \delta),$$
$$\Lambda_n^{\mathrm{sep}}(\mathcal{D}, \varphi, 2\delta) \leq e^{n \operatorname{Var}(\varphi,\delta)} \Lambda_n^{\mathrm{span}}(\mathcal{D}, \varphi, \delta).$$

The *pressure of φ on $\mathcal{D}$ at scale ε* is

$$P(\mathcal{D}, \varphi, \varepsilon; f) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \Lambda_n^{\mathrm{sep}}(\mathcal{D}, \varphi, \varepsilon),$$

and the *pressure of φ on $\mathcal{D}$* is

$$P(\mathcal{D}, \varphi; f) = \lim_{\varepsilon \to 0} P(\mathcal{D}, \varphi, \varepsilon).$$

More precisely, this is the *upper capacity topological pressure*, and one could also consider the lower capacity pressure, obtained by taking a liminf. Then one would write $\overline{P}$ and $\underline{P}$ to distinguish the two. Since lower pressure will play no role in this paper, we choose the more streamlined notation and terminology. The above definition appears in [20, §2.1] and is a non-stationary version of the usual notion of upper capacity pressure [46]. For a set $Z \subset X$, we let $P(Z, \varphi, \varepsilon; f) := P(Z \times \mathbb{N}, \varphi, \varepsilon; f)$, and thus $P(Z, \varphi; f)$ denotes the usual upper capacity pressure.

When $\varphi = 0$ the above definition gives the *entropy of $\mathcal{D}$*:

$$h(\mathcal{D}, \varepsilon; f) = h(\mathcal{D}, \varepsilon) := P(\mathcal{D}, 0, \varepsilon) \text{ and } h(\mathcal{D}) = \lim_{\varepsilon\to 0} h(\mathcal{D}, \varepsilon). \tag{2.2}$$

We sometimes write $P(f, \varphi)$ for the topological pressure of the whole space $P(X, \varphi; f)$, particularly when we need to compare pressure for two different maps f and g.

We let $\mathcal{M}(f)$ denote the set of f-invariant Borel probability measures and $\mathcal{M}_e(f)$ the set of ergodic f-invariant Borel probability measures. The variational principal for pressure [58, Theorem 9.10] states that if X is a compact metric space and f is continuous, then

$$P(f, \varphi) = \sup_{\mu \in \mathcal{M}(f)} \left\{ h_\mu(f) + \int \varphi \, d\mu \right\} = \sup_{\mu \in \mathcal{M}_e(f)} \left\{ h_\mu(f) + \int \varphi \, d\mu \right\}.$$

A measure achieving the supremum is an *equilibrium state*, and these are the objects whose existence and uniqueness we wish to study.

2.2. **Expansivity and tail entropy.** Given a homeomorphism $f \colon X \to X$ and $\varepsilon > 0$, consider for each $x \in X$ and $\varepsilon > 0$ the set

$$\Gamma_\varepsilon(x) := \{y \in X \,:\, d(f^k x, f^k y) < \varepsilon \text{ for all } n \in \mathbb{Z}\}$$

is the *(bi-infinite) Bowen ball of x of size ε*. Note that f is expansive if and only if there exists $\varepsilon > 0$ so that $\Gamma_\varepsilon(x) = \{x\}$ for all $x \in X$.

For systems that fail to be expansive, it is useful to consider the *tail entropy* of f at scale $\varepsilon > 0$ is

$$h_f^*(\varepsilon) = \sup_{x\in X} \lim_{\delta\to 0} \limsup_{n\to\infty} \Lambda_n^{\mathrm{span}}(\Gamma_\varepsilon(x) \times \mathbb{N}, 0, \delta; f). \tag{2.3}$$

This quantity was introduced in [8]; equivalent definitions can also be formulated using open covers [42].

The map f is entropy-expansive if $h_f^*(\varepsilon) = 0$ for some $\varepsilon > 0$, and is asymptotically h-expansive if $h_f^*(\varepsilon) \to 0$ as $\varepsilon \to 0$. See [12, 15] for connections between these notions and the theory of symbolic extensions. An interesting result of [12] is that positive tail entropy rules out the existence of a principal symbolic extension, and thus symbolic dynamics fails in a strong way for such systems.

For our purposes, the key property of tail entropy is that given a collection $\mathcal{D} \subset X \times \mathbb{N}$, it allows us to control $h(\mathcal{D}, \delta)$ in terms of $h(\mathcal{D}, \varepsilon)$ for some $0 < \delta < \varepsilon$. The following is proved in §11.

Lemma 2.2. *Given any $\mathcal{D} \subset X \times \mathbb{N}$ and $0 < \delta < \varepsilon$, we have*

$$h(\mathcal{D}, \delta; f) \leq h(\mathcal{D}, \varepsilon; f) + h_f^*(\varepsilon).$$

In particular, $h(\mathcal{D}; f) \leq h(\mathcal{D}, \varepsilon; f) + h_f^*(\varepsilon)$.

2.3. **Obstructions to expansivity, specification, and regularity.** It was shown by Bowen [10] that (X, f, φ) has a unique equilibrium state whenever (X, f) has expansivity and specification, and φ has a certain regularity property. We require the results from [26], which give existence and uniqueness in the presence of 'obstructions to specification and regularity' and 'obstructions to expansivity'. The idea is that if these obstructions have smaller pressure than the whole system, then existence and uniqueness holds.

2.3.1. *Expansivity.* In our examples, expansivity does not hold, so we introduce a suitable measurement of the size of the non-expansive points, introduced in [25, 26].

Definition 2.3. *For $f\colon X \to X$ the set of non-expansive points at scale ε is $\mathrm{NE}(\varepsilon) := \{x \in X : \Gamma_\varepsilon(x) \neq \{x\}\}$. An f-invariant measure μ is almost expansive at scale ε if $\mu(\mathrm{NE}(\varepsilon)) = 0$. Given a potential φ,* *the pressure of obstructions to expansivity at scale* ε *is*

$$\begin{aligned}\mathrm{P}^{\perp}_{\exp}(\varphi,\varepsilon) &= \sup_{\mu\in\mathcal{M}_e(f)}\left\{h_\mu(f)+\int\varphi\,d\mu\;:\;\mu(\mathrm{NE}(\varepsilon))>0\right\}\\ &= \sup_{\mu\in\mathcal{M}_e(f)}\left\{h_\mu(f)+\int\varphi\,d\mu\;:\;\mu(\mathrm{NE}(\varepsilon))=1\right\}.\end{aligned}$$

We define a scale-free quantity by

$$\mathrm{P}^{\perp}_{\exp}(\varphi)=\lim_{\varepsilon\to 0}\mathrm{P}^{\perp}_{\exp}(\varphi,\varepsilon).$$

2.3.2. *Specification.* The following specification property was introduced in [25].

Definition 2.4. *A collection of orbit segments* $\mathcal{G}\subset X\times\mathbb{N}$ *has* specification at scale ε *if there exists* $\tau\in\mathbb{N}$ *such that for every* $\{(x_j,n_j):0\le j\le k\}\subset\mathcal{G}$*, there is a point* x *in*

$$\bigcap_{j=0}^{k} f^{-(m_{j-1}+\tau)}B_{n_j}(x_j,\varepsilon),$$

where $m_{-1}=-\tau$ *and* $m_j=\left(\sum_{i=0}^{j}n_i\right)+j\tau$ *for each* $j\ge 0$.

The above definition says that there is some point x whose trajectory shadows each of the (x_i,n_i) in turn, taking a transition time of exactly τ iterates between each one. The numbers m_j for $j\ge 0$ are the time taken for x to shadow (x_0,n_0) up to (x_j,n_j).

It is sometimes convenient to consider collections $\mathcal{G}$ in which only long orbit segments have specification, and this motivates the following definition.

Definition 2.5. *A collection of orbit segments* $\mathcal{G}\subset X\times\mathbb{N}$ *has* tail specification at scale ε *if there exists* $N_0\in\mathbb{N}$ *so that the collection* $\mathcal{G}_{\ge N_0}:=\{(x,n)\in\mathcal{G}\mid n\ge N_0\}$ *has specification at scale* ε.

2.3.3. *Regularity.* We require a regularity condition for the potential φ on the collection $\mathcal{G}$, inspired by the Bowen condition [10], which was introduced in [24, 26]

Definition 2.6. *Given* $\mathcal{G}\subset X\times\mathbb{N}$*, a potential* φ *has the* Bowen property on $\mathcal{G}$ at scale ε *if*

$$V(\mathcal{G},\varphi,\varepsilon):=\sup\{|S_n\varphi(x)-S_n\varphi(y)|:(x,n)\in\mathcal{G},y\in B_n(x,\varepsilon)\}<\infty.$$

We say φ *has the* Bowen property on $\mathcal{G}$ *if there exists* $\varepsilon>0$ *so that* φ *has the Bowen property on* $\mathcal{G}$ *at scale* ε.

Note that if $\mathcal{G}$ has the Bowen property at scale ε, it has it for all smaller scales. Thus, if φ has the Bowen property on $\mathcal{G}$, it has it for all sufficiently small scales.

2.4. **General results on uniqueness of equilibrium states.** The tool we use to prove existence and uniqueness of equilibrium states and are stated as Theorems B and D from [26], which generalizes results from [24, 25].

We give two formulations of this abstract result. The first is a simpler statement that is sufficient for the Mañé examples, and is a consequence of the more complicated second statement. The more general formulation is necessary for the application to the Bonatti-Viana examples. The basic idea is to find a collection of orbit segments $\mathcal{G} \subset X \times \mathbb{N}$ that satisfies specification and the Bowen property, and that is sufficiently large in an appropriate sense. To make this notion of largeness precise, we need the following definition. We denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 2.7. *A* decomposition *for* (X, f) *consists of three collections* $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset X \times \mathbb{N}_0$ *and three functions* $p, g, s \colon X \times \mathbb{N} \to \mathbb{N}_0$ *such that for every* $(x, n) \in X \times \mathbb{N}$*, the values* $p = p(x, n)$*,* $g = g(x, n)$*, and* $s = s(x, n)$ *satisfy* $n = p + g + s$*, and*

$$(x, p) \in \mathcal{P}, \quad (f^p(x), g) \in \mathcal{G}, \quad (f^{p+g}(x), s) \in \mathcal{S}. \tag{2.4}$$

Given a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ *and* $M \in \mathbb{N}$*, we write* $\mathcal{G}^M$ *for the set of orbit segments* (x, n) *for which* $p \leq M$ *and* $s \leq M$*.*

Without loss of generality, we always assume $X \times \{0\} \subset \mathcal{P} \cap \mathcal{G} \cap \mathcal{S}$ to allow for 'trivial' decompositions. That is, an (x, n) which belongs to one of the collections $\mathcal{P}, \mathcal{G}$ or $\mathcal{S}$ or transitions directly from $\mathcal{P}$ to $\mathcal{S}$ can be assigned an obvious 'trivial' decomposition. We say that $\mathcal{S}$ (respectively $\mathcal{P}$) is trivial if we can take $s(x, n) = 0$ (respectively $p(x, n) = 0$) for every $(x, n) \in X \times \mathbb{N}$.

The following version of our abstract result from [26] is suitable for application to the Mañé examples.

Theorem 2.8. *Let* X *be a compact metric space and* $f \colon X \to X$ *a homeomorphism. Let* $\varphi \colon X \to \mathbb{R}$ *be a continuous potential function. Suppose that* $P^{\perp}_{\exp}(\varphi) < P(\varphi)$*, and that* $X \times \mathbb{N}$ *admits a decomposition* $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ *with the following properties:*

(1) $\mathcal{G}$ *has specification at any scale;*
(2) φ *has the Bowen property on* $\mathcal{G}$*;*
(3) $P(\mathcal{P} \cup \mathcal{S}, \varphi) < P(\varphi)$*.*

Then there is a unique equilibrium state for φ*.*

For the Bonatti–Viana examples we need another version of our abstract result from [26], where we work at a fixed scale instead of requiring specification at all scales. The price we pay is that we must ask for specification for each collection $\mathcal{G}^M$. Some extra complications associated with not being able to take a limit $\varepsilon \to 0$ enter the statement, and we explain these after the result.

Theorem 2.9. *Let X be a compact metric space and $f\colon X \to X$ a homeomorphism. Let $\varphi\colon X \to \mathbb{R}$ be a continuous potential function. Suppose there exists $\varepsilon > 0$ such that $P^{\perp}_{\exp}(\varphi, 100\varepsilon) < P(\varphi)$ and $X \times \mathbb{N}$ admits a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ with the following properties:*

(1) For each $M \geq 0$, $\mathcal{G}^M$ has tail specification at scale ε;
(2) φ has the Bowen property at scale 100ε on $\mathcal{G}$;
(3) $P(\mathcal{P} \cup \mathcal{S}, \varphi, \varepsilon) + \mathrm{Var}(\varphi, 100\varepsilon) < P(\varphi)$.

Then there is a unique equilibrium state for φ.

We make some comments on this formulation of the result:

(1) The transition time τ for specification for $\mathcal{G}^M$ is allowed to depend on M. Note that if $\mathcal{G}$ had specification at all scales, then a simple argument based on modulus of continuity of f shows that (1) is true for any ε. Thus, considering $\mathcal{G}^M$ for all M at a fixed scale in some sense stands in for controlling $\mathcal{G}$ at all scales. The Bonatti-Viana example is a situation where we do not expect to find $\mathcal{G}$ with specification at all scales, but where specification for $\mathcal{G}^M$ for all M at a fixed scale is verifiable.
(2) There are two scales present in the theorem: ε and 100ε. We require specification at scale ε, while expansivity and the Bowen property are controlled at the larger scale 100ε. There is nothing fundamental about the constant 100, but it is essential that expansivity and the Bowen property are controlled at a larger scale than specification. This is because every time we use specification in our argument to estimate an orbit, we move distance up to ε away from our original orbit, and we need to control expansivity and regularity properties for orbits after multiple applications of the specification property.
(3) The $\mathrm{Var}(\varphi, 100\varepsilon)$ term appears because we must control points that are distance up to 100ε from a separated set for $\mathcal{P} \cup \mathcal{S}$. Clearly, this term vanishes if we are allowed to take ε arbitrarily small, as in Theorem 2.8.

3. Perturbations of Anosov Diffeomorphisms

In this section, we collect some more background material about weak forms of hyperbolicity, and perturbations of Anosov diffeomorphisms. We also establish a pressure estimate for C^0 perturbations of Anosov diffeomorphisms that plays a key role in our results.

3.1. **Weak forms of hyperbolicity.** Let M be a compact manifold. Recall that a diffeomorphism $f\colon M \to M$ is Anosov if there is a Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^u$ such that E^s is uniformly contracting and E^u is uniformly expanding. (That is, there exists $\ell \in \mathbb{N}$ such that for all unit vectors $v \in E^s$ and $w \in E^u$, we have $\|Df^\ell(v)\| \leq \frac{1}{2}$ and $\|Df^\ell(w)\| \geq 2$.)

We will study diffeomorphisms that are not Anosov but still possess a weaker form of hyperbolicity called a dominated splitting.

A Df-invariant vector bundle $E \subseteq TM$ has a *dominated splitting* if

$$E = E_1 \oplus \cdots \oplus E_k,$$

where each subbundle E_i is Df-invariant with constant dimension, and there exists an integer $\ell \geq 1$ with the following property: for every $x \in M$, all $i = 1, \dots, (k-1)$, and every pair of unit vectors $u \in E_1(x) \oplus \cdots \oplus E_i(x)$ and $v \in E_{i+1}(x) \oplus \cdots \oplus E_k(x)$, it holds that

$$\frac{|Df_x^\ell(u)|}{|Df_x^\ell(v)|} \leq \frac{1}{2}.$$

(See for example [5, Appendix B, Section 1] for properties of systems with a dominated splitting.)

A diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *partially hyperbolic* if there exists a dominated splitting $TM = E^s \oplus E^c \oplus E^u$ where E^s is uniformly contracting, E^u is uniformly expanding, and at least one of E^s and E^u is non-trivial.

For $f\colon M \to M$ partially hyperbolic we know there exist f-invariant foliations W^s and W^u tangent to E^s and E^u respectively that we call the *stable and unstable foliations* [47, Theorem 4.8]. There may or may not be foliations tangent to either E^c, $E^s \oplus E^c$, or $E^c \oplus E^u$. When such exist we denote these by W^c, W^{cs}, and W^{cu} and refer to these as the center, center-stable, and center-unstable foliations respectively. For $x \in M$ we let $W^\sigma(x)$ be the leaf of the foliation $\sigma \in \{s, u, c, cs, cu\}$ containing x when this is defined. Given $\eta > 0$, we write $W^\sigma_\eta(x)$ for the set of points in $W^\sigma(x)$ that can be connected to x via a path along $W^\sigma(x)$ with length at most η.

Suppose W^1, W^2 are foliations of M with the property that $TM = TW^1 \oplus TW^2$. The standard notion of local product structure for

W^1, W^2 says that for every $x, y \in M$ that are close enough to each other, the local leaves $W^u_{\mathrm{loc}}(x)$ and $W^s_{\mathrm{loc}}(y)$ intersect in exactly one point. We give a slightly non-standard definition of local product structure which additionally keeps track of the scales involved. We say that W^1, W^2 have a *local product structure at scale $\eta > 0$ with constant* $\kappa \geq 1$ if for every $x, y \in M$ with $\varepsilon := d_W(x, y) < \eta$, the leaves $W^1_{\kappa\varepsilon}(x)$ and $W^2_{\kappa\varepsilon}(y)$ intersect in a single point.

We will consider splittings $TM = E^{cs} \oplus E^{cu}$ where E^{cs} and E^{cu} are close to the stable and unstable distributions of an Anosov map f. In §3.6, we give precise descriptions of the relevant constants for the examples we study.

3.2. **Constants associated to Anosov maps.** In order to give a precise description of the class of examples to which our methods apply, we need to recall some constants associated to an Anosov map f. First, we will consider a constant $C = C(f)$ arising from the Anosov shadowing lemma [49, Theorem 1.2.3], [35].

Lemma 3.1 (Anosov Shadowing Lemma)**.** *Let f be an Anosov diffeomorphism. There exists $C = C(f)$ so that if $2\eta > 0$ is an expansivity constant for f, then every $\frac{\eta}{C}$-pseudo-orbit $\{x_n\}$ for f can be η-shadowed by an orbit $\{y_n\}$ for f.*

The other constant that will be important for us is a constant $L = L(f)$ associated with the Gibbs property for the measure of maximal entropy for f. More precisely, let $f\colon M \to M$ be a topologically mixing Anosov diffeomorphism, and let $h = h_{\mathrm{top}}(f)$ be its topological entropy. Recall that f is expansive and has the specification property [7]. For any $\eta > 0$ that is smaller than the expansivity constant for f, Bowen showed [10, Lemma 3] that there is a constant $L = L(f, \eta)$ so that

$$\Lambda^{\mathrm{sep}}_n(X \times \mathbb{N}, 0, \eta; f) \leq Le^{nh} \tag{3.1}$$

for every n. The constant L can be determined explicitly in terms of the transition time in the specification property.

3.3. **Partition sums for C^0 perturbations.** Let $f\colon M \to M$ be an Anosov diffeomorphism of a compact manifold. Using the Anosov shadowing lemma, we show that there is a C^0-neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$, there is a natural map from g to f given by sending a point x to a point whose f-orbit shadows the g-orbit of x. It is a folklore result that this map is a semi-conjugacy when $\mathcal{U}$ is sufficiently small, and a version of this result is proved in [16, Proposition 4.1]. This allows us to control the partition sums of g at large enough scales from above, and the pressure at all scales from below.

Lemma 3.2. *Let f be an Anosov diffeomorphism. Let $C = C(f)$ be the constant from the Anosov shadowing lemma, and $3\eta > 0$ be an expansivity constant for f. If $g \in \mathrm{Diff}(M)$ is such that $d_{C^0}(f, g) < \eta/C$, then:*

(i) $P(g, \varphi) \geq P(f, \varphi) - \mathrm{Var}(\varphi, \eta)$;
(ii) $\Lambda_n^{\mathrm{sep}}(\varphi, 3\eta; g) \leq \Lambda_n^{\mathrm{sep}}(\varphi, \eta; f) e^{n \mathrm{Var}(\varphi, \eta)}$.

It follows from (ii) that

$$P(g, \varphi, 3\eta) \leq P(f, \varphi) + \mathrm{Var}(\varphi, \eta). \tag{3.2}$$

However, it may be that $P(g, \varphi)$ is significantly greater than $P(g, \varphi, 3\eta)$ due to the appearance of entropy at smaller scales for g (note that g need not be expansive, even though f is). Nonetheless, we can obtain an upper bound on $P(g, \varphi)$ which involves the tail entropy; Lemma 2.2 admits a simple generalization to pressure, yielding $P(g, \varphi) \leq P(g, \varphi, \varepsilon) + h_g^*(\varepsilon) + \mathrm{Var}(\varphi, \varepsilon)$. Together with (3.2) this gives the bound

$$P(g, \varphi) \leq P(f, \varphi) + h_g^*(3\eta) + 2\,\mathrm{Var}(\varphi, 3\eta). \tag{3.3}$$

The pressure of g, and consequently the tail entropy term, can be arbitrarily large for a C^0 perturbation of f. For example, f can be perturbed continuously in a neighborhood of a fixed point to create a whole disc of fixed points, and then composed with a homeomorphism of this disc that has arbitrarily large entropy.

3.4. **Pressure estimates.** The examples that we consider in §4-§6 are obtained as C^0-perturbations of Anosov maps, where the perturbation is made inside a small neighborhood of a fixed point. Our strategy is to apply the abstract uniqueness results of Theorems 2.8 and 2.9 by taking $\mathcal{G}$ to be the set of orbit segments that consistently spend 'enough' time outside this neighborhood, while $\mathcal{P}, \mathcal{S}$ are orbit segments spending nearly all their time near the fixed point (see §5 and §7 for details). In this section we give an estimate on the pressure carried by such orbit segments. First, we fix the following data.

- Let $f\colon M \to M$ be a transitive Anosov diffeomorphism of a compact manifold, with topological entropy $h = h_{\mathrm{top}}(f)$.
- Let q be a fixed point for f.
- Let 3η be an expansivity constant for f.
- Let $C = C(f)$ be the constant from the shadowing lemma.
- Let $L = L(f, \eta)$ be a constant so that (3.1) holds.

Now we choose g, $\mathcal{C}$, and φ:

- Let $g\colon M \to M$ be a diffeomorphism with $d_{C^0}(f,g) < \eta/C$.
- Let $\rho < 3\eta$.
- Let $r > 0$, and let $\mathcal{C} = \mathcal{C}(q,r) = \{(x,n) \in M \times \mathbb{N} : S_n^g \chi(x) < nr\}$, where χ_q is the indicator function of $M \setminus B(q,\rho)$.
- Let φ be any continuous function.

The following pressure estimate on $\mathcal{C}$ is proved in §10.

Theorem 3.3. *Under the assumptions above, we have*

$$P(\mathcal{C},\varphi;g) \leq h_g^*(5\eta) + (1-r) \sup_{x \in B(q,\rho)} \varphi(x) + r(\sup_{x \in M} \varphi(x) + h + \log L - \log r).$$

In practice, we will take r small and consider maps g with $h_g^*(5\eta)$ small, so that $P(\mathcal{C},\varphi;g)$ is close to $\varphi(q)$.

3.5. **Obstructions to expansivity.** In addition to the properties described in the previous section, the examples we study in §4–§8 will have the following expansivity property:

[E] there exist $\varepsilon > 0$, $r > 0$, and fixed points q, q' such that for $x \in X$, if there exists a sequence $n_k \to \infty$ with $\frac{1}{n_k} S_{n_k}^g \chi_q(x) \geq r$, and a sequence $m_k \to \infty$ with $\frac{1}{m_k} S_{m_k}^{g^{-1}} \chi_{q'}(x) \geq r$, then $\Gamma_\varepsilon(x) = \{x\}$.

Let g be as in the previous section, and suppose $\varepsilon > 0$ and $r > 0$ are such that **[E]** holds. Then $\mathcal{C} = \mathcal{C}(r)$ from above has the following property, which is proved in §10.

Theorem 3.4. *Under the above assumptions, we have the pressure estimate $P_{\exp}^\perp(\varphi,\varepsilon) \leq P(\mathcal{C}(q,r) \cup \mathcal{C}(q',r),\varphi)$.*

For the classes of systems that we study, we will apply our general uniqueness theorems by finding suitable decompositions $(\mathcal{P},\mathcal{G},\mathcal{S})$ which satisfy $P(\mathcal{P} \cup \mathcal{S},\varphi) \leq P(\mathcal{C},\varphi)$, in addition to the other hypotheses of these theorems. Combined with Theorems 3.3 and 3.4, this will allow us to get uniqueness under the condition that the quantity appearing in Theorem 3.3 is less than $P(\varphi;g)$. This leads to the bounds (4.4) and (6.8) in Theorems 4.1 and 6.1, respectively.

3.6. **Cone estimates and local product structure.** Let $F^1, F^2 \subset \mathbb{R}^d$ be subspaces such that $F^1 \cap F^2 = \{0\}$ (we do not assume that $F^1 + F^2 = \mathbb{R}^d$). Let $\measuredangle(F^1,F^2) := \min\{\measuredangle(v,w) : v \in F^1, w \in F^2\}$, and consider the quantity $\bar\kappa(F^1,F^2) := (\sin \measuredangle(F^1,F^2))^{-1} \geq 1$. Some elementary trigonometry shows that

$$\|v\| \leq \bar\kappa(F^1,F^2) \text{ for every } v \in F^1 \text{ with } d(v,F^2) \leq 1, \tag{3.4}$$

or equivalently,

$$\|v\| \leq \bar\kappa(F^1,F^2) d(v,F^2) \text{ for every } v \in F^1. \tag{3.5}$$

The quantity $\bar{\kappa}$ can also be characterized as the norm of the projection from $F^1 \oplus F^2$ onto F^1 along F^2, see Figure 1.

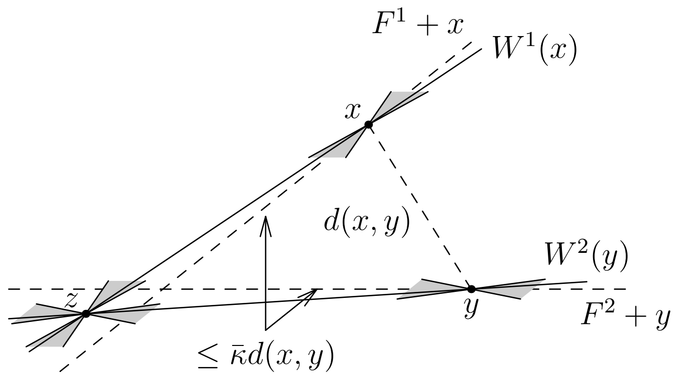

FIGURE 1. Local product structure estimates

Given $\beta \in (0,1)$ and $F^1, F^2 \subset \mathbb{R}^d$, the *$\beta$-cone of F^1 and F^2* is

$$C_\beta(F^1, F^2) = \{v + w : v \in F^1, w \in F^2, \|w\| < \beta \|v\|\}.$$

Lemma 3.5. *Let W^1, W^2 be any foliations of $F^1 \oplus F^2$ with C^1 leaves such that $T_x W^1(x) \subset C_\beta(F^1, F^2)$ and $T_x W^2(x) \subset C_\beta(F^2, F^1)$, and let $\bar{\kappa} = \bar{\kappa}(F^1, F^2)$. Then for every $x, y \in F^1 \oplus F^2$ the intersection $W^1(x) \cap W^2(y)$ consists of a single point z. Moreover,*

$$\max\{d_{W^1}(x,z), d_{W^2}(y,z)\} \leq \frac{1+\beta}{1-\beta} \bar{\kappa} d(x,y).$$

We prove Lemma 3.5 in §11 following the standard proof of local product structure. For the Mañé and Bonatti–Viana examples we will consider foliations on $\mathbb{T}^d$ whose lifts to $\mathbb{R}^d$ satisfy the hypotheses of Lemma 3.5. Uniqueness of the intersection point on $\mathbb{T}^d$ follows from restricting to sufficiently small local leaves.

4. Results for Mañé's examples

4.1. **Mañé's example.** We now review the class of robustly transitive diffeomorphisms originally considered by Mañé [39].

Fix $d \geq 3$ and let $A \in SL(d, \mathbb{Z})$ be a hyperbolic toral automorphism with only one eigenvalue outside the unit circle and all eigenvalues real, positive, simple, and irrational. Let λ_u be the unique eigenvalue greater than 1 and λ_s be the largest of the other eigenvalues. Let f_A be the automorphism of $\mathbb{T}^d$ determined by the matrix A, and let $h = h_{\text{top}}(f_A)$ be the topological entropy.

The Mañé class of examples are C^0 perturbations of f_A, and we denote them by f_0. We describe how the perturbation is constructed, taking care to control the size of the perturbation and to build in necessary uniform control on how cone fields behave under this perturbation. Control on the cone fields is essential to ensure that local product structure and tail entropy estimates apply at a scale which is 'compatible' with the C^0 size of the perturbation in order to apply our pressure estimates.

To this end, let $F^u, F^c, F^s \subset \mathbb{R}^d$ be the eigenspaces corresponding to (respectively) λ_u, λ_s, and all eigenvalues smaller than λ_s, and let $F^{cs} = F^c \oplus F^s$.[1] Let $\omega_{cs,u} = \measuredangle(F^{cs}, F^u)$ and $\omega_{c,s} = \measuredangle(F^c, F^s)$. Fix $\eta > 0$ such that

$$5\eta \text{ is an expansivity constant for } f_A; \tag{4.1}$$

$$\eta < \tfrac{1}{42} \sin^2 \omega_{cs,u} \sin \omega_{c,s} (\max\{\|A\|, \|A^{-1}\|\})^{-1}. \tag{4.2}$$

Let q be a fixed point for f_A, and fix $0 < \rho < 3\eta$. We carry out a perturbation in a ρ-neighbourhood of q. Let $\mathcal{F}^{u,c,s}$ be the foliations of $\mathbb{T}^d$ by leaves parallel to $F^{u,c,s}$. The leaves of $\mathcal{F}^s$, $\mathcal{F}^c$, and $\mathcal{F}^u$ are dense in $\mathbb{T}^d$ since all eigenvalues are irrational. Let $0 < \beta < \min(\frac{1}{3}, \sin \omega_{cs,u})$ be such that (4.2) continues to hold when η is replaced by

$$\eta(1+\beta)\frac{(\sin \omega_{cs,u}) + \beta}{(\sin \omega_{cs,u}) - \beta},$$

and consider the cones

$$C^s_\beta = C_\beta(F^s, F^{cu}), \qquad C^c_\beta = C_\beta(F^c, F^s \oplus F^u),$$
$$C^u_\beta = C_\beta(F^u, F^{cs}), \qquad C^{cs}_\beta = C_\beta(F^{cs}, F^u).$$

Outside of $B(q, \rho)$, we set f_0 to be equal to f_A. Inside $B(q, \rho)$, the fixed point q undergoes a pitchfork bifurcation in the direction of $\mathcal{F}^c$; see [39] for details. The perturbation is carried out so that

- $\mathcal{F}^c$ is still an invariant foliation for f_0, and we write $E^c = T\mathcal{F}^c$;
- writing $\beta' = \beta \sin \omega_{cs,u} \sin \omega_{c,s}$, the cones C^u_β and $C^s_{\beta'}$ are invariant and uniformly expanding under Df_0 and Df_0^{-1}, respectively; in particular, they contain Df_0-invariant distributions E^s and E^u that integrate to f_0-invariant foliations W^s and W^u.
- $E^{cs} := E^c \oplus E^s$ integrates to a foliation W^{cs}. This holds because $E^s \subset C^s_{\beta'}$ guarantees that $E^{cs} \subset C^{cs}_\beta$, see §5.1.
- $d_{C_0}(f_A, f_0) < \eta$.

[1] We use the notation F^c for the eigenspace corresponding to λ_s because when we perturb the map this will become a center direction that experiences both contraction and expansion.

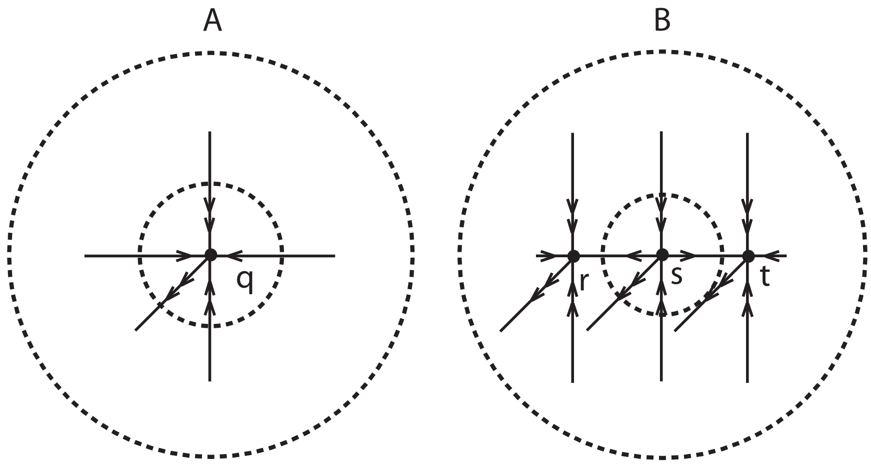

FIGURE 2. Mañé's construction

Thus, f_0 is partially hyperbolic with $T\mathbb{T}^d = E^s \oplus E^c \oplus E^u$ and both E^s and E^{cu} integrate to foliations. We will see in Lemma 5.3 that $\mathbb{T}^d$ has a local product structure at scale 5η for the foliations W^u and W^{cs}.

The index of q changes during the perturbation, and we may also assume that for any point in $\mathbb{T}^d \setminus B(q, \rho/2)$ the contraction in the direction E^c is λ_s. Inside $B(q, \rho/2)$, the perturbed map experiences some weak expansion in the direction E^c, and two new fixed points are created on $W^c(q)$, see Figure 2. Let $\lambda_c = \lambda_c(f_0) > 1$ be the greatest expansion which occurs in the center direction. We can carry out the construction so that λ_c is arbitrarily close to 1.

The numbers $\rho > 0$ and $\lambda_c > 1$ are the two pieces of information we require about the map f_0. Outside $B(q, \rho)$, the maps f_0 and f_A are identical, and we can carry out the construction so there exists a constant K so that both $f_A(B(q, \rho)) \subset B(q, K\rho)$ and $f_0(B(q, \rho)) \subset B(q, K\rho)$. Thus the C^0 distance between f_0 and f_A is at most $K\rho$. In particular, by choosing ρ small, we can ensure that $d_{C^0}(f_0, f_A) < \eta/C$ where $C = C(f_A)$ is the constant from the Shadowing Lemma. This allows us to apply Lemma 3.2 to f_0, or to a C^1-perturbation of f_0.

4.2. C^1 **perturbations of Mañé's example.** We now consider diffeomorphisms g in a C^1 neighborhood of f_0. Let $\mathcal{U}_0 \subset \mathrm{Diff}^1(\mathbb{T}^d)$ be a C^1-neighborhood of f_0 such that the following is true for every $g \in \mathcal{U}_0$.

- $d_{C^0}(g, f_A) < \eta/C$, where $C = C(f_A)$ is the constant provided by Lemma 3.1.
- g is partially hyperbolic with $T\mathbb{T}^d = E^s_g \oplus E^c_g \oplus E^u_g$, where $E^\sigma_g \subset C^\sigma_\beta$ for each $\sigma \in \{s, c, u, cs\}$.

- The distribution E^c_g integrates to a foliation W^c_g.
- Each of the leaves $W^{cs}_g(x)$ and $W^u_g(x)$ is dense for every $x \in \mathbb{T}^d$.
- $\|Dg\| \leq 2\|A\|$ and $\|Dg^{-1}\| \leq 2\|A^{-1}\|$.

For the C^1 perturbations, partial hyperbolicity (with $E^\sigma_g \subset C^\sigma_\beta$) and integrability are provided by [33, Theorem 6.1]; density of the leaves was shown in [51]. In particular, the conditions above are satisfied on a C^1-neighborhood of f_0, so $\mathcal{U}_0$ is non-empty.

Given $g \in \mathcal{U}_0$, consider the following two quantities:

$$\lambda_c(g) = \sup\{\|Dg|_{E^c(x)}\| : x \in B(q, \rho/2)\},$$
$$\lambda_s(g) = \sup\{\|Dg|_{E^c(x)}\| : x \in X \setminus B(q, \rho/2)\}.$$

Note that $\lambda_s(f_0) < 1 < \lambda_c(f_0)$, and these quantities vary continuously as g varies via C^1 perturbation. In particular, there is a C^1-open neighborhood of f_0 on which $\lambda_s(g) < 1 < \lambda_c(g)$. Consider the quantity

$$\gamma = \gamma(g) := \frac{\ln \lambda_c(g)}{\ln \lambda_c(g) - \ln \lambda_s(g)} > 0.$$

Note that $\gamma(g) \to 0$ as $\lambda_c(g) \to 1$ (as long as $\lambda_s(g) \not\to 1$), and a simple calculation shows that for any $r > \gamma$,

$$\lambda_c(g)^{1-r} \lambda_s(g)^r < 1. \tag{4.3}$$

Roughly, this implies uniform contraction in the center-stable direction along any orbit segment of length n that spends at least rn iterates outside $B(q, \rho/2)$ (see Lemma 5.5 for a precise formulation of this statement).

Now we can state a precise condition on g and φ that guarantees existence of a unique equilibrium state. As a reminder, the constant $L = L(f_A, \eta)$ is the one given at (3.1).

Theorem 4.1. *Given $g \in \mathcal{U}_0$ as above, let $\gamma = \gamma(g)$, and let $\varphi \colon \mathbb{T}^d \to \mathbb{R}$ be Hölder continuous. If*

$$(1-\gamma) \sup_{x \in B(q,\rho)} \varphi(x) + \gamma(\sup_{x \in \mathbb{T}^d} \varphi(x) + h + \log L - \log \gamma) < P(g, \varphi), \tag{4.4}$$

then φ has a unique equilibrium state with respect to g.

The ingredients required to prove Theorem 4.1 are established in the next section, and the proof is completed in §5.5. First, we explain how Theorem 4.1 implies Theorem A. Let $\mathcal{U}$ be the set of all diffeomorphisms $g \in \mathcal{U}_0$ such that (4.4) is satisfied. Note that all the quantities in (4.4) vary continuously under C^1 perturbations of g, so $\mathcal{U}$ is C^1-open. It only remains to show that $\mathcal{U}$ is non-empty, which we do by showing that we can always find a Mañé example $f_0 \in \mathcal{U}$.

Recall from Lemma 3.2(i) that $P(g,\varphi) > P(f_A,\varphi) - \mathrm{Var}(\varphi,\eta)$. Moreover, we have

$$(1-\gamma) \sup_{x\in B(q,\rho)} \varphi(x) \leq (1-\gamma)\varphi(q) + \mathrm{Var}(\varphi,\rho).$$

Thus to prove (4.4) it suffices to verify that

$$(1-\gamma)\varphi(q)+\gamma(\sup_{\mathbb{T}^d}\varphi+h+\log L-\log\gamma)+2\,\mathrm{Var}(\varphi,\eta) < P(f_A,\varphi). \tag{4.5}$$

Thus, we show that f_0 can be chosen so that (4.5) holds. Given a hyperbolic toral automorphism f_A and a Hölder potential $\varphi\colon \mathbb{T}^d \to \mathbb{R}$, it follows from [10] that φ has a unique equilibrium state (with respect to f_A), which is positive on every open set. In particular, the Dirac measure δ_q at the fixed point q is invariant but not an equilibrium state for φ, so

$$\varphi(q) = h_{\delta_q}(f_A) + \int \varphi\, d\delta_q < P(f_A,\varphi).$$

We now choose η small enough so that

$$2\,\mathrm{Var}(\varphi,\eta) < P(\varphi,f_A) - \varphi(q),$$

and $\bar\gamma > 0$ small enough so that

$$\bar\gamma(\sup_{\mathbb{T}^d}\varphi - \varphi(q) + h + \log L - \log\bar\gamma) < P(\varphi,f_A) - \varphi(q) - 2\,\mathrm{Var}(\varphi,\eta).$$

We now choose our Mañé example f_0 with ρ small enough so that $d(f_0,f_A) < \eta/C$, and λ_c close enough to 1 that $\gamma(f_0) < \bar\gamma$. Then (4.4) holds for f_0, and thus $\mathcal{U}$ is non-empty. This establishes Theorem A from Theorem 4.1.

Another consequence of Theorem 4.1 is the following 'bounded range' theorem.

Theorem 4.2. *Let $\mathcal{U}_0 \subset \mathrm{Diff}(\mathbb{T}^d)$ be as above, and suppose $g \in \mathcal{U}_0$ is such that for $L = L(f_A,\eta)$, $h = h_{\mathrm{top}}(f_A)$, and $\gamma = \gamma(g)$, we have*

$$\gamma(\log L + h - \log\gamma) < h. \tag{4.6}$$

Then writing $D = h - \gamma(\log L + h - \log\gamma) > 0$, every Hölder continuous potential φ with the bounded range hypothesis $\sup\varphi - \inf\varphi < D$ has a unique equilibrium state. In particular, (4.6) is a criterion for g to have a unique measure of maximal entropy.

Proof. If $\sup\varphi - \inf\varphi < D := h - \gamma(\log L + h - \log\gamma)$, then we can use (4.6) to verify (4.4), as follows:

$$\begin{aligned}(1-\gamma)\sup_{x\in B(q,\rho)}\varphi(x) &+ \gamma(\sup_{x\in\mathbb{T}^d}\varphi(x) + h + \log L - \log\gamma)\\ &= (1-\gamma)\sup_{B(q,\rho)}\varphi + \gamma(\sup_{\mathbb{T}^d}\varphi) + h_{\mathrm{top}}(f_A) - D\\ &\le \sup_{\mathbb{T}^d}\varphi + h_{\mathrm{top}}(f_A) - D\\ &< \inf\varphi + h_{\mathrm{top}}(f_A) \le P(f_A,\varphi).\end{aligned}$$

Thus Theorem 4.1 applies. □

5. Technique for Mañé's examples

The strategy for proving Theorem 4.1 is as follows. We consider the collection $\mathcal{G}$ of orbit segments (x,n) for which (x,i) spends at least γi iterates outside of $B(q,\rho)$ for all $i \le n$. These orbit segments experience uniform contraction in the E^{cs} direction (this is made precise in Lemma 5.5). Using the local product structure (Lemma 5.3) this will allow us to prove specification and the Bowen property for such orbit segments. Then the estimate (4.4), together with Theorems 3.3 and 3.4, will let us bound the pressure of obstructions to expansivity and specification away from $P(g,\varphi)$.

5.1. **Local product structure.** We require local product structure for g at scale 5η repeatedly through this section. We establish this here, beginning with some consequences of the bound (4.2). Let $\kappa' = \bar\kappa(F^{cs},F^u) = (\sin\omega_{cs,u})^{-1}$ and $\kappa'' = \bar\kappa(F^c,F^s) = (\sin\omega_{c,s})^{-1}$ (see (3.4)) and let $\kappa = 2\kappa'$. Let $M = \max\{\|A\|,\|A^{-1}\|\}$, so (4.2) gives

$$21\eta\kappa\kappa''\kappa' M < 1. \tag{5.1}$$

Note that $M \ge \lambda_u > 4$ and so (5.1) implies

$$10\eta\kappa\kappa''\kappa' < \tfrac{1}{4}, \tag{5.2}$$

which we will use in the proof of Lemma 5.3. Moreover $\kappa \ge 2$ and $\kappa',\kappa'' \ge 1$, so $\kappa\kappa''\kappa' M \ge 8$, and (5.1) also gives the bound

$$20\eta\kappa\kappa''\kappa'\max(\|A\|,\|A^{-1}\|) + 5\eta \le 21\eta\kappa\kappa''\kappa' M < 1,$$

which we will use in the proof of Lemma 5.9. By the choice of β following (4.2), we similarly have

$$10\eta\kappa(1+\beta)\frac{1+\kappa'\beta}{1-\kappa'\beta}\kappa''\kappa' < \frac{1}{4}, \tag{5.3}$$

$$20\eta\kappa(1+\beta)\frac{1+\kappa'\beta}{1-\kappa'\beta}\kappa''\kappa' \max(\|A\|, \|A^{-1}\|) + 5\eta < 1, \tag{5.4}$$

which will be used in the proofs of Lemmas 5.3 and 5.9. Since $\kappa'', \kappa' \geq 1$ we have

$$(1+\beta)\frac{1+\kappa'\beta}{1-\kappa'\beta}\kappa''\kappa' \geq 1.$$

In [39] it is shown that the leaves W^{cs} exist for the maps we are considering. This will be used in the next lemma.

Lemma 5.1. *With $\beta' = \beta \sin\omega_{cs,u} \sin\omega_{c,s} = \beta/(\kappa''\kappa')$ and $E^c = F^c$, $E^s \subset C^s_{\beta'}$, we have $E^{cs} \subset C^{cs}_{\beta}$.*

Proof. It suffices to show that $v + w \in C^{cs}_\beta$ for every $v \in E^c = F^c$ and $w \in C^s_{\beta'}$. Let $w_s \in F^s$, $w_c \in F^c$, $w_u \in F^u$ be such that $w = w_s + w_c + w_u$, then $\|w_c + w_u\| < \beta'\|w_s\|$. From (3.5) we have

$$\begin{aligned}\|w_u\| &\leq \bar\kappa(F^{cs}, F^u) d(w_u, F^{cs}) \leq \kappa'\|w_u + w_c\|, \\ \|w_s\| &\leq \bar\kappa(F^c, F^s) d(w_s, F^c) \leq \kappa''\|w_s + w_c + v\|,\end{aligned}$$

and so

$$\begin{aligned}\|(v+w)_u\| = \|w_u\| &\leq \kappa'\|w_u + w_c\| \leq \kappa'\beta'\|w_s\| \\ &\leq \kappa'\beta'\kappa''\|w_s + w_c + v\| = \beta\|(v+w)_{cs}\|. \quad \square\end{aligned}$$

For local product structure, we apply Lemma 3.5, together with the following lemma, which is an elementary observation.

Lemma 5.2. *Given $\varepsilon > 0$, let $\gamma\colon [0,1] \to \mathbb{T}^d$ be a path such that $d(\gamma(0), \gamma(1)) < \varepsilon$, and let $\tilde\gamma$ be a lifting of γ to $\mathbb{R}^d$. If the length of γ is less than $1 - \varepsilon$, then $d(\tilde\gamma(0), \tilde\gamma(1)) < \varepsilon$.*

We now establish local product structure at scale 5η for maps $g \in \mathcal{U}_0$. The key ingredients that allow us to do this are the assumptions that $E^\sigma_g \subset C^\sigma_\beta$ for each $\sigma \in \{s, c, u, cs\}$ and the choice of β in §4.1.

Lemma 5.3. *Every $g \in \mathcal{U}_0$ as in §4.2 has a local product structure for W^{cs}_g, W^u_g at scale 5η with constant $\kappa = 2\kappa'$.*

Proof. Let $\widetilde{W}^{cs}$ and $\widetilde{W}^u$ be the lifts of W^{cs}_g, W^u_g to $\mathbb{R}^d$. Given $x, y \in \mathbb{T}^d$ with $\varepsilon := d(x,y) < 5\eta$, let $\tilde x, \tilde y \in \mathbb{R}^d$ be lifts of x, y with $\varepsilon = d(\tilde x, \tilde y) < 5\eta$. By Lemma 3.5 the intersection $\widetilde{W}^{cs}(x) \cap \widetilde{W}^u(y)$ has a unique point

$\tilde{z}$, which projects to $z \in \mathbb{T}^d$. Moreover, the leaf distances between $\tilde{x}, \tilde{z}$ and $\tilde{y}, \tilde{z}$ are at most $(\frac{1+\beta}{1-\beta})\kappa'\varepsilon$. Since $\beta < \frac{1}{3}$ this is less than $2\kappa' d(x,y)$, so z is in the intersection of the local leaves $(W_g^{cs})_{\kappa\varepsilon}(x)$ and $(W_g^u)_{\kappa\varepsilon}(x)$.

It remains to show that z is the only point in this intersection. Suppose z' is any other point in $(W_g^{cs})_{\kappa\varepsilon}(x) \cap (W_g^u)_{\kappa\varepsilon}(y)$, and let $\gamma\colon [0,1] \to \mathbb{T}^d$ be any path that goes from z to z' inside $W_g^{cs}(x)$, and then returns to z inside $W_g^u(y)$. Lifting γ to a path $\tilde{\gamma}$ in $\mathbb{R}^d$ we see that $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$ (otherwise $\tilde{W}^{cs}(\tilde{z}) \cap \tilde{W}^u(\tilde{z})$ would have more than one point where $\tilde{z}$ is the lift of z), and so by Lemma 5.2, γ must have length at least 1. Since this is true for any such γ, we have $d_{W_g^{cs}}(z,z') + d_{W_g^u}(z,z') \geq 1$. Thus one of the terms is at least $\frac{1}{2}$. If $d_{W_g^{cs}}(z,z') \geq \frac{1}{2}$, then (5.2) gives

$$10\eta\kappa \leq \tfrac{1}{2} \leq d_{W_g^{cs}}(z,z') \leq d_{W_g^{cs}}(z',x) + d_{W_g^{cs}}(z,x) \leq d_{W_g^{cs}}(z',x) + 5\eta\kappa,$$

hence $d_{W_g^{cs}}(z',x) \geq 5\eta\kappa$. The case $d_{W_g^u}(z,z')$ yields $d_{W_g^u}(z',y) \geq 5\eta\kappa$ by a similar argument. $\square$

5.2. **Specification.** We produce a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ such that $\mathcal{G}$ has specification. The main tool for establishing specification for mixing locally maximal hyperbolic sets $f\colon \Lambda \to \Lambda$ is that given $\delta > 0$ there is some $N \in \mathbb{N}$ such that for $x, y \in \Lambda$ and $n \geq N$ we have $f^n(W_\delta^u(x)) \cap W_\delta^s(y) \neq \emptyset$. We want to mimic the idea replacing the stable and unstable manifolds with centerstable and unstable manifolds.

From now on we fix $g \in \mathcal{U}_0$ and write $W^\sigma = W_g^\sigma$ for $\sigma = s, c, cs, u$. Recall that all leaves of W^u are dense in $\mathbb{T}^d$ by the definition of $\mathcal{U}_0$. This can be made uniform by the following lemma, which is proved in §11 and relies on the local product structure from Lemma 5.3.

Lemma 5.4. *For every $\delta > 0$ there is $R > 0$ such that for every $x, y \in \mathbb{T}^d$, we have $W_R^u(x) \cap W_\delta^{cs}(y) \neq \emptyset$.*

Because g is uniformly expanding along W^u, we see that for every $\delta > 0$ there is $N \in \mathbb{N}$ such that for every $x \in \mathbb{T}^d$ and $n \geq N$, we have $g^n(W_\delta^u(x)) \supset W_R^u(g^n x)$. Thus by Lemma 5.4 we have

$$g^n(W_\delta^u(x)) \cap W_\delta^{cs}(y) \neq \emptyset \text{ for every } x, y \in \mathbb{T}^d. \tag{5.5}$$

We now prove a lemma about contraction along W^{cs} for orbits which spend a uniform proportion of time away from the fixed point q. Let χ be the indicator function of $\mathbb{T}^d \setminus B(q,\rho)$, so

$$\frac{1}{i} S_i \chi(x) = \frac{1}{i} S_i^g \chi(x) = \frac{1}{i} \sum_{j=0}^{i-1} \chi(g^j(x))$$

is the proportion of time that an orbit segment (x, i) spends outside $B(q, \rho)$. Recall that

$$\gamma = \gamma(g) = \frac{\ln \lambda_c(g)}{\ln \lambda_c(g) - \ln \lambda_s(g)}.$$

Let $r > \gamma$ and let $\theta_r = \lambda_c(g)^{1-r}\lambda_s(g)^r$. By (4.3) we have $\theta_r < 1$.

Lemma 5.5. *Suppose $(x, n) \in \mathbb{T}^d \times \mathbb{N}$ is such that $\frac{1}{i}S_i^g\chi(x) \geq r$ for all $0 \leq i \leq n$, where r is as above.*

(a) For any $y \in B_n(x, \rho/2)$, we have $\|Dg^i|_{E^{cs}(y)}\| < (\theta_r)^i$ for all $0 \leq i \leq n$.
(b) For any $y, z \in W^{cs}_{\rho/2}(x)$, we have $d(f^i y, f^i z) \leq \theta_r^i d(y, z)$ for all $0 \leq i \leq n$.
(c) For $0 < \delta < \rho/2$, we have $W^{cs}_\delta(x) \subset B_n(x, \delta)$.

Proof. Given $0 \leq i \leq n$, the inequality $\frac{1}{i}S_i^g\chi(x) > r$ implies that the orbit segment (x, i) spends at least ir iterates outside of $B(q, \rho)$. It follows that (y, i) spends at least ir iterates outside of $B(q, \rho/2)$. By the definition of $\lambda_c(g)$ and $\lambda_s(g)$, it follows that

$$\|Dg^i|_{E^{cs}(y)}\| \leq \lambda_c^{i-ir}\lambda_s^{ir} = (\theta_r)^i.$$

This proves the first claim. It is an easy exercise to prove (b) using the uniform contraction estimate provided by (a), and (c) follows immediately from (b). $\square$

Now we define the decomposition. Given $g \in \mathcal{U}_0$, let $\gamma = \gamma(g)$ as above. Because the left-hand side of (4.4) varies continuously in γ, we can choose $r > \gamma$ such that (4.4) continues to hold with r in place of γ. Fixing this value of r, we consider the following collections of orbit segments:

$$\begin{aligned} \mathcal{G} &= \{(x, n) \in \mathbb{T}^d \times \mathbb{N} : S_i^g\chi(x) \geq ir \ \forall\ 0 \leq i \leq n\}, \\ \mathcal{P} &= \{(x, n) \in \mathbb{T}^d \times \mathbb{N} : S_n^g\chi(x) < nr\}. \end{aligned} \tag{5.6}$$

The collection $\mathcal{G}$ is chosen so that the centerstable manifolds are uniformly contracted along orbit segments from $\mathcal{G}$. These collections, together with a trivial collection for $\mathcal{S}$, define a decomposition of any point $(x, n) \in X \times \mathbb{N}$ as follows: let p be the largest integer in $\{0, ..., n\}$ such that $\frac{1}{p}S_p\chi(x) < r$; then $(x, p) \in \mathcal{P}$ and a short calculation shows that $(g^p(x), n - p) \in \mathcal{G}$. This last statement follows from the fact that if $\frac{1}{k}S_k\chi(g^p x) < r$ for some $0 \leq k \leq n - p$, then

$$\frac{1}{p+k}S_{p+k}\chi(x) = \frac{1}{p+k}\left(S_p\chi(x) + S_k\chi(g^p(x))\right) < r,$$

contradicting the maximality of p.

The reason that we can take a trivial collection for $\mathcal{S}$ is that for the Mañé examples, lack of uniform hyperbolicity is only manifested in the stable direction, and this is handled by the collection $\mathcal{P}$. In the Bonatti-Viana examples, we will require non-trivial collections $\mathcal{P}$ to handle non-uniform contraction, and $\mathcal{S}$ to handle non-uniform expansion.

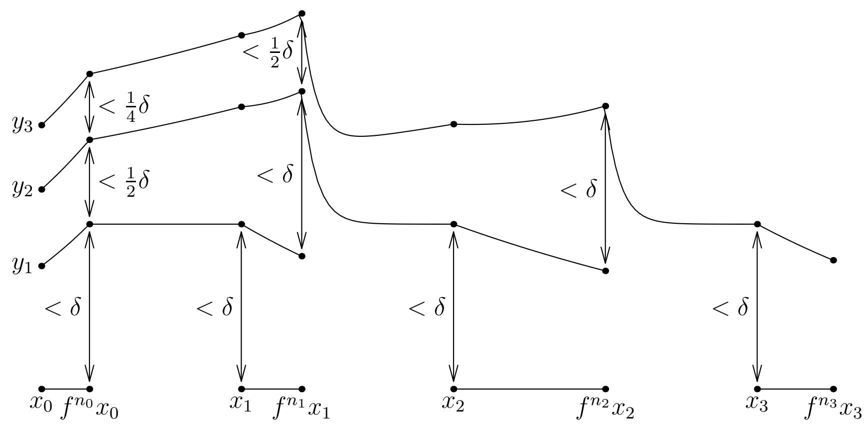

FIGURE 3. Specification for $\mathcal{G}$

Lemma 5.6. *The collection $\mathcal{G}$ has specification at any scale $\delta > 0$.*

Proof. For an arbitrary fixed $\delta > 0$, we prove specification at scale 3δ. The key property that allows us to transition from one orbit to another is (5.5). This property, together with uniform expansion on W^u, allows us to choose $\tau = \tau(\delta) \in \mathbb{N}$ such that

$$
\begin{aligned}
&g^\tau(W^u_\delta(x)) \cap W^{sc}_\delta(y) \neq \emptyset \text{ for all } x, y \in \mathbb{T}^d, \\
&d(g^{-\tau}y, g^{-\tau}z) < \frac{1}{2}d(y,z) \text{ for all } x \in \mathbb{T}^d \text{ and } y, z \in W^u_\delta(x).
\end{aligned} \tag{5.7}
$$

Now we show that $\mathcal{G}$ has specification with gluing time τ. Given any $(x_0, n_0), \dots, (x_k, n_k) \in \mathcal{G}$, we construct y_j iteratively such that (y_j, m_j) shadows $(x_0, n_0), \dots, (x_j, n_j)$, where $m_0 = n_0$, $m_1 = n_0 + \tau + n_1$, $\dots$, $m_k = (\sum_{i=0}^k n_i) + k\tau$. We also set $m_{-1} = -\tau$, see Figure 3.

Start by letting $y_0 = x_0$, and we choose $y_1, \dots, y_k$ iteratively so that

$$
\begin{array}{ccc}
g^{m_0}y_1 \in W^u_\delta(g^{m_0}y_0) & \text{and} & g^{m_0+\tau}y_1 \in W^{cs}_\delta(x_1) \\
g^{m_1}y_2 \in W^u_\delta(g^{m_1}y_1) & \text{and} & g^{m_1+\tau}y_2 \in W^{cs}_\delta(x_2) \\
\vdots & \vdots & \vdots \\
g^{m_{k-1}}y_k \in W^u_\delta(g^{m_{k-1}}y_{k-1}) & \text{and} & g^{m_{k-1}+\tau}y_k \in W^{cs}_\delta(x_k).
\end{array}
$$

That is, for $j \in \{0, \dots, k-1\}$, we let y_{j+1} be a point such that

$$
y_{j+1} \in g^{-m_j}(W^u_\delta(g^{m_j}y_j)) \cap g^{-(m_j+\tau)}(W^{cs}_\delta(x_{j+1})).
$$

Using the fact that $g^{m_j} y_{j+1}$ is in the unstable manifold of $g^{m_j} y_j$, and the fact that the distance is contracted by $\frac{1}{2}$ every time the orbit passes backwards through a 'transition', we obtain that

$$\begin{array}{rcl} d_{n_j}(g^{m_{j-1}+\tau} y_j, g^{m_{j-1}+\tau} y_{j+1}) & < & \delta \\ d_{n_{j-1}}(g^{m_{j-2}+\tau} y_j, g^{m_{j-2}+\tau} y_{j+1}) & < & \delta/2 \\ \vdots & & \vdots \\ d_{n_0}(y_j, y_{j+1}) & < & \delta/2^j. \end{array}$$

That is, $d_{n_{j-i}}(g^{m_{j-i-1}+\tau} y_j, g^{m_{j-i-1}+\tau} y_{j+1}) < \delta/2^i$ for each $i \in \{0, \dots, j\}$. This estimate, together with the fact that $g^{m_j+\tau}(y_{j+1}) \in B_{n_{j+1}}(x_{j+1}, \delta)$ from Lemma 5.5 gives that $d_{n_j}(g^{m_{j-1}+\tau} y_k, x_j) < 2\delta + \sum_{j=1}^{\infty} 2^{-j}\delta = 3\delta$. It follows that

$$y_k \in \bigcap_{j=0}^{k} g^{-(m_{j-1}+\tau)} B_{n_j}(x_j, 3\delta),$$

and thus $\mathcal{G}$ has specification at scale 3δ. □

5.3. **Verifying the Bowen property.** Let $\theta_u \in (0,1)$ be such that $\|Dg|_{E^u(x)}^{-1}\| \leq \theta_u$ for all $x \in \mathbb{T}^d$. Let $\theta_r \in (0,1)$ be the constant that appears in the previous subsection, and let κ be the constant associated with the local product structure of $E_g^{cs} \oplus E_g^u$ (see §3.1 and §5.1). Let $\varepsilon = \rho/(2\kappa)$.

Lemma 5.7. *Given $(x,n) \in \mathcal{G}$ and $y \in B_n(x, \varepsilon)$, we have*

$$d(g^k x, g^k y) \leq \kappa\varepsilon(\theta_r^k + \theta_u^{n-k}) \tag{5.8}$$

for every $0 \leq k \leq n$.

Proof. Using the local product structure, there exists $z \in W^{cs}_{\kappa\varepsilon}(x) \cap W^u_{\kappa\varepsilon}(y)$. Since g^{-1} is uniformly contracting on W^u, we get

$$d(g^k z, g^k y) \leq \theta_u^{n-k} d(g^n z, g^n y) \leq \theta_u^{n-k}\kappa\varepsilon,$$

and Lemma 5.5 gives

$$d(g^k x, g^k z) \leq \theta_r^k d(x,z) \leq \theta_r^k \kappa\varepsilon.$$

The triangle inequality gives (5.8). □

Lemma 5.8. *Any Hölder continuous φ has the Bowen property on $\mathcal{G}$ at scale ε.*

Proof. By Hölder continuity there are constants $K > 0$ and $\alpha \in (0,1)$ such that $|\varphi(x) - \varphi(y)| \leq K d(x,y)^\alpha$ for all $x, y \in \mathbb{T}^d$. Now given

$(x,n) \in \mathcal{G}$ and $y \in B_n(x,\varepsilon)$, Lemma 5.7 gives

$$\begin{aligned}|S_n\varphi(x) - S_n\varphi(y)| &\le K\sum_{k=0}^{n-1} d(g^k x, g^k y)^\alpha \le K\kappa\varepsilon \sum_{k=0}^{n-1} (\theta_u^{n-k} + \theta_r^k)^\alpha \\ &\le K\kappa\varepsilon \sum_{j=0}^{\infty} (\theta_u^{j\alpha} + \theta_r^{j\alpha}) =: V < \infty. \qquad \square\end{aligned}$$

5.4. **Expansivity.** We study the expansivity properties of $g \in \mathcal{U}_0$. Every such g is partially hyperbolic with one-dimensional center bundle, and thus [29, Proposition 6] tells us that g is entropy expansive; that is, $h_g^*(\alpha) = 0$ for sufficiently small $\alpha > 0$. We need to know that $h_g^*(5\eta) = 0$, where 5η is the scale of the local product structure for g. It is not immediate from the statement of [29, Proposition 6] that we can take $\alpha = 5\eta$, so we use similar ideas to give a self-contained proof. The main idea is to show that $\Gamma_{5\eta}(x)$ is contained in a single center leaf for any x.

Lemma 5.9. *For all $x \in \mathbb{T}^d$, $\Gamma_{5\eta}(x)$ is contained in a compact subset of $W_g^c(x)$.*

Proof. Given $x \in \mathbb{T}^d$, let $y \in \Gamma_{5\eta}(x)$, so that $d(g^n y, g^n x) < 5\eta$ for all $n \in \mathbb{Z}$. By Lemma 5.3, there is $z \in W^{cs}_{5\eta\kappa}(x) \cap W^u_{5\eta\kappa}(y)$. Applying g we see that

$$g(z) \in W^{cs}_{10\eta\kappa\|A\|}(gx) \cap W^u_{10\eta\kappa\|A\|}(gy).$$

Let γ be a path that follows W^{cs} from gx to gz and then follows W^u from gz to gy, such that the length of γ is at most $20\eta\kappa\|A\|$. Then $d(\gamma(0),\gamma(1)) < 5\eta$, and since

$$20\eta\kappa\|A\| < 1 - 5\eta$$

by (5.4), Lemma 5.2 implies that the lift of γ to $\mathbb{R}^d$ also has $d(\tilde\gamma(0),\tilde\gamma(1)) < 5\eta$. By the uniqueness part of Lemma 3.5, we conclude that gz is in fact the unique point in $W^{cs}_{5\eta\kappa}(gx) \cap W^u_{5\eta\kappa}(gy)$.

Iterating the above argument gives

$$g^n z \in W^{cs}_{5\eta\kappa}(g^n x) \cap W^u_{5\eta\kappa}(g^n y)$$

for all $n \ge 0$. Because $\|Dg|_{E^u}^{-1}\| \le \theta_u < 1$, we see that for $z \in W^u_{5\eta\kappa\theta_u^n}(y)$ for all $n \ge 0$, hence $z = y$.

Having shown that $y \in W^{cs}_{\eta\kappa}(x)$, we now show that in fact y lies in $W^c(x)$. To this end, let $\widetilde{W}^{cs}, \widetilde{W}^c, \widetilde{W}^s$ be the lifts of W^{cs}, W^c, W^s to $\mathbb{R}^d$, and let $\tilde x, \tilde y \in \mathbb{R}^d$ be lifts of x, y such that $\tilde y \in \widetilde{W}^{cs}_{5\eta\kappa}(\tilde x)$. Adopt coordinates in which $\tilde x = 0$, and let $\pi_{cs} \colon \mathbb{R}^d \to F^{cs}$ be projection along

F^u. Let $\hat{y} = \pi_{cs}\tilde{y}$ and note that $\tilde{y} = \hat{y} + y_u$ where $y_u \in F^u$, so by the definition of $\kappa' = \bar{\kappa}(F^{cs}, F^u)$ we have

$$\|\hat{y}\| \le \kappa'\|\hat{y} + y_u\| = \kappa' d(\tilde{x}, \tilde{y}) = \kappa' d(x, y). \tag{5.9}$$

Let $\widehat{W}^c = \pi_{cs}\widetilde{W}^c$ and $\widehat{W}^s = \pi_{cs}\widetilde{W}^s$. Given $v_s + v_c \in \widehat{W}^s$, there is $v_u \in E^u$ such that $v_s + v_c + v_u \in C^s_\beta$, and hence

$$\|v_c\| \le \kappa'\|v_c + v_u\| < \kappa'\beta\|v_s\|,$$

so we see that $\widehat{W}^s$ is contained in $C_{\kappa'\beta}(E^s, E^c)$. Similarly, $\widehat{W}^c$ is contained in $C_{\kappa'\beta}(E^c, E^s)$. Thus by Lemma 3.5 there is a unique point $\hat{z}$ in $\widehat{W}^s(\hat{y}) \cap \widehat{W}^c(0)$. (Recall that $0 = \tilde{x}$.) Moreover, $d_{\widehat{W}^s}(\hat{y}, \hat{z})$ and $d_{\widehat{W}^c}(0, \hat{z})$ are both bounded above by

$$\left(\frac{1 + \kappa'\beta}{1 - \kappa'\beta}\right) \kappa''\|\hat{y}\|,$$

where we recall that $\kappa'' = \bar{\kappa}(E^s, E^c)$. Let $\tilde{z}$ be the unique point on $W^{cs}(\tilde{x})$ such that $\pi\tilde{z} = \hat{z}$; we see that $\tilde{z}$ is the unique point of intersection of $\tilde{W}^s(\tilde{y})$ and $\tilde{W}^c(\tilde{x})$, and that moreover

$$d_{\widetilde{W}^s}(\tilde{y}, \tilde{z}), d_{\widetilde{W}^c}(\tilde{x}, \tilde{z}) \le (1 + \beta)\left(\frac{1 + \kappa'\beta}{1 - \kappa'\beta}\right) \kappa''\kappa' d(x, y),$$

where we use (5.9) to estimate $\|\hat{y}\|$. We have proved the following: given any $\tilde{x} \in \mathbb{R}^d$ and $\tilde{y} \in \tilde{W}^{cs}(x)$, there is a unique point $\tilde{z} \in \tilde{W}^s(\tilde{y}) \cap \tilde{W}^c(\tilde{x})$, and moreover we have

$$d_{\widetilde{W}^s}(\tilde{y}, \tilde{z}), d_{\widetilde{W}^c}(\tilde{x}, \tilde{z}) \le \tilde{\kappa} d(\tilde{x}, \tilde{y}), \tag{5.10}$$

where we put

$$\tilde{\kappa} = (1 + \beta)\left(\frac{1 + \kappa'\beta}{1 - \kappa'\beta}\right) \kappa''\kappa'.$$

Projecting to $\mathbb{T}^d$, we see that given $y \in W^{cs}_{5\eta\kappa}(x)$ there is a unique $z \in W^c_{5\eta\kappa\tilde{\kappa}}(x) \cap W^s_{5\eta\kappa\tilde{\kappa}}(y)$, where uniqueness follows as in Lemma 5.3 by using the inequality $5\eta\kappa\tilde{\kappa} < \frac{1}{4}$ (from (5.3)) and Lemma 5.2.

Then we have

$$g^{-1}(z) \in W^c_{10\eta\kappa\tilde{\kappa}\|A^{-1}\|}(g^{-1}x) \cap W^s_{10\eta\kappa\tilde{\kappa}\|A^{-1}\|}(g^{-1}y),$$

and if in addition $y \in \Gamma_{5\eta}(x)$, then arguing as in the first paragraph of the proof we see that in fact

$$g^{-1}(z) \in W^c_{5\eta\kappa\tilde{\kappa}}(g^{-1}x) \cap W^s_{5\eta\kappa\tilde{\kappa}}(g^{-1}y),$$

where this time we use the inequality $20\eta\kappa\tilde{\kappa}\|A^{-1}\| + 5\eta < 1$ from (5.4). Iterating gives a similar result for $g^{-n}z$, and since W^s is uniformly expanded by g^{-1} the same argument as before shows that in fact we must have $y = z$, so $y \in W^c_{10\eta\kappa\tilde{\kappa}}(x)$, which completes the proof of Lemma 5.9. $\square$

We now show that there is no tail entropy at scale 5η.

Lemma 5.10. *Every diffeomorphism $g \in \mathcal{U}_0$ has $h^*_g(5\eta) = 0$.*

Proof. To estimate $\Lambda^{\mathrm{sep}}_n(\Gamma_{5\eta}(x) \times \mathbb{N}, 0, \delta; g)$, let $A_n \subset \Gamma_{5\eta}(x)$ be a maximal (n,δ)-separated subset. Let $k(n) = \#A_n$, and since by Lemma 5.9 we have $\Gamma_{5\eta}(x)$ is contained in a compact part of a one-dimensional center leaf, we can order the elements of A_n as $a_1, a_2, \dots, a_{k(n)}$ so that $a_{i\pm1}$ are the two points nearest to a_i for each i. For each $0 \le j < n$ and $1 \le i < k(n)$, let b^j_i be the distance from $f^j(a_i)$ to $f^j(a_{i+1})$ along the leaf $W^c_{10\eta\kappa}(g^j x)$. Note that for each j we have $\sum_i b^j_i \le 10\eta\kappa$.

On the other hand, the points a_i are (n,δ)-separated, so for every i there is some $0 \le j < n$ such that $b^j_i \ge d(f^j a_i, f^j a_{i+1}) \ge \delta$. In particular, for every i we have $\sum_{j=0}^{n-1} b^j_i \ge \delta$, and it follows that

$$(k(n)-1)\delta \le \sum_{i=1}^{k(n)-1} \sum_{j=0}^{n-1} b^j_i = \sum_{j=0}^{n-1} \sum_{i=1}^{k(n)-1} b^j_i \le 10\eta\kappa n.$$

Thus $k(n)$ grows at most linearly in n, which completes the proof of Lemma 5.10. $\square$

Lemma 5.11. *For any $r > \gamma$ and $\varepsilon = \rho/2$, the diffeomorphism g satisfies Condition **[E]** from §3.5.*

Proof. Given $x \in \mathbb{T}^d$, Lemma 5.9 shows that $\Gamma_\varepsilon(x) \subset W^{cs}_{\varepsilon\kappa}(x)$. It follows from Pliss' Lemma [50] that if $m_k \to \infty$ is such that $\frac{1}{m_k} S^{g^{-1}}_{m_k}\chi(x) \ge r$ for every k, then for every $r' \in (\gamma, r)$ there exists $m'_k \to \infty$ such that for every k and every $0 \le j \le m'_k$, we have $\frac{1}{j} S^{g^{-1}}_j (g^{-m'_k + j} x) \ge r'$. Thus $g^{-m'_k} x$ has the property that

$$\tfrac{1}{m} S^g_m(g^{-m'_k} x) \ge r' \text{ for all } 0 \le m \le m'_k,$$

so we can apply Lemma 5.5 and conclude that

$$\Gamma_\varepsilon(x) \subset g^{m'_k}(W^{cs}_{\varepsilon\kappa}(g^{-m'_k} x)) \subset B(x, \varepsilon\kappa\theta_{r'}^{m'_k})$$

Since $m'_k \to \infty$ and $\theta_{r'} < 1$, this implies that $\Gamma_\varepsilon(x) = \{x\}$. $\square$

5.5. **Proof of Theorem 4.1.** We have now done all the work to show that if $g \in \mathcal{U}_0$ and $\varphi\colon \mathbb{T}^d \to \mathbb{R}$ satisfy the hypotheses of Theorem 4.1, then the conditions of Theorem 2.8 are satisfied, and hence there is a unique equilibrium state for $(\mathbb{T}^d, g, \varphi)$. We recall how this is done; this will complete the proof of Theorem 4.1.

Since $\gamma = \gamma(g)$ satisfies (4.4), we may choose $r > \gamma$ such that (4.4) still holds with γ replaced by r, and define the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ as in (5.6). The following facts are shown in the previous sections.

- $\mathcal{G}$ has specification at all scales (Lemma 5.6).
- φ has the Bowen property on $\mathcal{G}$ at scale $\varepsilon = \frac{\rho}{2\kappa}$ (Lemma 5.8).
- $P(\mathcal{P}, \varphi; g)$ admits the following upper bound (Theorem 3.3):

$$h_g^*(5\eta) + (1-r) \sup_{x \in B(q,\rho)} \varphi(x) + r(\sup_{x \in \mathbb{T}^d} \varphi(x) + h + \log L - \log r).$$

- $h_g^*(5\eta) = 0$ (Lemma 5.10), so (4.4) gives $P(\mathcal{P}, \varphi) < P(g, \varphi)$.
- By Theorem 3.4 and Lemma 5.11, $P_{\exp}^{\perp}(\rho/2) \leq P(\mathcal{P}, \varphi)$.

Putting these ingredients together, we see that under the conditions of Theorem 4.1, all the hypotheses of Theorem 2.8 are satisfied for the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$. This completes the proof of Theorem 4.1. □

6. Results for the Bonatti-Viana example

We now review the Bonatti-Viana construction in [6] as well as some of the additional facts about invariant foliations in [17].

Let $B \in \mathrm{SL}(4, \mathbb{Z})$ induce a hyperbolic toral automorphism on the 4-torus and let B have four distinct real eigenvalues

$$0 < \lambda_1 < \lambda_2 < 1/3 < 3 < \lambda_3 < \lambda_4.$$

By possibly replacing B with a power of itself we may assume that the induced hyperbolic toral automorphism f_B has at least three fixed points, say q, q', q''. A neighborhood of one of the fixed points, say q'', will be left unperturbed to ensure robust transitivity. A deformation will be done around q and q'. Around q we will deform in the stable direction and around q' in the unstable direction. These deformations will be C^0 small, but C^1 large.

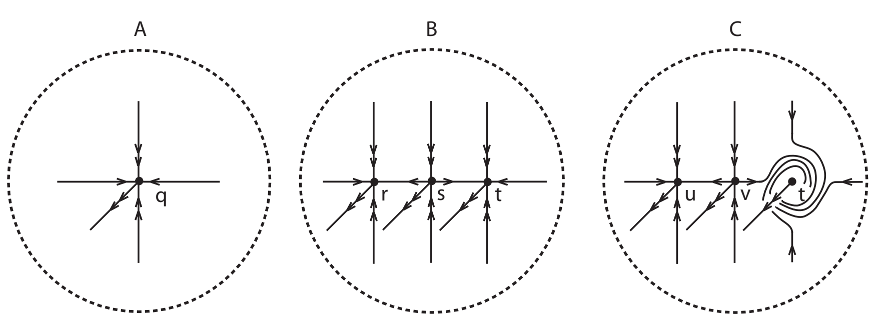

Figure 4. Bonatti–Viana construction

Let F^s, F^u be the two-dimensional subspaces of $\mathbb{R}^d$ corresponding to contracting and expanding eigenvalues of B, respectively. Let $\kappa = 2\bar{\kappa}(F^s, F^u)$, where $\bar{\kappa}$ is as in (3.4). Let $\eta > 0$ be sufficiently small that

$$5\eta\kappa < \tfrac{1}{4}. \tag{6.1}$$

Fixing $\rho > 0$, we consider the scales $\rho' = 5\rho$ and $\rho'' = 300\kappa\rho'$. We assume that ρ is sufficiently small that $\rho'' < 5\eta$. The role of these scales is as follows:

(1) All perturbations take place in the balls $B(q, \rho)$ and $B(q', \rho)$;
(2) The scale ρ' is chosen to ensure that at this scale the center-stable (resp. center-unstable) leaves are contracted by g (resp. g^{-1});
(3) The scale ρ'' is the distance that points need to be away from q and q' to guarantee uniform contraction/expansion estimates at a large enough scale to verify the hypotheses of Theorem 2.9.

The deformation around the points q and q' is done in two steps. We first describe the deformation around q. In the first step, we perform a DA-type deformation around q in the stable direction λ_2 similar to the one done for the Mañé examples. The stable index of q changes from 2 to 1 and two new fixed points q_1 and q_2 are created just as before. The second step is done by deforming the diffeomorphism in a neighborhood of q_2 so that the contracting eigenvalues become complex; see Figure 4. We use a deformation such that the E^{cs} is uniformly contracting outside $B(q, \rho)$.

Note the creation of fixed points with different indices prevents the topologically transitive map from being Anosov. These non-real eigenvalues also forbid the existence of a one-dimensional invariant subbundle inside E^{cs}. So the resulting map $\hat{f}$ has a splitting $E^{cs} \oplus E^u$.

To finish the construction take the deformation just made on f_B near q and repeat it on $\hat{f}^{-1}$ in the neighborhood of q'. We obtain a map f_1 that is robustly transitive, not partially hyperbolic, and has a dominated splitting $T\mathbb{T}^4 = E^{cs} \oplus E^{cu}$ with $\dim E^{cs} = \dim E^{cu} = 2$ (see [6] for proofs of these facts). Put $\beta \in (0, \frac{1}{3})$ such that (6.1) continues to hold with η replaced by

$$\eta(1+\beta)\frac{(\sin \omega_{s,u}) + \beta}{(\sin \omega_{s,u}) - \beta},$$

and assume the perturbation is such that $E^{cs} \subset C_\beta(F^s, F^u)$ and $E^{cu} \subset C_\beta(F^u, F^s)$. In particular, f_1 has local product structure at scale 5η with constant κ, exactly as in Lemma 5.3.

Let $C = C(f_B)$ be the constant provided by Lemma 3.1. Outside $B(q, \rho) \cup B(q', \rho)$, the maps f_0 and f_B are identical, and we can carry out the construction so there exists a constant K so that both $f_B(B(q,\rho)) \subset B(q, K\rho)$ and $f_1(B(q,\rho)) \subset B(q, K\rho)$ (and similarly for q'). Thus the C^0 distance between f_1 and f_B is at most $K\rho$. In particular, by choosing ρ small, we can ensure that $d_{C^0}(f_1, f_B) < \eta/C$. This allows us to apply Lemma 3.2 to f_1, or to a perturbation of f_1.

As in the previous section, we now consider diffeomorphisms g in a C^1 neighborhood of f_1. In [17], it is shown that for $g \in \text{Diff}(\mathbb{T}^4)$ sufficiently close to f_1, there are invariant foliations tangent to E^{cs}_g and E^{cu}_g respectively. Furthermore, the argument of Lemma 6.1 and 6.2 of [6] shows that each leaf of each foliation is dense in the torus. The existence of foliations was not known when [6] was written, but these arguments apply with only minor modification now that the existence result has been established by [17]. Thus we can consider a C^1-neighborhood $\mathcal{V}_0$ of f_1 such that the following is true for every $g \in \mathcal{V}_0$.

- $d_{C^0}(g, f_B) < \eta/C$.
- g has a dominated splitting $T\mathbb{T}^4 = E^{cs}_g \oplus E^{cu}_g$, with $\dim E^{cs}_g = \dim E^{cu}_g = 2$ and E^{cs}_g, E^{cu}_g tangent to $C_\beta(F^s, F^u)$ and $C_\beta(F^u, F^s)$ respectively.
- The distributions E^{cs}_g, E^{cu}_g integrate to foliations W^{cs}_g, W^{cu}_g.
- Each of the leaves $W^{cs}_g(x)$ and $W^{cu}_g(x)$ is dense for every $x \in \mathbb{T}^4$.

Given $g \in \mathcal{V}_0$, consider the quantities

$$\begin{aligned}
\lambda_{cs}(g) &= \sup\{\|Dg|_{E^{cs}(x)}\| : x \in \mathbb{T}^4\}, \\
\lambda_{cu}(g) &= \inf\{\|Dg|^{-1}_{E^{cs}(x)}\|^{-1} : x \in \mathbb{T}^4\}, \\
\lambda_{s}(g) &= \sup\{\|Dg|_{E^{cs}(x)}\| : x \in \mathbb{T}^4 \setminus B(q,\rho)\}, \\
\lambda_{u}(g) &= \inf\{\|Dg|^{-1}_{E^{cs}(x)}\|^{-1} : x \in \mathbb{T}^4 \setminus B(q',\rho)\}.
\end{aligned} \tag{6.2}$$

We also need the quantity

$$\lambda_c(g) = \max\{\lambda_{cs}(g), \lambda_{cu}(g)^{-1}\}.$$

Note that by the construction of f_1 we have

$$\begin{aligned}
\lambda_s(f_1) < 1 < \lambda_{cs}(f_1), \\
\lambda_{cu}(f_1) < 1 < \lambda_u(f_1),
\end{aligned}$$

and by continuity these inequalities hold for C^1-perturbations of f_1. We assume that $\mathcal{V}_0$ is chosen such that

$$(1+\beta)\left(\frac{\lambda_c(g) - \lambda_s(g)}{1 - \lambda_s(g)}\right) < 2 \tag{6.3}$$

for every $g \in \mathcal{V}_0$. Similarly to before, let

$$\gamma(g) = \max\left\{\frac{\ln \lambda_{cs}(g)}{\ln \lambda_{cs}(g) - \ln \lambda_s(g)}, \frac{\ln \lambda_{cu}(g)}{\ln \lambda_{cu}(g) - \ln \lambda_u(g)}\right\}$$

A simple calculation as in (4.3) shows that for any $r > \gamma$, we have

$$\lambda_{cs}^{1-r}\lambda_s^r < 1, \tag{6.4}$$

$$\lambda_{cu}^{1-r}\lambda_u^r > 1, \tag{6.5}$$

so that in particular, writing

$$\theta_r(g) = \min(\lambda_{cs}^{1-r}\lambda_s^r, \lambda_{cu}^{-(1-r)}\lambda_u^{-r}), \tag{6.6}$$

we have $\theta_r(g) < 1$ for all $r > \gamma(g)$. As in §4, Theorem B will follow from the next result. For notational convenience, we write

$$Q = B(q, \rho'' + \rho) \cup B(q', \rho'' + \rho) \tag{6.7}$$

for a suitably large neighborhood of the region where the C^0-perturbation occurs.

Theorem 6.1. *Given $g \in \mathcal{V}_0$ as above, let $\gamma = \gamma(g)$, $\lambda_c = \lambda_c(g)$, $V = \operatorname{Var}(\varphi, 300\rho')$, and let $\varphi\colon \mathbb{T}^4 \to \mathbb{R}$ be Hölder continuous. If*

$$6\log\lambda_c + (1-\gamma)\sup_Q \varphi + \gamma(\sup_{\mathbb{T}^4}\varphi + \log L + h - \log\gamma) + V < P(g, \varphi), \tag{6.8}$$

then φ has a unique equilibrium state with respect to g.

The main differences between (6.8) and the corresponding conditions in Theorem 4.1 are the presence of the terms $6\log\lambda_c$ and V. The first of these terms is due to the fact that the Bonatti-Viana example may not be entropy expansive, and in particular we may have $h_g^*(5\eta) > 0$; we will see that this quantity is bounded by $6\log\lambda_c$. The V term comes from the finite scale version of our general theorem 2.9, and comes from the fact that we establish specification for suitable orbit collections at scale $3\rho'$.

The argument for deriving Theorem B from Theorem 6.1 is essentially identical to the argument after Theorem 4.1 deriving Theorem A: first one observes that condition (6.8) is satisfied for a C^1-open set $\mathcal{V} \subset \mathcal{V}_0$. Then one argues that this set is non-empty because (6.8) can be verified with an analogue of (4.5), adding the term $6\log\lambda_c$, which can be made arbitrarily small.

We also have the following analogue of Theorem 4.2, which is proved exactly as that result is.

Theorem 6.2. *Let $\mathcal{V}_0 \subset \mathrm{Diff}(\mathbb{T}^4)$ be as above, and suppose $g \in \mathcal{V}_0$ is such that for $L = L(f_B)$, $h = h_{\mathrm{top}}(f_B)$, $\gamma = \gamma(g)$, and $\lambda_c = \lambda_c(g)$ we have*

$$6 \log \lambda_c + \gamma(\log L + h - \log \gamma) < h. \tag{6.9}$$

Then writing

$$D = h - 6 \log \lambda_c - \gamma(\log L + h - \log \gamma) > 0,$$

every Hölder continuous potential φ with the bounded range hypothesis $\sup \varphi - \inf \varphi < D$ *has a unique equilibrium state. In particular,* (6.9) *is a sufficient criterion for g to have a unique measure of maximal entropy.*

7. Technique for Bonatti-Viana examples

The arguments in this section are similar to the proof of Theorem 4.1 in §5, although there are a number of complications. There are three important differences:

- we need to control φ in neighborhoods around two fixed points;
- verifying specification is more subtle because of the lack of a uniformly expanding direction;
- we must obtain an upper bound for the tail entropy, which may be nonzero.

7.1. **Specification.** This time there are three issues that go into the specification property: We must control the size of $W^{cs}_\delta(x)$ under iteration, the size of $W^{cu}_\delta(x)$ under iteration, and to transition from one orbit to the next we use the following fact, which has the same proof as Lemma 5.4.

Lemma 7.1. *For every $\delta > 0$ there is $R > 0$ such that for all $x, y \in \mathbb{T}^4$, we have $W^{cu}_R(x) \cap W^{cs}_\delta(y) \neq \emptyset$.*

Although the leaves $W^{cu}(x)$ are not expanding at every point, and the leaves $W^{cs}(x)$ are not contracting at every point, we nevertheless see expansion and contraction if we look at a scale suitably large relative to ρ. More precisely, consider the quantities $\theta_{cs} = \frac{4}{5} + \frac{1}{5}\lambda_s(g) < 1$ and $\theta_{cu} = \frac{4}{5} + \frac{1}{5}\lambda_u(g)^{-1} < 1$. Let d_{cs} and d_{cu} be the metrics induced on leaves W^{cs} and W^{cu}. Then we have the following result.

Lemma 7.2. *If $x \in \mathbb{T}^4$ and $y \in W^{cs}(x)$ are such that $d_{cs}(x, y) > \rho'$, then $d_{cs}(gx, gy) < \theta_{cs} d_{cs}(x, y)$. Similarly, if $y \in W^{cu}(x)$ and $d_{cu}(x, y) > \rho'$, then $d_{cu}(g^{-1}x, g^{-1}y) < \theta_{cu} d_{cu}(x, y)$.*

Proof. We give the proof for W^{cs}; the proof for W^{cu} is analogous. Given a path σ on $\mathbb{T}^4$, write $\ell(\sigma)$ for the length of σ. Let σ be a path from x to y in $W^{cs}(x)$ such that $\ell(\sigma) = d_{cs}(x, y)$. Decompose σ as the disjoint union of paths σ_i where $\ell(\sigma_i) \in [\rho', 2\rho']$. Clearly it suffices to show that $\ell(g\sigma_i) < \theta_{cs}\ell(\sigma_i)$ for each i.

Because $\ell(\sigma_i) \leq 2\rho' \leq 1-2\rho$, the curve σ_i has at most one connected component that intersects $B(q, \rho)$; this follows from Lemma 5.2. Let ℓ_1 be the length of this component; because this component lies in $W^{cs}(x)$, which is tangent to $C_\beta(F^{cs}, F^u)$, we have $\ell_1 \leq 2\rho(1 + \beta)$. Let $\ell_2 = \ell(\sigma_i) - \ell_1$. Let v be a tangent vector to the curve σ at the point $p \in \mathbb{T}^4$. If $p \in B(q, \rho)$ then we have $\|Dg(v)\| \leq \lambda_c(g)\|v\|$, while if $p \notin B(q, \rho)$ then $\|Dg(v)\| \leq \lambda_s(g)\|v\|$. Thus we obtain

$$\begin{aligned}\ell(g\sigma_i) &\leq \lambda_c\ell_1 + \lambda_s\ell_2 = (\lambda_c - \lambda_s)\ell_1 + \lambda_s\ell(\sigma_i)\\ &\leq (\lambda_c - \lambda_s)2\rho(1 + \beta) + \lambda_s\ell(\sigma_i) < 4(1 - \lambda_s)\rho + \lambda_s\ell(\sigma_i),\end{aligned}$$

where the last inequality uses (6.3). Since $\rho = \frac{1}{5}\rho' \leq \frac{1}{5}\ell(\sigma_i)$, this gives

$$\ell(g\sigma_i) < \tfrac{4}{5}(1 - \lambda_s)\ell(\sigma_i) + \lambda_s\ell(\sigma_i) = \theta_{cs}\ell(\sigma_i).$$

Summing over i gives $d_{cs}(gx, gy) \leq \ell(g\sigma) < \theta_{cs}\ell(\sigma) = \theta_{cs}d_{cs}(x, y)$. The proof for d_{cu} is similar. ☐

The following is an immediate consequence of Lemmas 7.2 and 7.1.

Lemma 7.3. *For every $R > \rho'$ and $x \in \mathbb{T}^4$, we have*

$$\begin{aligned}g(W^{cs}_R(x)) &\subset W^{cs}_{\theta_{cs}R}(gx),\\ g^{-1}(W^{cu}_R(x)) &\subset W^{cu}_{\theta_{cu}^{-1}R}(g^{-1}x).\end{aligned}$$

In particular, there is $\tau_0 \in \mathbb{N}$ such that for every $x, y \in \mathbb{T}^4$ we have

$$g^{\tau_0}(W^{cu}_{\rho'}(x)) \cap W^{cs}_{\rho'}(y) \neq \emptyset. \tag{7.1}$$

Remark 7.4. *Lemma 7.3, and in particular* (7.1)*, gives us the ability to glue* any two *orbit segments together at scale ρ'. The ability to specify two, rather than finitely many orbits together is a weak version of the specification property that appears in the literature. However, our results require the ability to glue finitely many orbit segments together, and for this we have to find an appropriate decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$.*

Now let $\rho'' := 300\kappa\rho'$, where $\kappa = 2\bar{\kappa}(F^{cs}, F^u)$ is the constant arising in the local product structure of W^{cs}, W^{cu}, as before. Let χ be the indicator function of $\mathbb{T}^4 \setminus B(q, \rho''+\rho)$ and χ' be the indicator function of $\mathbb{T}^4 \setminus B(q', \rho''+\rho)$. Thus the Birkhoff averages of χ, χ' give the proportion of time an orbit spends away from the fixed points q, q'. The choice of scale is to ensure uniform estimates on $W^{cs}_{\rho''}$ and $W^{cu}_{\rho''}$ for points x for which $\chi(x) = 1$ and $\chi'(x) = 1$.

From now on we fix $r > \gamma(g)$, in practice $\gamma(g)$ will be very small and r will be slightly larger than $\gamma(g)$, and consider the following collection of orbit segments:

$$\mathcal{G} = \{(x, n) : \tfrac{1}{i} S_i\chi(x) \geq r \text{ and } \tfrac{1}{i} S_i\chi'(f^{n-i}x) \geq r \text{ for all } 0 \leq i \leq n\}.$$

We will show that $\mathcal{G}^M$ has specification at scale $3\rho'$. To get a decomposition we consider $\mathcal{G}$ together with the collections

$$\mathcal{P} = \{(x, n) \in \mathbb{T}^4 \times \mathbb{N} : \tfrac{1}{n} S_n\chi(x) < r\},$$
$$\mathcal{S} = \{(x, n) \in \mathbb{T}^4 \times \mathbb{N} : \tfrac{1}{n} S_n\chi'(x) < r\}.$$

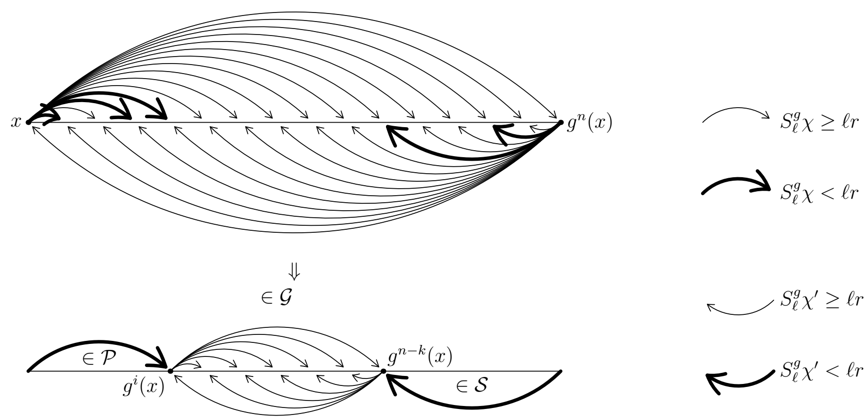

Figure 5. Decomposition for Bonatti-Viana

Lemma 7.5. *The collections $\mathcal{P}, \mathcal{G}, \mathcal{S}$ form a decomposition for g.*

Proof. Let $(x, n) \in X \times \mathbb{N}$. Let $0 \leq i \leq n$ be the largest integer so $\frac{1}{i} S_i\chi(x) < r$, and $0 \leq k \leq n$ be the largest integer so $\frac{1}{k} S_k\chi'(g^{n-k}x) < r$. A short calculation shows that $\frac{1}{\ell} S_\ell\chi(g^i x) \geq r$ for $0 \leq \ell \leq n - i$, and $\frac{1}{\ell} S_\ell\chi'(g^{n-k-\ell}x) \geq r$ for $0 \leq \ell \leq n-k$, see Figure 5. Thus, if we assume that $i + k < n$, letting $j = n - (i + k)$, we have

$$(x, i) \in \mathcal{P}, \qquad (g^i x, j) \in \mathcal{G}, \qquad (g^{i+j}x, k) \in \mathcal{S}.$$

If $i + k \geq n$, we can choose a decomposition with $j = 0$. $\square$

As a reminder $\mathcal{G}^M$ is the set of orbit segments (x, n) for which $p \leq M$ and $s \leq M$. Orbit segments in $\mathcal{G}^M$ satisfy the following analogue of Lemma 5.5.

Lemma 7.6. *Let $\nu = \lambda_c/\theta_r$. For every $M \in \mathbb{N}_0$, $(x, n) \in \mathcal{G}^M$, and $0 \leq i \leq n$, we have*

(a) $\|Dg^i|_{E^{cs}(y)}\| \le \nu^{2M}\theta_r^i$ *for* $y \in B_n(x,\rho'')$*;*
(b) $\|Dg^{-i}|_{E^{cu}(g^n y)}\| \le \nu^{2M}\theta_r^i$ *for* $y \in B_n(x,\rho'')$*;*
(c) $d_{cs}(g^i y, g^i z) \le \nu^{2M}\theta_r^i d_{cs}(y,z)$ *when* $y \in B_n(x,\rho')$ *and* $z \in W^{cs}_{2\rho'}(y)$*;*
(d) $d_{cu}(g^{n-i}y, g^{n-i}z) \le \nu^{2M}\theta_r^i d_{cu}(y,z)$ *when* $y \in B_n(g^{-n}x,\rho')$ *and* $z \in W^{cu}_{2\rho'}(y)$*.*

Proof. We prove (a). Given $(x,n) \in \mathcal{G}^M$ and $0 \le i \le n$, we have $S_i\chi(x) > ir - 2M$, and so the orbit segment (x,i) spends greater than $ir - 2M$ iterates outsides $B(q,4\rho')$, and thus (y,i) spends greater than $ir-2M$ iterates outsides $B(q,\rho)$. It follows that

$$\begin{aligned}\|Dg^i|_{E^{cs}(y)}\| &\le \lambda_c^{i-(ir-2M)}\lambda_s^{ir-2M}\\ &= \lambda_c^{i(1-r)}\lambda_s^{ir}\lambda_c^{2M}\lambda_s^{-2M} = (\theta_r)^i\nu^{2M}.\end{aligned}$$

For (c), note that if $y \in B_n(x,\rho')$ and $z' \in W^{cs}_{2\rho'}(y)$, then $z' \in B_n(x,3\rho')$. Thus, the uniform derivative estimate of (a) applies to all points in $W^{cs}_{2\rho'}(y)$, and it is an easy exercise to use this to obtain the statement of (c). The proof of (b) is similar to (a), and (d) follows. □

We are now in a position to give the main lemma of this subsection regarding the specification property. The facts that drive our proof are:

- For any $x \in \mathbb{T}^d$ and $n \in \mathbb{N}$, from Lemma 7.3 we have $W^{cs}_{\rho'}(x) \subset B_n(x,\rho')$ and $g^{-n}(W^{cu}_{\rho'}(g^n x)) \subset B_n(x,\rho')$;
- If $(x,n) \in \mathcal{G}^M$ and $y,z \in B_n(x,3\rho')$ and $g^n z \in W^{cu}(g^n y)$, then Lemma 7.6 (c) gives $d_n(y,z) \le \nu^{2M} d_{cu}(g^n y, g^n z)$ and $d_{cu}(y,z) \le \nu^{2M}\theta_r^n d_{cu}(g^n y, g^n z)$.

A key part of the specification proof for the Mañé example is the fact that we can obtain uniform backward contraction along each orbit segment by choosing τ large. In this setting we no longer have this option because E^{cu} is not uniformly expanding. Thus we take another route; we observe that we get uniform backwards contraction along W^{cu} near $(x,n) \in \mathcal{G}^M$ as long as n is sufficiently large, so everything will work provided all the orbit segments we glue together are sufficiently long. This yields tail specification for $\mathcal{G}^M$, which is sufficient for Theorem 2.9 to apply.

More precisely, given M we take $N = N(M)$ such that $\theta_r\nu^{2M}\lambda_c^{\tau_0} < \frac{1}{2}$, where τ_0 is as in (7.1). Then we let $\mathcal{G}^M_{\ge N} := \{(x,n) \in \mathcal{G}^M \mid n \ge N\}$.

Lemma 7.7. *For every* M*, let* $N = N(M)$ *be as above. Then* $\mathcal{G}^M_{\ge N}$ *has specification at scale* $3\rho'$*.*

Proof. We follow the proof of Lemma 5.6, making the necessary adjustments as described above. Write $\tau = \tau_0$, so that (7.1) gives $g^\tau(W^{cu}_{\rho'}(x)) \cap W^{cs}_{\rho'}(y) \ne \emptyset$ for every $x,y \in \mathbb{T}^4$.

For every $(x,n) \in \mathcal{G}^M_{\geq N}$ and $y,z \in g^{-(n+\tau)}(g^\tau(W^{cu}_{\rho'}(x)))$, our choice of N gives

$$d(y,z) < \tfrac{1}{2} d(g^{n+\tau}y, g^{n+\tau}z). \tag{7.2}$$

Now we show that $\mathcal{G}^M_{\geq N}$ has specification with gluing time τ. Given any $(x_0,n_0),\dots,(x_k,n_k) \in \mathcal{G}^M$ with $n_i \geq N$, we construct y_j iteratively such that (y_j,m_j) shadows $(x_0,n_0),\dots,(x_j,n_j)$, where $m_0 = n_0$, $m_1 = n_0 + \tau + n_1$, $\dots$, $m_k = (\sum_{i=0}^k n_i) + k\tau$. We also set $m_{-1} = -\tau$.

Start by letting $y_0 = x_0$, and we choose $y_1,\dots,y_k$ iteratively so that

$$\begin{array}{ccc} g^{m_0}y_1 \in W^{cu}_{\rho'}(g^{m_0}y_0) & \text{and} & g^{m_0+\tau}y_1 \in W^{cs}_{\rho'}(x_1) \\ g^{m_1}y_2 \in W^{cu}_{\rho'}(g^{m_1}y_1) & \text{and} & g^{m_1+\tau}y_2 \in W^{cs}_{\rho'}(x_2) \\ \vdots & \vdots & \vdots \\ g^{m_{k-1}}y_k \in W^{cu}_{\rho'}(g^{m_{k-1}}y_{k-1}) & \text{and} & g^{m_{k-1}+\tau}y_k \in W^{cs}_{\rho'}(x_k). \end{array}$$

That is, for $j \in \{0,\dots,k-1\}$, we let y_{j+1} be a point such that

$$y_{j+1} \in g^{-m_j}(W^{cu}_{\rho'}(g^{m_j}y_j)) \cap g^{-(m_j+\tau)}(W^{cs}_{\rho'}(x_{j+1})).$$

Using the fact that $g^{m_j}y_{j+1}$ is in the unstable manifold of $g^{m_j}y_j$ together with the estimate (7.2), we obtain that

$$\begin{array}{rcl} d_{n_j}(g^{m_{j-1}+\tau}y_j, g^{m_{j-1}+\tau}y_{j+1}) & < & \rho' \\ d_{n_{j-1}}(g^{m_{j-2}+\tau}y_j, g^{m_{j-2}+\tau}y_{j+1}) & < & \rho'/2 \\ \vdots & & \vdots \\ d_{n_0}(y_j, y_{j+1}) & < & \rho'/2^j. \end{array}$$

That is, $d_{n_{j-i}}(g^{m_{j-i-1}+\tau}y_j, g^{m_{j-i-1}+\tau}y_{j+1}) < \rho'/2^i$ for each $i \in \{0,\dots,j\}$. This estimate, together with the fact that $g^{m_j+\tau}(y_{j+1}) \in B_{n_{j+1}}(x_{j+1},\rho')$ from Lemma 7.3 gives that $d_{n_j}(g^{m_{j-1}+\tau}y_k, x_j) < 2\rho' + \sum_{j=1}^\infty 2^{-j}\rho' = 3\rho'$. It follows that

$$y_k \in \bigcap_{j=0}^k g^{-(m_{j-1}+\tau)} B_{n_j}(x_j, 3\delta),$$

and thus $\mathcal{G}^M_{\geq N}$ has specification at scale $3\rho'$. □

We remark that the proof of Lemma 7.7 can be adapted to show that the entire collection $\mathcal{G}^M$ has specification, and not just its tail $\mathcal{G}^M_{\geq N}$. The idea is to fix $N = N(M)$ as before and then choose $(\bar{x},\bar{n}) \in \mathcal{G}$ with $\bar{n} > N$; one can find an orbit that shadows (x_1,n_1), $(\bar{x},\bar{n})$, (x_2,n_2), $(\bar{x},\bar{n}),\dots$, using the uniform contraction estimates along $(\bar{x},\bar{n})$ to get specification with $\tau = 2\tau_0 + N(M)$. The book-keeping in this argument is messier than in the lemma above, but the essential ideas are the same.

7.2. **Bowen property.** Let $\theta_r \in (0,1)$ be the constant that appears in the previous subsection, and let κ be the constant associated with the local product structure of $E^{cs}_g \oplus E^{cu}_g$ (see §3.1).

Lemma 7.8. *Given $(x,n) \in \mathcal{G}$ and $y \in B_n(x, 300\rho')$, we have*

$$d(g^k x, g^k y) \le \kappa 600\rho'(\theta_r^{n-k} + \theta_r^k) \tag{7.3}$$

for every $0 \le k \le n$.

Proof. Using the local product structure at scale $300\rho'$, there exists $z \in W^{cs}_{\kappa 300\rho'}(x) \cap W^{cu}_{\kappa 300\rho'}(y) = W^{cs}_{\rho''}(x) \cap W^{cu}_{\rho''}(y)$. By Lemma 7.6, we get

$$d(g^k z, g^k y) \le \theta_r^{n-k} d(g^n z, g^n y) \le \theta_r^{n-k}\kappa 300\rho',$$

and

$$d(g^k x, g^k z) \le \theta_r^k d(x,z) \le \theta_r^k \kappa 300\rho'.$$

The triangle inequality gives (7.3). ☐

Lemma 7.9. *Any Hölder continuous φ has the Bowen property on $\mathcal{G}$ at scale $300\rho'$.*

Proof. By Hölder continuity, there are constants $K > 0$ and $\alpha \in (0,1)$ such that $|\varphi(x) - \varphi(y)| \le K d(x,y)^\alpha$ for all $x, y \in \mathbb{T}^d$. Now given $(x,n) \in \mathcal{G}$ and $y \in B_n(x, 300\rho')$, Lemma 7.8 gives

$$\begin{aligned} |S_n\varphi(x) - S_n\varphi(y)| &\le K \sum_{k=0}^{n-1} d(g^k x, g^k y)^\alpha \le K\kappa 600\rho' \sum_{k=0}^{n-1} (\theta_r^{n-k} + \theta_r^k)^\alpha \\ &\le K\kappa 600\rho' \sum_{j=0}^{\infty} (\theta_r^{j\alpha} + \theta_r^{j\alpha}) =: V < \infty. \end{aligned}$$ ☐

7.3. **Expansivity.** We want to obtain a bound on h^*_g (the tail entropy of g). We note that by the results of [32], the tail entropy may be positive. We first need the following estimate on α-dense sets in local leaves of W^{cu}.

Lemma 7.10. *Let $\varepsilon > \delta > 0$ be such that $2\varepsilon\|Dg\| < 1 - 2\varepsilon$. Given $n \in \mathbb{N}$, and $x, z \in \mathbb{T}^4$ such that $d_n(x,z) < \varepsilon$, we have*

$$\Lambda_n^{\mathrm{span}}(W^{cu}_\varepsilon(z) \cap B_n(x,\varepsilon), 0, \delta; g) \le 4\varepsilon^2\delta^{-2}(1+\beta)^2\lambda_c^{2n}. \tag{7.4}$$

Proof. First we claim that $W^{cu}_\varepsilon(z) \cap B_k(x,\varepsilon) \subset g^{-k}(W^{cu}_{2\varepsilon}(g^k z))$. This follows by induction; it is true for $k = 0$, and given the result for k, we see that any $z' \in W^{cu}_\varepsilon(z) \cap B_{k+1}(x,\varepsilon)$ has $g^k(z') \in W^{cu}_{2\varepsilon}(g^k z)$ by the inductive hypothesis, and so

$$g^{k+1}(z') \in W^{cu}_{2\varepsilon\|Dg\|}(g^{k+1}z) \cap B(g^{k+1}x, \varepsilon) \subset W^{cu}_{2\varepsilon}(g^{k+1}z),$$

where the last inclusion follows because $g^{k+1}z \in B(g^{k+1}x, \varepsilon)$ and $2\varepsilon\|Dg\|$ is not enough distance to wrap all the way around the torus and enter $B(g^{k+1}x, \varepsilon)$ again.

Now fix $\alpha > 0$ small (an exact estimate will come later). Recall that $W^{cu}_{2\varepsilon}(g^n z)$ is the graph of a function from F^{cu} to F^{cs} with norm less than β. The projection of $W^{cu}_{2\varepsilon}(g^n z)$ to F^{cu} along F^{cs} is contained in a ball of radius 2ε, so $B_{2\varepsilon}(0)$ in F^{cu} has an α-dense subset in the d_n-metric with cardinality less than or equal to $4\varepsilon^2\alpha^{-2}$. Projecting this set back to $W^{cu}_{2\varepsilon}(g^n z)$ along F^{cu} gives $E \subset W^{cu}_{2\varepsilon}(g^n z)$ that is $(1+\beta)\alpha$-dense.

Consider the set $g^{-n}(E) \subset W^{cu}(z)$. Given any $y \in W^{cu}_\varepsilon(z) \cap B_n(x, \varepsilon)$, we have $g^n(y) \in W^{cu}_{2\varepsilon}(g^n z)$ and so there is $z' \in E$ such that $d_{cu}(g^n y, g^n z') < (1+\beta)\alpha$. Since g^{-1} expands distances along W^{cu} by at most λ_c, we have $d_n(y, z') < (1+\beta)\alpha\lambda_c^n$. Putting $\alpha = \delta(1+\beta)^{-1}\lambda_c^{-n}$ we see that $g^{-n}(E)$ is an (n,δ)-spanning set for $W^{cu}_\varepsilon(z) \cap B_n(x,\varepsilon)$, and moreover

$$\#g^{-n}(E) \le 4\varepsilon^2\alpha^{-2} \le 4\varepsilon^2\delta^{-2}(1+\beta)^2\lambda_c^{2n},$$

which gives (7.4) and completes the proof of Lemma 7.10. □

Our next lemma obtains an estimate on the tail entropy by applying Lemma 7.10 with $\varepsilon = 5\eta$; for this we need η small enough that $10\eta(\|Dg\| + 1) < 1$.

Lemma 7.11. *For every $g \in \mathcal{V}_0$ we have $h_g^*(5\eta) \le 6\log\lambda_c$.*

Proof. Given $x \in \mathbb{T}^4$ and $\delta > 0$, we estimate $\Lambda_n^{\mathrm{span}}(\Gamma_{5\eta}(x), 0, 2\delta; g)$ for $n \in \mathbb{N}$. To do this, we start by fixing

$$\alpha = \alpha(n) = \frac{\delta}{\kappa\lambda_c^n} \tag{7.5}$$

where κ is from the local product structure. Let $E \subset \Gamma_{5\eta}(x)$ be an α-dense set with cardinality

$$\#E \le (10\eta/\alpha)^4 = (10\eta)^4\kappa^4\lambda_c^{4n}\delta^{-4};$$

note that such a set exists because $\Gamma_{5\eta}(x)$ is contained in $x + [-5\eta, 5\eta]^4$.

Now we have $W^{cu}_{\kappa\alpha}(z) \subset W^{cu}_{5\eta}(z)$ for each $z \in E$, so by Lemma 7.10, there is an (n,δ)-spanning set E_z for $W^{cu}_{\kappa\alpha}(z) \cap B_n(x, 5\eta)$ with

$$\#E_z \le (10\eta)^2\delta^{-2}(1+\beta)^2\lambda_c^{2n}.$$

Let $E' = \bigcup_{z\in E} E_z$, then we have

$$\#E' \le (10\eta)^6\delta^{-6}(1+\beta)^2\kappa^4\lambda_c^{6n}.$$

We claim that E' is $(n, 2\delta)$-spanning for $\Gamma_{5\eta}(x)$, which will complete the proof of Lemma 7.11. To see this, take any $y \in \Gamma_{5\eta}(x)$, and observe that because E is α-dense in $B(x, 5\eta)$, there is $z = z(y) \in E \cap B(y, \alpha)$.

By the local product structure there is $\bar{z} = \bar{z}(y) \in W^{cs}_{\kappa\alpha}(y) \cap W^{cu}_{\kappa\alpha}(z)$. Notice that because distance expansion along W^{cu} is bounded above by λ_c for each iteration of g, we have

$$d_n(y, \bar{z}) < \kappa\alpha\lambda_c^n = \delta. \tag{7.6}$$

By our choice of E_z, there is $z' \in E_z$ such that $d_n(z', \bar{z}) < \delta$. Thus $d_n(y, z') < 2\delta$, as desired. It follows that

$$\Lambda_n^{\mathrm{span}}(\Gamma_{5\eta}(x), 0, 2\delta; g) \leq (10\eta)^6\delta^{-6}(1+\beta)^2\kappa^4\lambda_c^{6n},$$

hence $h_g^*(5\eta) \leq 6\log\lambda_c$, which proves Lemma 7.11. □

Lemma 7.12. *For every $r > \gamma(g)$ and $\varepsilon = 300\rho'$, the diffeomorphism g satisfies Condition **[E]**.*

Proof. We follow the same ideas as in Lemmas 5.9 and 5.11. Suppose $x \in \mathbb{T}^d$, $r > 0$, and $n_k, m_k \to \infty$ are such that

$$\tfrac{1}{n_k} S^g_{n_k}\chi(x) \geq r, \qquad \tfrac{1}{m_k} S^{g^{-1}}_{m_k}\chi'(x) \geq r \tag{7.7}$$

for every k. Our goal is to show that $\Gamma_\varepsilon(x) = \{x\}$.

First we fix $r' \in (\gamma, r)$ and observe that by Pliss' lemma [50] there are $m_k', n_k' \to \infty$ such that

$$\begin{aligned} S^g_m\chi'(g^{-m_k'}x) &\geq mr' \text{ for every } 0 \leq m \leq m_k', \\ S^{g^{-1}}_n\chi(g^{n_k'}x) &\geq nr' \text{ for every } 0 \leq n \leq n_k'. \end{aligned} \tag{7.8}$$

As in the proof of Lemma 7.6, for every $y \in B_{m_k'}(g^{-m_k'}x, \rho'')$ and $z \in g^{n_k'}B_{n_k'}(x, \rho'')$, we now have

$$\begin{aligned} \|Dg^m(y)|_{E^{cs}}\| &\leq \theta_{r'}^m \text{ for every } 0 \leq m \leq m_k', \\ \|Dg^{-n}(z)|_{E^{cu}}\| &\leq \theta_{r'}^m \text{ for every } 0 \leq n \leq n_k', \end{aligned} \tag{7.9}$$

where $\theta_{r'} < 1$ is as in (6.6).

Now let $x' \in \Gamma_\varepsilon(x)$. By the local product structure, there is a unique point $x'' \in W^{cu}_{\kappa\varepsilon}(x) \cap W^{cs}_{\kappa\varepsilon}(x')$. The same argument as in the first paragraph of the proof of Lemma 5.9 shows that for every $n \in \mathbb{Z}$ we have

$$g^n(x'') \in W^{cu}_{\kappa\varepsilon}(g^nx) \cap W^{cs}_{\kappa\varepsilon}(g^nx'). \tag{7.10}$$

In particular, for each $k \in \mathbb{N}$ we can apply (7.9) with z a point along the W^{cu}-geodesic from $g^{n_k'}x$ to $g^{n_k'}x''$, and deduce that

$$d_{cu}(x, x'') \leq \theta_{r'}^{n_k'} d_{cu}(g^{n_k'}x, g^{n_k'}x'') \leq \theta_{r'}^{n_k'}\kappa\varepsilon.$$

Sending $k \to \infty$ gives $d_{cu}(x, x'') = 0$ and hence $x'' = x$ since $x'' \in W^{cu}_{\kappa\varepsilon}(x)$. Now by (7.10) we have $g^nx \in W^{cs}_{\kappa\varepsilon}(g^nx')$ for all $n \in \mathbb{Z}$, and for each $k \in \mathbb{N}$ we can apply (7.9) with y a point along the W^{cs}-geodesic from $g^{-m'_k}x$ to $g^{-m'_k}x'$, obtaining

$$d_{cs}(x,x') \leq \theta_{r'}^{m'_k} d_{cs}(g^{-m'_k}x, g^{-m'_k}x') \leq \theta_{r'}^{m'_k}\kappa\varepsilon.$$

Again, as k increases we get $d_{cs}(x,x') = 0$ hence $x' = x$, which completes the proof of Lemma 7.12. $\square$

7.4. **Verification of Theorem 6.1.** We have now done all the work to show that if $g \in \mathcal{V}_0$ and $\varphi\colon \mathbb{T}^4 \to \mathbb{R}$ satisfy the hypotheses of Theorem 6.1, then the conditions of Theorem 2.9 are satisfied, and hence there is a unique equilibrium state for $(\mathbb{T}^d, g, \varphi)$. We recall how this is done; this will complete the proof of Theorem 6.1. We define the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ as in Lemma 6.6. The following facts are shown in the previous sections.

- $\mathcal{G}^M$ has tail specification at scale $3\rho'$ (Lemma 7.7), so condition (1) of Theorem 2.9 holds.
- φ has the Bowen property on $\mathcal{G}$ at scale $300\rho'$ (Lemma 7.9), so condition (2) of Theorem 2.9 holds.
- $P(\mathcal{P}\cup\mathcal{S}, \varphi, 5\eta) = \max\{P(\mathcal{P}, \varphi, 5\eta), P(\mathcal{S}, \varphi, 5\eta)\}$ and both collections satisfy the hypotheses of Theorems 3.3 and 10.1, and thus we have the upper bound

  $$(1-r)\sup_{x\in Q}\varphi(x) + r(\sup_{x\in X}\varphi(x) + \log L + h - \log(r)),$$

  and r can be chosen arbitrarily close to γ.
- $h_g^*(5\eta) < 6\log\lambda_c$ (Lemma 7.11), so by Theorem 3.3, $P(\mathcal{P}\cup\mathcal{S}, \varphi)$ is bounded above by

  $$6\log\lambda_c + (1-r)\sup_{x\in Q}\varphi(x) + r(\sup_{x\in X}\varphi(x) + \log L + h - \log(r))$$

- Thus, the assumption (6.8) gives that

  $$P(\mathcal{P}\cup\mathcal{S}, \varphi) + \operatorname{Var}(\varphi, 300\rho') < P(g, \varphi),$$

  which verified condition (3) of Theorem 2.9.
- Expansivity at scale $300\rho'$: by Theorem 3.4 and Lemma 5.11, $P_{\exp}^{\perp}(\varphi, 300\rho') \leq P(\mathcal{P}\cup\mathcal{S}, \varphi) < P(g, \varphi)$.

Putting these ingredients together, we see that under the conditions of Theorem 6.1, all the hypotheses of Theorem 2.9 are satisfied for the decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$. This completes the proof of Theorem 6.1.

8. SRB measures

Following the definition in [3, Chapter 13], an *SRB measure* for a C^2 diffeomorphism f is an ergodic invariant measure μ that is hyperbolic (non-zero Lyapunov exponents) and has absolutely continuous conditional measures on unstable manifolds. For uniformly hyperbolic systems, an SRB measure can be obtained as the unique equilibrium state for the *geometric potential* $\varphi(x) = -\log J^u_f(x) = -\log|\det Df(x)|_{E^u(x)}|$ (see [60] for a nice overview). In this section, we prove similar results for the Mañé and Bonatti–Viana examples. The existence of a unique SRB measure for the Mañé and Bonatti-Viana examples follows from [6, 1]. However, Theorems 8.2 and 8.4 below are the first results that characterize the SRB measure for these examples as a unique equilibrium state. Note that this yields an independent proof of the uniqueness of the SRB measure as a corollary.

8.1. **Preliminaries.**

Given a C^2 diffeomorphism g on a d-dimensional manifold and $\mu \in \mathcal{M}_e(g)$, let $\lambda_1 < \cdots < \lambda_s$ be the Lyapunov exponents of μ, and let d_i be the multiplicity of λ_i, so that $d_i = \dim E_i$, where for a Lyapunov regular point x for μ we have

$$E_i(x) = \{0\} \cup \{v \in T_xM : \lim_{n\to\pm\infty} \tfrac{1}{n} \log \|Dg^n_x(v)\| = \lambda_i\} \subset T_xM.$$

Let $k = k(\mu) = \max\{1 \leq i \leq s(\mu) : \lambda_i \leq 0\}$, and let $\lambda^+(\mu) = \sum_{i>k} d_i(\mu)\lambda_i(\mu)$ be the sum of the positive Lyapunov exponents, counted with multiplicity.

The Margulis–Ruelle inequality [3, Theorem 10.2.1] gives $h_\mu(g) \leq \lambda^+(\mu)$, and it was shown by Ledrappier and Young [37] that equality holds if and only if μ has absolutely continuous conditionals on unstable manifolds. In particular, we see that for any ergodic invariant measure μ, we have

$$h_\mu(g) - \lambda^+(\mu) \leq 0, \tag{8.1}$$

with equality if and only if μ is absolutely continuous on unstable manifolds. Thus an ergodic measure μ is an SRB measure if and only if it is hyperbolic and equality holds in (8.1).

First we prove a general result on non-negativity of pressure for the geometric potential associated to a foliation. Let M be a compact Riemannian manifold and let W be a C^0 foliation of M with C^1 leaves. Suppose that there is $\delta > 0$ such that

$$\sup_{x\in M} m_{W(x)}(W_\delta(x)) < \infty, \tag{8.2}$$

where $m_{W(x)}$ denotes volume on the leaf $W(x)$ with the induced metric.

Lemma 8.1. *Let W be a foliation of M as above, with $\delta > 0$ such that* (8.2) *holds. Let $f\colon M \to M$ be a diffeomorphism and let $\psi(x) = -\log|\det Df(x)|_{T_xW(x)}|$. Then $P(f,\psi) \geq 0$.*

Proof. Note that ψ is continuous because f is C^1 and W is C^0. Thus for every $\varepsilon > 0$, there is $\delta > 0$ such that $d(x,y) < \delta$ implies

$$|\psi(x) - \psi(y)| < \varepsilon. \tag{8.3}$$

Decreasing δ if necessary, we can assume that (8.2) holds. Now for every $x \in M$ and every $y \in B_n(x,\delta)$, we have

$$|\det Df^n(y)|_{T_yW(y)}| \geq e^{-\varepsilon n} e^{-S_n\psi(x)}. \tag{8.4}$$

Writing $B_n^W(x,\delta)$ for the connected component of $W(x) \cap B_n(x,\delta)$ containing x, we get

$$m_{W(f^n x)}(f^n B_n^W(x,\delta)) \geq e^{-\varepsilon n} e^{-S_n\psi(x)} m_{W(x)} B_n^W(x,\delta).$$

Since $f^n B_n^W(x,\delta) \subset W_\delta(f^n x)$, we write C for the quantity in (8.2) and get

$$m_{W(x)} B_n^W(x,\delta) \leq C e^{\varepsilon n} e^{S_n\psi(x)} \tag{8.5}$$

for every x, n.

Now let V be a local leaf of W. Given $n \in \mathbb{N}$, let Z_n be a maximal (n,δ)-separated subset of V. Then $V \subset \bigcup_{x\in Z_n} B_n^W(x,\delta)$, and so (8.5) gives

$$m_V(V) \leq \sum_{x\in Z_n} m_V B_n^W(x,\delta) \leq \sum_{x\in Z_n} C e^{\varepsilon n} e^{S_n\psi(x)} \leq C e^{\varepsilon n} \Lambda_n(\psi,\delta).$$

We conclude that $P(f,\psi) \geq P(f,\psi,\delta) \geq -\varepsilon$, and since $\varepsilon > 0$ was arbitrary this shows that $P(f,\psi) \geq 0$. $\square$

We note that (8.2) holds for the unstable foliation W^u of the Mañé family and the center-unstable foliation W^{cu} of the Bonatti–Viana foliation, because both lie within cones of width β around linear foliations, and so $2\varepsilon(1+\beta)$ and $2\varepsilon(1+\beta)^2$ give the appropriate upper bounds. Thus Lemma 8.1 applies to both families of examples.

8.2. **SRB for Mañé.** Given $g \in \mathcal{U}_0$ and q, ρ, γ, h, L as in §4.1, let

$$J_g^u(x) := |\det Dg(x)|_{E^u(x)}|,$$

and $\varphi^{\text{geo}} = -\log J_g^u$. Note that φ^{geo} is Hölder continuous because the map g is C^2 and the distribution E^u is Hölder. The Hölder continuity of E^u follows from the argument of §6.1 of Brin and Stuck [13]. (The result there is stated for uniformly hyperbolic diffeomorphisms, but the argument extends unproblematically to the case of absolute partial hyperbolicity, which covers our setting.)

In this section, we build up a proof of the following theorem.

Theorem 8.2. *Let g be a C^2 diffeomorphism in $\mathcal{U}_0$ such that*

$$\gamma(h + \log L - \log \gamma) < \left(\frac{\sup_{x \in \mathbb{T}^d} \varphi^{geo}(x)}{\inf_{x \in \mathbb{T}^d} \varphi^{geo}(x)} \right) h. \tag{8.6}$$

Then the following are true for the geometric potential $\varphi^{geo} = -\log J^u_g$.

(1) $t = 1$ is the unique root of the function $t \mapsto P(t\varphi^{geo})$.
(2) There is $\xi > 0$ such that $t\varphi^{geo}$ has a unique equilibrium state μ_t for each $t \in (-\xi, 1 + \xi)$.
(3) μ_1 is the unique SRB measure for g.

Note that since E^u is uniformly expanding, we have $\sup \varphi^{\text{geo}} < 0$, hence the ratio in (8.6) is less than 1. We can carry out the Mañé construction to make this ratio as close to 1 as we like, and then (8.6) is satisfied as long as γ, λ_c are chosen sufficiently close to 0 and 1. In particular, (8.6) is satisfied for a non-empty C^1-open set of diffeomorphisms in $\mathcal{U}_0$.

8.2.1. *Proof of uniqueness.* Note that because $\sup \varphi^{\text{geo}} < 0$, the function $t \mapsto P(t\varphi^{\text{geo}})$ is a convex strictly decreasing function from $\mathbb{R} \to \mathbb{R}$, so it has a unique root. It remains to show that this root occurs at $t = 1$, that we have uniqueness of the equilibrium state for all t in a neighborhood of $[0, 1]$, and that μ_1 is the unique SRB measure.

We prove the uniqueness results first; in §8.2.2 we will prove that $P(\varphi^{\text{geo}}) = 0$ and that μ_1 is the unique SRB measure.

To get a unique equilibrium state for $t\varphi^{\text{geo}}$, it suffices to show that the quantity

$$P^*(g, t) = (1 - \gamma) \sup_{B(q,\rho)} t\varphi^{\text{geo}} + \gamma(\sup_{\mathbb{T}^d} t\varphi^{\text{geo}} + h + \log L - \log \gamma) \tag{8.7}$$

from (4.4) satisfies $P^*(g, t) < P(t\varphi^{\text{geo}})$ for all $t \in [0, 1]$ and then apply Theorem 4.1. Note that since the equality is strict it will then continue to hold for all t in a neighborhood of $[0, 1]$.

Let $\lambda_u = \min_x \|Dg(x)|_{E^u}\|$ and $\nu_u = \max_x \|Dg(x)|_{E^u}\|$; then the ratio in (8.6) is $\frac{\log \lambda_u}{\log \nu_u}$. Note that for all $t \in [0, 1]$ we have

$$P^*(g, t) \le -t \log \lambda_u + \gamma(h + \log L - \gamma), \tag{8.8}$$

as illustrated in Figure 6.

At $t = 0$ we have $P(t\varphi^{\text{geo}}) = h$, and since $\inf \varphi^{\text{geo}} \ge -\log \nu_u$ we have

$$P(t\varphi^{\text{geo}}) \ge \max(h - t \log \nu_u, 0),$$

where we use the fact that $P(t\varphi^{\text{geo}})$ is nonincreasing in t and that $P(\varphi^{\text{geo}}) \ge 0$ by Lemma 8.1.

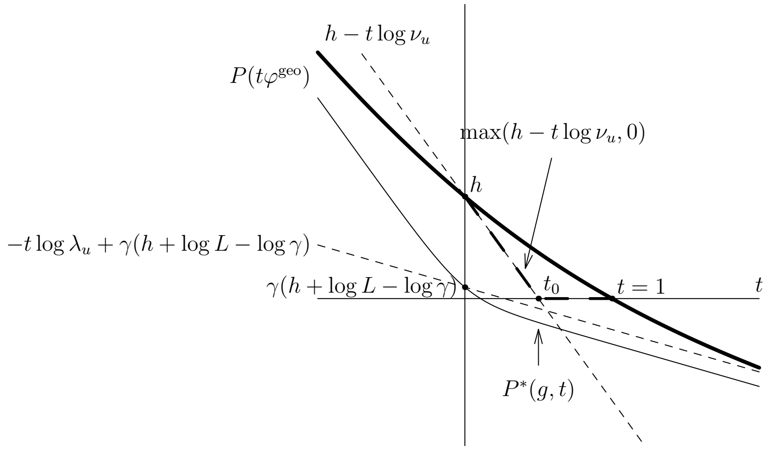

FIGURE 6. Estimating $P^*(g,t)$ and $P(t\varphi^{\rm geo})$.

Let $t_0 = h/\log\nu_u$. It suffices to show that

$$-t\log\lambda_u + \gamma(h + \log L - \log\gamma) < h - t\log\nu_u \tag{8.9}$$

for all $t \in [0, t_0]$, since the case $t = t_0$ implies that $P^*(g,t) < 0$ for all $t \in [t_0, 1]$. To see (8.9), we observe that (8.6) gives

$$\begin{aligned}-t\log\lambda_u + \gamma(h + \log L - \log\gamma) &< -t\log\lambda_u + \frac{\log\lambda_u}{\log\nu_u}h \\ &= \frac{\log\lambda_u}{\log\nu_u}(h - t\log\nu_u) < P(t\varphi^{\rm geo}).\end{aligned}$$

By (8.8), we have $P^*(g,t) < P(t\varphi^{\rm geo})$, so Theorem 4.1 gives the uniqueness part of Theorem 8.2.

8.2.2. *Lyapunov exponents for the Mañé family.* It remains to show that $P(g, \varphi^{\rm geo}) = 0$ and that the unique equilibrium state is in fact the unique SRB measure. We achieve this by studying the Lyapunov exponents of g and using the characterization of the SRB measure given at (8.1). Recall that $P(\varphi^{\rm geo}) \geq 0$ by Lemma 8.1, so we focus our attention on the upper bound.

Let μ be ergodic, and let $\{(\lambda_i(\mu), d_i(\mu)) : 1 \leq i \leq s\}$ be the Lyapunov spectrum of μ. Recall that $E^{cs} \oplus E^u$ is Dg-invariant, so for every μ-regular x the Oseledecs decomposition is a sub-splitting of $E^{cs} \oplus E^u$, and thus $d_s(\mu) = 1$. Note that it immediately follows that

$$\int \varphi^{\rm geo}\, d\mu = -\lambda_s(\mu).$$

Thus,

$$\int \varphi^{\text{geo}}\, d\mu \geq -\lambda^+(\mu) \tag{8.10}$$

and if $\lambda_{s-1}(\mu) < 0$ it immediately follows that $\int \varphi^{\text{geo}}\, d\mu = -\lambda^+(\mu)$. Let $\mathcal{M}_* \subset \mathcal{M}_e(g)$ be the set of ergodic μ such that μ is hyperbolic and $\lambda_{s-1}(\mu) < 0$.

Lemma 8.3. *If $\mu \in \mathcal{M}_e(g) \setminus \mathcal{M}_*$, then*

$$h_\mu(g) - \lambda^+(\mu) \leq P^*(\varphi).$$

Proof. If $\mu \in \mathcal{M}_e(g) \setminus \mathcal{M}_*$, then either μ is not hyperbolic, or $\lambda_{s-1}(\mu) > 0$. Then there exists a set $Z \subset M$ with $\mu(Z) = 1$ so that for each $z \in Z$, there exists $v \in E^{cs}_z$ with

$$\lim_{n\to\infty} \tfrac{1}{n} \log \|Dg^n_z(v)\| \geq 0.$$

Taking $r > \gamma$ such that $\theta_r < 1$, we see that each $z \in Z$ is contained in the set

$$A^+ = \{x : \text{there exists } K(x) \text{ so } \tfrac{1}{n} S^g_n \chi(x) < r \text{ for all } n > K(x)\},$$

where we recall that χ is the characteristic function of the neighborhood of the perturbation. To see this, suppose that $z \notin A^+$. Then there exists $n_k \to \infty$ with $\frac{1}{n_k} S^g_{n_k} \chi(z) \geq \gamma$. By lemma 5.5, this gives

$$\|Dg^n_z(v)\| \leq \|Dg^{n_k}|_{E^{cs}(z)}\| < (\theta_r)^{n_k},$$

and thus

$$\lim_{n_k\to\infty} \tfrac{1}{n_k} \log \|Dg^{n_k}_z(v)\| \leq \log \theta_r < 0,$$

which is a contradiction. Thus, $\mu(A^+) = 1$. It follows that

$$h_\mu(g) - \lambda^+(\mu) \leq h_\mu(g) + \int \varphi^{\text{geo}}\, d\mu \leq P(\mathcal{C}, \varphi^{\text{geo}}) \leq P^*(g, 1).$$

where the first inequality uses (8.10), the second uses Theorem 10.3, and the third uses Theorem 3.3. □

It follows from Lemma 8.3 and Lemma 8.1 that

$$P(g, \varphi^{\text{geo}}) = \sup \left\{ h_\mu(g) + \int \varphi^{\text{geo}}\, d\mu \,:\, \mu \in \mathcal{M}_* \right\}. \tag{8.11}$$

Now, for every $\mu \in \mathcal{M}_*$, we have $\int \varphi^{\text{geo}}\, d\mu = -\lambda^+(\mu)$, and thus

$$h_\mu(g) + \int \varphi^{\text{geo}}\, d\mu = h_\mu(g) - \lambda^+(\mu) \leq 0 \tag{8.12}$$

by (8.1). Together with (8.11) this gives $P(g, \varphi^{\text{geo}}) \leq 0$, and we conclude that $P(g, \varphi^{\text{geo}}) = 0$.

To complete the proof of Theorem 8.2 it only remains to show that the unique equilibrium state μ_1 is in fact an SRB measure for g, and that there are no other SRB measures. For the first claim, we observe that $\mu_1 \in \mathcal{M}_*$ implies that μ_1 is hyperbolic, and since $P(g, \varphi^{\mathrm{geo}}) = 0$, (8.12) gives $h_{\mu_1}(g) - \lambda^+(\mu_1) = 0$, so μ_1 is an SRB measure.

To see that there is no other SRB measure, we observe that if $\nu \neq \mu_1$ is any ergodic measure, then $h_\nu(g) - \lambda^+(\nu) \leq h_\nu(g) - \int \varphi^{\mathrm{geo}}\, d\nu < P(g, \varphi^{\mathrm{geo}}) = 0$ by (8.10) and the uniqueness of μ_1 as an equilibrium measure. This completes the proof of Theorem 8.2.

8.3. **SRB measures for the Bonatti-Viana family.** We follow the same basic strategy that we followed for the Mañé family to show that the SRB measure is the unique equilibrium measure for a suitable geometric potential. It is a folklore result that a C^2 diffeomorphism with a dominated splitting has Hölder continuous distributions, but no proof, or even statement, of this fact is available in the literature to the best of our knowledge[2]. We sidestep the issue of Hölder continuity of the distributions by presenting a direct proof that the geometric potential $\varphi^{\mathrm{geo}} := -\log J^{cu}_g$ has the Bowen property on $\mathcal{G}$. An advantage of this approach is that it is suitable for generalization to non-uniformly hyperbolic settings. The main idea is Lemma 8.6 below, which gives contraction estimates analogous to Lemma 7.8 for the action of Dg on the Grassmannian.

8.3.1. *Additional control on the construction of g.* In order for our estimates on Dg to apply, we must specify the construction of g more carefully than in previous sections to obtain more refined information on how the unstable foliation for g sits in the cone around the linear foliation for f_B.

Let $\lambda_1 < \lambda_2 < \frac{1}{3} < 3 < \lambda_3 < \lambda_4$ be the eigenvalues of the toral automorphism f_B in the Bonatti–Viana construction, and let F^u and F^s be the unstable and stable eigenspaces for f_B.

Fixing F^u and F^s as above, we can represent a linear map $A\colon \mathbb{R}^4 \to \mathbb{R}^4$ in the form

$$\begin{bmatrix} A^{ss} & A^{su} \\ A^{us} & A^{uu} \end{bmatrix}$$

[2]For diffeomorphisms of surfaces, this result is given in [52]. The idea of proof for the general folklore result is to modify the C^r section theorem from Hirsch, Pugh and Shub [33].

so that for $\sigma_1, \sigma_2 \in \{s, u\}$, $A^{\sigma_1\sigma_2} \colon F^{\sigma_1} \to F^{\sigma_2}$ is such that $A|_{F^{\sigma_1}} = A^{\sigma_1 s} + A^{\sigma_2 u}$. Given $\lambda, \xi, K > 0$, let

$$\mathcal{L}_{F^s,F^u}(\lambda, \xi, K) = \{A \colon \mathbb{R}^4 \to \mathbb{R}^4 \text{ linear} \mid \|(A^{uu})^{-1}\|, \|A^{ss}\| < \lambda, \\ \|A^{us}\|, \|A^{su}\| < \xi, \|A^{uu}\| < K\}.$$

Note that $B \in \mathcal{L}_{F^s,F^u}(\max(\lambda_2, \lambda_3^{-1}), 0, \lambda_4)$, where B is the linear map for the original toral automorphism. We return to the construction of the Bonatti-Viana diffeomorphism f_1. For a fixed λ_h and λ_p with $\lambda_h < 1 < \lambda_p$ and $\beta, \xi, K > 0$, to be specified shortly, we can carry out the construction so that in addition to the condition that $E^{cs} \subset C_\beta(F^s, F^u)$ and $E^{cu} \subset C_\beta(F^u, F^s)$, we impose the additional constraints:

- $Df_1(x) \in \mathcal{L}_{F^s,F^u}(\lambda_h, \xi, K)$ when $x \notin Q$;
- $Df_1(x) \in \mathcal{L}_{F^s,F^u}(\lambda_p, \xi, K)$ when $x \in Q$.

The interpretation of these constants is that λ_h gives uniform contraction/expansion estimates outside the perturbation where the map is hyperbolic, while λ_p controls the dynamics inside the perturbation. We now take a C^1-neighborhood $\mathcal{V}_1$ of f_1 such that the following is true for every $g \in \mathcal{V}_1$:

- $d_{C^0}(g, f_B) < \eta/C$.
- g has a dominated splitting $T\mathbb{T}^4 = E^{cs}_g \oplus E^{cu}_g$, with E^{cs}_g, E^{cu}_g tangent to $C_\beta(F^s, F^u)$ and $C_\beta(F^u, F^s)$ respectively.
- The distributions E^{cs}_g, E^{cu}_g integrate to foliations W^{cs}_g, W^{cu}_g.
- Each of the leaves $W^{cs}_g(x)$ and $W^{cu}_g(x)$ is dense for every $x \in \mathbb{T}^4$.
- $Dg(x) \in \mathcal{L}_{F^s,F^u}(\lambda_h, \xi, K)$ when $x \notin Q$;
- $Dg(x) \in \mathcal{L}_{F^s,F^u}(\lambda_p, \xi, K)$ when $x \in Q$;
- Properties (6.3)–(6.5) hold with r as in (8.13) below.

The first four conditions are the same as those that determine $\mathcal{V}_0$ in §6; the fifth and sixth conditions clearly persist under C^1 perturbation of f_1, and the seventh condition is verified below, so $\mathcal{V}_1$ is non-empty.

We now discuss how the constants $\lambda_h, \lambda_p, \beta, \xi, K$ should be chosen in the construction. We choose λ_h, λ_p and $r > 0$ such that $\max(\lambda_2, \lambda_3^{-1}) < \lambda_h < 1 < \lambda_p$ and

$$r(\log L + h - \log r) + 6 \log \lambda_p + h \left(\frac{\log(\lambda_3 \lambda_p^2) - \log \lambda_4}{\log \lambda_3 + \log \lambda_4} \right) < 0, \tag{8.13}$$

and moreover $\lambda_h^r \lambda_p^{1-r} < 1$. We let $K = 2\lambda_4$. Writing

$$\nu(\lambda, \xi) = \lambda^2 \left(\frac{1}{1 - \xi\beta\lambda} + \frac{\xi(\xi + 2\beta\lambda_4)}{(1 - \xi\beta\lambda)^2} \right) \tag{8.14}$$

and $\nu_\sigma = \nu(\lambda_\sigma, \xi)$ for $\sigma = h, p$, we take ξ, β small enough such that

$$\nu_h^r \nu_p^{1-r} < 1. \tag{8.15}$$

Note that (8.13) and (8.15) can be satisfied by taking λ_p close enough to 1 and r, ξ, β sufficiently small, and the Bonatti–Viana construction can be carried out for any choice of the parameters λ_p, ξ, β.

Note that the quantities $\lambda_{cs}, \lambda_{cu}, \lambda_s, \lambda_u$ from (6.2) can be estimated in terms of λ_h, λ_p, and as $\xi, \beta \to 0$ we get $\lambda_{cs}, \lambda_{cu}^{-1} \to \lambda_p$ and $\lambda_s, \lambda_u^{-1} \to \lambda_h$. In particular, given $r > 0$ with $\lambda_h^r \lambda_p^{1-r} < 1$, we can choose ξ, β sufficiently small that every $g \in \mathcal{V}_1$ satisfies (6.3)–(6.5).

Finally, we observe that we can choose $\mathcal{V}_1$ such that $\sup_{x\in\mathbb{T}^4} \varphi^{\mathrm{geo}}(x) \approx \log \lambda_c - \log \lambda_4$ and $\inf_{x\in\mathbb{T}^4} \varphi^{\mathrm{geo}}(x) \approx -(\log \lambda_3 + \lambda_4)$, so that

$$\sup \varphi^{\mathrm{geo}} + \mathrm{Var}(\varphi^{\mathrm{geo}}, 300\rho') \le 2 \sup \varphi^{\mathrm{geo}} - \inf \varphi^{\mathrm{geo}} \approx 2 \log \lambda_c + \log \lambda_3 - \log \lambda_4,$$

and from (8.13) we can assume that for every $g \in \mathcal{V}_1$ we have

$$r(\log L + h - \log r) + 6 \log \lambda_c < h \left(\frac{2 \sup \varphi^{\mathrm{geo}} - \inf \varphi^{\mathrm{geo}}}{\inf \varphi^{\mathrm{geo}}} \right). \tag{8.16}$$

Note the similarity in form between (8.16) and (8.6).

Theorem 8.4. *Given any C^2 diffeomorphism $g \in \mathcal{V}_1$, the following are true for the geometric potential $\varphi^{geo} = -\log |\det Dg(x)|_{E^{cu}}|$:*

(1) $t = 1$ is the unique root of the function $t \mapsto P(t\varphi^{geo})$.
(2) There is $\xi > 0$ such that $t\varphi^{geo}$ has a unique equilibrium state μ_t for each $t \in (-\xi, 1 + \xi)$.
(3) μ_1 is the unique SRB measure for g.

The remainder of this section is devoted to the proof of Theorem 8.4. The only difference between this setting and the setting of Theorem 6.1 is that here we do not know that $t\varphi^{\mathrm{geo}}$ is Hölder. Thus we must prove directly that φ^{geo} (and hence $t\varphi^{\mathrm{geo}}$) has the Bowen property on $\mathcal{G}$, where $\mathcal{G}$ is as in §6. We do this in §8.3.2 below. Then in §7.4 we prove the first and third claims in Theorem 8.4.

Here we show that the pressure bound (6.8) holds for $t\varphi^{\mathrm{geo}}$ for all $t \in [0, 1]$. Indeed, using the fact that (8.6) and (8.16) have the same structure, we see that the same argument as in the proof of Theorem 8.2 immediately gives

$$\begin{aligned} P^*(g, t) := (2 \sup - \inf)(t\varphi^{\mathrm{geo}}) + 6 \log \lambda_c + r(\log L + h - \log r) \\ < \max(h - t \inf \varphi^{\mathrm{geo}}, 0) \le P(t\varphi^{\mathrm{geo}}). \end{aligned} \tag{8.17}$$

Moreover, the quantity in (6.8) is bounded above by $P^*(g, t)$, so we have the necessary pressure gap.

8.3.2. *The Bowen property for the geometric potential.* The usual approach to the geometric potential in the uniformly hyperbolic case is to argue that the unstable distribution is Hölder continuous (i.e. the section $x \mapsto E^u(x)$ is Hölder continuous), and thus the map from $\varphi^{\text{geo}}(x) = -\log J^u(x)$ is Hölder. This approach is captured on the following commutative diagram:

$$\begin{tikzcd} & G \arrow[dr, "\psi"] & \\ M \arrow[ur, "E^u"] \arrow[rr, "\varphi^{\text{geo}}"'] & & \mathbb{R} \end{tikzcd}$$

where G is the appropriate Grassmannian bundle over M, and ψ sends $W \in G$ to $-\log|\det dg(x)|_W|$. Note that all we need for ψ to be Hölder continuous is for the map g to be $C^{1+\alpha}$. Thus, the question of regularity of φ^{geo} reduces to the question of regularity for $E^u \colon M \to G$.

In our setting, where $\varphi^{\text{geo}}(x) = -\log J^{cu}(x)$, we obtain refined estimates on $E^{cu} \colon \mathbb{T}^4 \to G$ for good orbit segments, which allow us to establish the Bowen property on these segments.

More precisely, we let G_2 denote the Grassmannian bundle of 2-planes in $\mathbb{R}^4$ over the torus. Since the underlying manifold is the torus, this is a product bundle, and we can identify G_2 with $\mathbb{T}^4 \times \mathrm{Gr}(2, \mathbb{R}^4)$, where $\mathrm{Gr}(2, \mathbb{R}^4)$ is the space of planes through the origin in $\mathbb{R}^4$. The map g induces dynamics on G_2 by the formula

$$(x, V) \to (g(x), Dg(V)).$$

We consider the subset G_2^β of G_2 which corresponds to the space of 2-dimensional subspaces of $\mathbb{R}^4$ lying in C_β. We introduce some convenient formalism to make this notion precise. Let U_β be the set of linear maps $L \colon F^u \to F^s$ such that $\|L\| < \beta$. Now let $G_2^\beta = \mathbb{T}^4 \times U_\beta$, so that $(x, L) \in G_2^\beta$ can be identified with the 2-dimensional subspace of $T_x\mathbb{T}^4$ given by the graph of L:

$$\Gamma(x, L) := (\mathrm{Id} + L)F^u = \{v + Lv \in T_x\mathbb{T}^4 \mid v \in F^u\}.$$

Note that for every $x \in \mathbb{T}^4$ we have $E^{cu}(x) = \Gamma(x, L)$ for some $L \in U_\beta$, so we can view E^{cu} as a map from $\mathbb{T}^4$ to G_2^β. Equip G_2^β with the metric

$$d((x, L), (x', L')) = d(x, x') + \|L - L'\|.$$

Now consider the map $\psi\colon G_2^\beta \to \mathbb{R}$ given by $\psi(x,L) = |\det dg(x)|_{\Gamma(x,L)}|$. Then $\varphi^{\text{geo}} = \psi \circ E^{cu}$; we have the following commutative diagram:

$$\begin{tikzcd} & G_2^\beta \arrow[dr, "\psi"] & \\ \mathbb{T}^4 \arrow[ur, "E^{cu}"] \arrow[rr, "\varphi^{\text{geo}}"'] & & \mathbb{R} \end{tikzcd}$$

Our strategy is to carry out the argument of Lemma 7.9 with respect to the induced dynamics on G_2^β. In light of the above discussion, we need the following two results.

Lemma 8.5. *The function ψ is Lipschitz.*

Lemma 8.6. *Given $\theta > \max(\nu_h^r \nu_p^{1-r}, \theta_r)$, there is $C \in \mathbb{R}$ such that for every $(x,n) \in \mathcal{G}$, $y \in B_{300\rho'}(x,n)$, and $0 \le k \le n$, we have*

$$d(E^{cu}(g^k x), E^{cu}(g^k y)) \le C(\theta^k + \theta^{n-k}).$$

In Lemma 8.6 we identify a 2-dimensional subspace $E \subset T_x\mathbb{T}^4$ with the pair $(x,L) \in G_2^\beta$ such that $E = \Gamma(x,L)$.

We start by proving Lemma 8.5, which is elementary; then we prove Lemma 8.6, which requires some linear algebra estimates that we state as Lemma 8.7 below (a proof of these is given in §11).

Proof of Lemma 8.5. Given $v,w \in \mathbb{R}^4$, let $A(v,w)$ denote the square of the area of the parallelogram spanned by v,w. Note that $A(v,w)$ is a smooth function of v,w since one can show that $A(v,w) = \sum_{i<j}(v_i w_j - w_i v_j)^2$.

Let $v_1, v_2 \in \mathbb{R}^4$ be a basis for L^u. Given $(x,L) \in G_2^\beta$, we have

$$\psi(x,L) = \frac{1}{2}\log\left|\frac{A(Dg(x)(\mathrm{Id}+L)v_1, Dg(x)(\mathrm{Id}+L)v_2)}{A((\mathrm{Id}+L)v_1, (\mathrm{Id}+L)v_2)}\right|.$$

Since the ratio is bounded away from 0 on G_2^β, we see that ψ is C^1, since each of A, Dg, and $\mathrm{Id}+L$ is C^1. In particular, it is Lipschitz. $\square$

Proof of Lemma 8.6. Given $(x,n) \in \mathcal{G}$ and $y \in B_{300\rho'}(x,n)$, we use the local product structure to choose $z \in W^{cs}_{300\rho'\kappa}(x) \cap W^{cu}_{300\rho'\kappa}(y)$. Then

$$d(E^{cu}(g^k x), E^{cu}(g^k y)) \le d(E^{cu}(g^k x), E^{cu}(g^k z)) + d(E^{cu}(g^k z), E^{cu}(g^k y)).$$

Let $\bar\lambda_j = \lambda_c$ when $g^j x \in Q$ (inside the perturbation) and $\bar\lambda_j = \max(\lambda_s, \lambda_u^{-1})$ when $g^j x \notin Q$. Following the computations in the proof of Lemma 7.6, we have

$$(8.18)\qquad \begin{aligned} d(g^k x, g^k z) &\le 300\rho'\kappa \prod_{j=0}^{k-1} \bar\lambda_j \le \theta^k, \\ d(g^k z, g^k y) &\le 300\rho'\kappa \prod_{j=k+1}^{n} \bar\lambda_j \le \theta^{n-k}. \end{aligned}$$

Since unstable manifolds are C^1, we get

$$d(E^{cu}(g^k z), E^{cu}(g^k y) \le C d(g^k z, g^k y) \le C300\rho'\kappa\theta^{n-k}$$

for some uniform constant C, and so to prove Lemma 8.6 it suffices to get a constant C' such that

$$(8.19)\qquad d(E^{cu}(g^k x), E^{cu}(g^k z)) \le C'\theta^k.$$

We rely on a linear algebra estimate, which is proved in §11.5. Given a linear map $A\colon \mathbb{R}^4 \to \mathbb{R}^4$ with $A(C_\beta) \subset C_\beta$, let $\hat A$ denote the induced map on U_β; that is, the graph of $\hat A(L)$ is the image under A of the graph of L. A formula for $\hat A$ is derived in §11.5.

Lemma 8.7. *Given $A \in \mathcal{L}_{F^s,F^u}(\lambda, \xi, 2\lambda)$ and $L, L' \in U_\beta$, we have $\|\hat A L - \hat A L'\| \le \nu(\lambda, \xi)\|L - L'\|$.*

For $0 \le k \le n$, let $\bar\nu_k = \max(\lambda_c, \nu_p)$ if $g^k x \in Q$, and $\bar\nu_k = \max(\lambda_s, \nu_h)$ if $g^k x \notin Q$.

Let $A_k\colon U_\beta \to U_\beta$ be the map induced by $Dg(g^k x)$, and $B_k\colon U_\beta \to U_\beta$ the map induced by $Dg(g^k z)$. By Lemma 8.7, A_k, B_k are $\bar\nu_k$-contractions on U_β (that is, they are Lipschitz with constant $\bar\nu_k$). Moreover, since g is C^2 there is a constant C' depending only on g such that

$$(8.20)\qquad d_{C^0}(A_k, B_k) \le C' d(g^k x, g^k z) \le C' 300\rho\kappa' \prod_{j=0}^{k-1} \bar\nu_j,$$

where the second inequality uses (8.18). Thus we can apply the following lemma, which is proved in §11.5.

Lemma 8.8. *Let (D, d) be a metric space. For each $j \ge 0$ let $F_j, G_j\colon D \to D$ be Lipschitz maps. Suppose we have a constant $C > 0$ and a sequence $\lambda_0, \lambda_1, \dots \ge \lambda_{\min} > 0$ such that for each $k \ge 0$*

(1) $d_{C^0}(F_k, G_k) \le C \prod_{j-0}^{k-1} \lambda_j$;
(2) λ_k is a Lipschitz constant for F_k and G_k.

Let $\xi, \eta \in D$ and set $\xi_k = F_{k-1} \circ \cdots \circ F_0(\xi)$ and $\eta_k = G_{k-1} \circ \cdots \circ G_0(\eta)$. Then we have

$$d(\xi_k, \eta_k) \leq \left(d(\xi, \eta) + \frac{Ck}{\lambda_{\min}} \right) \prod_{j=0}^{k-1} \lambda_j. \tag{8.21}$$

Applying Lemma 8.8 to the maps $A_k, B_k \colon U_\beta \to U_\beta$, the sequence $\bar{\nu}_k$, and $\xi = E^{cu}(x)$, $\eta = E^{cu}(z)$, we obtain

$$d(E^{cu}(g^k x), E^{cu}(g^k z)) \leq \left(d(E^{cu}(x), E^{cu}(z)) + \frac{Ck}{\bar{\lambda}_h} \right) \prod_{j=0}^{k-1} \bar{\nu}_j,$$

and since $\prod_{j=0}^{k-1} \bar{\nu}_j \leq (\nu_h^r \nu_p^{1-r})^k$ by the definition of $\mathcal{G}$, the fact that $\nu_h^r \nu_p^{1-r} < \theta$ proves (8.19), which completes the proof of Lemma 8.6. □

Combining Lemmas 8.5 and 8.6 gives the Bowen property for φ^{geo} on $\mathcal{G}$, just as in Lemma 7.9.

8.3.3. *Lyapunov exponents for Bonatti-Viana examples.* It follows from Lemma 8.1 that $P(g, \varphi) \geq 0$, so to complete the proof of Theorem 8.4 we need to prove that $P(g, \varphi) \leq 0$ and that the unique equilibrium state μ obtained in the previous section has $\lambda^+(\mu) = \int \varphi^{\text{geo}}\, d\mu$, so that we have equality in (8.1).

To this end, let μ be ergodic, and let $\{(\lambda_i(\mu), d_i(\mu)) : 1 \leq i \leq s\}$ be the Lyapunov spectrum of μ, where $s \leq 4$ is the number of distinct exponents. Recall that $E^{cs} \oplus E^{cu}$ is Dg-invariant, so for every μ-regular x the Oseledets decomposition is a sub-splitting of $E^{cs} \oplus E^{cu}$. We split the possibilities into three cases:

(Case 1) If μ has exactly two positive Lyapunov exponents, or exactly one positive Lyapunov exponent with multiplicity 2, in which case

$$\int \varphi^{\text{geo}}\, d\mu = -\lambda^+(\mu);$$

(Case 2) If the Lyapunov exponents associated to E^{cu} and at least one of the Lyapunov exponents associated to E^{cs} are non-negative, in which case

$$\int \varphi^{\text{geo}}\, d\mu \geq -\lambda^+(\mu); \tag{8.22}$$

(Case 3) There is at most one positive Lyapunov exponent associated to E^{cu}. In this case, for μ-almost every x,

$$|\det Dg(x)|_{E^{cu}(x)}| \leq |\det Dg(x)|_{E^u(x)}|,$$

where $E^u(x)$ is the subspace in the Oseledecs decomposition which corresponds to the largest Lyapunov exponent, and a short calculation yields (8.22).

Let $\mathcal{M}_* \subset \mathcal{M}_e(g)$ be the set of ergodic μ such that μ is hyperbolic and belongs to Case 1) above. Let $P^*(g,t)$ be as in (8.17).

Lemma 8.9. *If $\mu \in \mathcal{M}_e(g) \setminus \mathcal{M}_*$, then*

$$h_\mu(g) - \lambda^+(\mu) \leq P^*(g,1).$$

Proof. Suppose that $\mu \in \mathcal{M}_e(g) \setminus \mathcal{M}_*$, and that either μ belongs to Case 1) and is not hyperbolic, or belongs to Case 2) above. Then there exists a set $Z \subset M$ with $\mu(Z) = 1$ so that for each $z \in Z$, there exists $v \in E^{cs}_z$ with

$$\lim_{n\to\infty} \tfrac{1}{n} \log \|Dg^n_z(v)\| \geq 0.$$

Thus with $r > \gamma$, we have $z \in A^+$, where

$$A^+ = \{x : \text{there exists } K(x) \text{ so } \tfrac{1}{n} S^g_n \chi(x) < r \text{ for all } n > K(x)\}.$$

To see this, suppose that $z \notin A^+$. Then there exists $n_k \to \infty$ with $\frac{1}{n_k} S^g_{n_k} \chi(z) \geq \gamma$. By lemma 5.5, this gives

$$\|Dg^{n_k}_z(v)\| \leq \|Dg^{n_k}|_{E^{cs}(z)}\| \leq (\theta_r)^{n_k},$$

and thus

$$\lim_{n_k\to\infty} \tfrac{1}{n_k} \log \|Dg^{n_k}_z(v)\| \leq \log \theta_r < 0,$$

which is a contradiction. Thus, $\mu(A^+) = 1$. It follows that

$$h_\mu(g) - \lambda^+(\mu) \leq h_\mu(g) + \int \varphi^{\text{geo}}\, d\mu \leq P(\mathcal{C}, \varphi^{\text{geo}}) \leq P^*(g,1),$$

where the first inequality uses (8.10), the second uses Theorem 10.3, and the third uses Theorem 3.3.

Now suppose μ belongs to case 3) above, and thus there is a non-positive exponent associated to E^{cu}. We can run a very similar argument to show that $\mu(A^-) > 0$, where

$$A^- = \{x : \text{there exists } K(x) \text{ so } \tfrac{1}{n} S^{g^{-1}}_n \chi(x) < r \text{ for all } n > K(x)\}.$$

The key point is that there exists a set $Z \subset M$ with $\mu(Z) = 1$ so that for each $z \in Z$, there exists $v \in E^{cu}_z$ with

$$\lim_{n\to-\infty} \tfrac{1}{n} \log \|Dg^{-n}_z(v)\| \geq 0.$$

It follows that $z \in A^-$, because otherwise there exists $n_k \to \infty$ with $\frac{1}{n_k} S^{g^{-1}}_{n_k} \chi(z) \geq \gamma$, and thus by lemma 5.5, we have

$$\|Dg^{-n_k}_z(v)\| \leq \|Dg^{-n_k}|_{E^{cs}(z)}\| \leq (\theta_r)^{n_k},$$

and thus

$$\lim_{n_k\to-\infty} \tfrac{1}{n_k}\log\|Dg_z^{-n_k}(v)\| \le \log\theta_r < 0,$$

which is a contradiction. Thus, $\mu(A^-)=1$. Again, it follows that

$$h_\mu(g)-\lambda^+(\mu) \le h_\mu(g)+\int \varphi^{\mathrm{geo}}\,d\mu \le P(\mathcal{C},\varphi^{\mathrm{geo}}) \le P^*(g,1).$$

where the first inequality uses (8.22), the second uses Theorem 10.3, and the third uses Theorem 3.3. □

It follows from Lemma 8.3 and Lemma 8.1 that

$$P(g,\varphi^{\mathrm{geo}}) = \sup\left\{h_\mu(g)+\int \varphi^{\mathrm{geo}}\,d\mu \,:\, \mu\in\mathcal{M}_*\right\}. \tag{8.23}$$

Now, for every $\mu\in\mathcal{M}_*$, we have $\int\varphi^{\mathrm{geo}}\,d\mu = -\lambda^+(\mu)$, and thus

$$h_\mu(g)+\int\varphi^{\mathrm{geo}}\,d\mu = h_\mu(g)-\lambda^+(\mu)\le 0. \tag{8.24}$$

Together with (8.23) this gives $P(g,\varphi^{\mathrm{geo}})\le 0$, hence $P(g,\varphi^{\mathrm{geo}})=0$. The rest of the proof of Theorem 8.4 is identical to that of Theorem 8.2.

9. Large Deviations and Multifractal Analysis

9.1. **Large deviations.** We give a precise version of our large deviations result Theorem D.

Definition 9.1. *Let m be an equilibrium measure for a potential φ (with respect to f). We say that m satisfies the* upper level-2 large deviations principle *if for any weak*-closed and convex subset A of $\mathcal{M}_f(X)$, then*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log m(\mathcal{E}_n^{-1}(A)) \le \sup_{\nu\in A\cap\mathcal{M}_f(X)}\left(h_\nu(f)+\int\varphi\,d\nu - P(\varphi)\right), \tag{9.1}$$

where $\mathcal{E}_n = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}$ is the nth empirical measure associated to x.

By the contraction principle [28], this implies the upper inequality of the level-1 large deviations principle for any continuous observable $\psi\colon X\to\mathbb{R}$. That is, for any $\varepsilon>0$, (9.1) implies

$$\limsup_{n\to\infty}\frac{1}{n}\log m\left\{x : \left|\frac{1}{n}S_n\psi(x)-\int\psi\,dm\right|\ge\varepsilon\right\}\le -q(\varepsilon),$$

where the rate function q is given by

$$q(\varepsilon) := P(\varphi) - \sup_{\substack{\nu\in\mathcal{M}_f(X)\\ \left|\int\psi\,dm - \int\psi\,d\nu\right|\geq\varepsilon}} \left(h_\nu(f) + \int \varphi\,d\nu \right). \tag{9.2}$$

It is shown in [26, Proposition 6.18] that the equilibrium measures provided by Theorem 2.8 satisfy the weak upper Gibbs property

$$m(B_n(x,\varepsilon)) \leq K e^{-nP(\varphi)+S_n\varphi(x)+n\operatorname{Var}(\varphi,\varepsilon)} \tag{9.3}$$

for every $\varepsilon > 0$; this condition immediately yields the upper large deviations bound (9.1) by applying general theory due to Pfister and Sullivan [48]. The following is proved in [26, Theorem D].

Theorem 9.2. *Suppose that the hypotheses of Theorem 2.8 are satisfied, and let μ be the unique equilibrium measure provided. Then μ satisfies the upper level-2 large deviations principle.*

By the contraction principle, μ also satisfies the upper inequality of the level-1 large deviations principle for any continuous observable ψ. The unique equilibrium measures obtained in this paper (including the SRB measures) for the Mañé family and their C^1 perturbations are obtained by verifying the hypotheses of Theorem 2.8, so Theorem 9.2 immediately applies.

For the Bonatti–Viana examples, we used Theorem 2.9, which does not allow us to take $\varepsilon \to 0$ in (9.3), and so does not immediately imply the upper large deviations principle. When $\varphi = 0$, (9.3) becomes the usual upper Gibbs bound, so the results from [48] do yield the upper level-2 large deviations principle for the measure of maximal entropy in the Bonatti–Viana family. However, the same question for equilibrium states for non-zero potentials remains open, as does the question of lower large deviations bounds in both the Mañé and Bonatti–Viana classes.

9.2. **Multifractal analysis.** Let g be a C^2 diffeomorphism in $\mathcal{U}_0$ as in §8.2, and for each $t \in [0,1]$, let μ_t be the unique equilibrium state for $t\varphi^{\text{geo}}$ as given by Theorem 8.2. In particular, μ_0 is the unique MME and μ_1 is the unique SRB measure.

Let g be a C^2 Mañé example as in §8.2, and for each $t \in [0,1]$ let μ_t be the unique equilibrium state for $t\varphi^{\text{geo}}$ as given by Theorem 8.2. In particular, μ_0 is the unique MME and μ_1 is the unique SRB measure. It follows from Lemma 5.10 that the entropy map $\mu \mapsto h_g(\mu)$ is upper semicontinuous, hence by Remark 4.3.4 of [36], uniqueness of the equilibrium state implies that the function $t \mapsto P(t\varphi)$ is differentiable on

$(-\varepsilon, 1+\varepsilon)$, with derivative $\chi^+(\mu_t)$, where we write $\chi^+(\mu) = \int \varphi \, d\mu$ for the largest Lyapunov exponent of μ.

It follows from Lemma 5.10 that the entropy map $\mu \mapsto h_g(\mu)$ is upper semicontinuous, hence uniqueness of the equilibrium state implies that the function $t \mapsto P(t\varphi)$ is differentiable on $(-\varepsilon, 1+\varepsilon)$, with derivative $\chi^+(\mu_t)$, where we write $\chi^+(\mu) = \int \varphi \, d\mu$ for the largest Lyapunov exponent of μ. This has immediate consequences for multifractal analysis. Given $\chi \in \mathbb{R}$, let

$$\begin{aligned} K_\chi &= \{x \in \mathbb{T}^d \mid \lim_{n\to\infty} \tfrac{1}{n} \log \|Dg^n|_{E^u(x)}\| = \chi\} \\ &= \{x \in \mathbb{T}^d \mid \lim_{n\to\infty} \tfrac{1}{n} S_n\varphi(x) = -\chi\} \end{aligned}$$

be the set of points whose largest Lyapunov exponent exists and is equal to χ. The following is a direct consequence of Theorem 8.2 and [21, Corollary 2.9].

Theorem 9.3. *Let g and μ_t be as in Theorem 8.2. Let $\chi_0 = \chi^+(\mu_0)$ and $\chi_1 = \chi^+(\mu_1)$. Then for every $\chi \in [\chi_1, \chi_0]$, we have*

$$\begin{aligned} h_{\mathrm{top}}(K_\chi, g) &= \inf\{P(t\varphi) + t\chi \mid t \in \mathbb{R}\} \\ &= \sup\{h_\mu(g) \mid \mu \in \mathcal{M}_f(X), \chi^+(\mu) = \chi\} \\ &= \sup\{h_\mu(g) \mid \mu \in \mathcal{M}_f^e(K_\chi)\}, \end{aligned}$$

where $h_{\mathrm{top}}(K_\chi, g)$ is topological entropy defined as a dimension characteristic in the sense of Bowen [9]*.Moreover, the infimum in the first line is achieved for some $t \in [0,1]$, and for this t we have $h_{\mathrm{top}}(K_\chi, g) = h_{\mu_t}(g)$.*

The analogue of Theorem 9.3 for the Bonatti–Viana family follows by the same argument for those $g \in \mathcal{V}_1$ with zero tail entropy (this can be achieved if g is constructed to be C^∞), but is left open for those $g \in \mathcal{V}_1$ with positive tail entropy; in that case, the entropy map may not be upper semicontinuous and [21] cannot be applied directly.

10. Pressure estimates

10.1. **Proof of Theorem 3.3.** Theorem 3.3 is a direct consequence of the following result and Lemma 2.2.

Theorem 10.1. *Under the assumptions of Theorem 3.3, we have*

$$h(\mathcal{C}, 5\eta; g) \le r(h_{\mathrm{top}}(f) + \log L - \log r). \tag{10.1}$$

Moreover, at any scale $\delta > 0$, we have

$$P(\mathcal{C}, \varphi, \delta; g) \le (1-r) \sup_{x \in B(q,\rho)} \varphi(x) + r \sup_{x\in M} \varphi(x) + h(\mathcal{C}, \delta; g). \tag{10.2}$$

Proof. The pressure estimate (10.2) follows from the observation that for every $(x,n) \in \mathcal{C}$ we have $f^k x \in B(q,\rho)$ for at least $(1-r)n$ values of $k \in \{0,1,\ldots,n-1\}$, and so

$$S_n^g\varphi(x) \le (1-r)n \sup_{x\in B(q,\rho)} \varphi(x) + rn \sup_{x\in M} \varphi(x);$$

this yields the partition sum estimate

$$\Lambda_n(\mathcal{C},\varphi,\delta;g) \le \Lambda_n(\mathcal{C},0,\delta;g)\exp(n\{(1-r)\sup_{x\in B(q,\rho)}\varphi(x) + r\sup_{x\in M}\varphi(x)\}),$$

which implies (10.2). Thus it only remains to obtain the entropy estimate (10.1). For each $(x,n) \in \mathcal{C}$, we partition its orbit into segments entirely in $B(q,\rho)$, and segments entirely outside $B(q,\rho)$. More precisely, let

$$\mathcal{Q} = \{(x,n) \,:\, g^k(x) \in B(q,\rho) \text{ for all } 0 \le k < n\}.$$

and note that $(x,n) \in \mathcal{Q}$ if and only if $x \in B_n(q,\rho)$. Given $(x,n) \in \mathcal{C}$, let $\{(x_i,n_i),(y_i,m_i) \,:\, i = 1,\ldots,\ell\}$ be the uniquely defined orbit segments such that

- $f^{n_i}(x_i) = y_i$ and $f^{m_i}(y_i) = x_{i+1}$;
- $(x_i,n_i) \in \mathcal{Q}$;
- (y_i,m_i) corresponds to an orbit segment in $X \setminus B(q,\rho)$.

Note that $\sum_{i=1}^{\ell} m_i = S_n^g\chi(x) < n\gamma$ (this corresponds to the total time the trajectory spends outside $B(q,\rho)$). Thus $\ell \le n\gamma$ (the worst possible bound corresponds to all $m_i = 1$). This defines a map

$$\pi\colon \mathcal{C} \to (\mathcal{Q}\times(M\times\mathbb{N}))^* = \bigcup_{\ell=1}^{\infty}(\mathcal{Q}\times(M\times\mathbb{N}))^*$$

with the property that

$$\pi(\mathcal{C}_n) \subset \bigcup_{\ell,\mathbf{m},\mathbf{n}} \prod_{i=1}^{\ell} \mathcal{Q}_{n_i} \times (M\times\{m_i\}) \tag{10.3}$$

where the union is over all $\mathbf{m} = (m_1,\ldots,m_\ell)$ and $\mathbf{n} = (n_1,\ldots,n_\ell)$ with $\sum(m_i+n_i) = n$ and $\sum m_i < \gamma n$.

Recall that L is the constant such that (3.1) holds. Since $d_{C_0}(f,g) < \delta$, using Lemma 3.2, we have

$$\Lambda_n(M,0,3\eta;g) \le \Lambda_n(M,0,\eta;f) \le Le^{nh} \tag{10.4}$$

Since $3\eta > \rho$, notice that $\Lambda_n(\mathcal{Q},0,3\eta;g) = 1$.

Let $E_n \subset \mathcal{C}_n$ be $(n,5\eta)$-separated, then going to the nearest element of η-separated sets of maximal cardinality in $\mathcal{Q}_{m_i}$ and $M\times\{n_i\}$, we get from (10.3) that

$$\begin{aligned}\Lambda_n(\mathcal{C},0,5\eta) &\le \sum_{\mathbf{m},\mathbf{n}} \prod_{i=1}^{\ell} \Lambda_{n_i}(\mathcal{Q},0,3\eta;g)\Lambda_{m_i}(M,0,3\eta;g) \\ &\le \sum_{\mathbf{m},\mathbf{n}} \prod_{i=1}^{\ell} L e^{m_i h} = L^\ell \sum_{\mathbf{m},\mathbf{n}} e^{(\sum m_i)h} \le L^{\gamma n} \sum_{\mathbf{m},\mathbf{n}} e^{\gamma n h}\end{aligned}$$

where the second inequality uses (10.4).

Now we observe that each choice of $\mathbf{m},\mathbf{n}$ is uniquely determined by choosing at most ℓ elements of $\{0,1,\dots,n-1\}$, which are the partial sums of m_i and n_i (the times when the trajectory enters or leaves $B(q,\rho)$). In particular since $\ell \le \gamma n$, it follows from Stirling's formula that the number of such $\mathbf{m},\mathbf{n}$ is at most

$$\sum_{k=1}^{\gamma n} \binom{n}{k} \le K e^{(-\gamma \log \gamma)n}$$

for some constant K (independent of n), and so we have

$$\Lambda_n(\mathcal{C},0,5\eta) \le L^{\gamma n} K e^{\gamma n(h - \log(\gamma))}.$$

This establishes (10.1) and completes the proof of Theorem 10.1. $\square$

Remark 10.2. *One could try to repeat the above argument directly to obtain an estimate on $h(\mathcal{C},\delta;g)$ for arbitrary $\delta > 0$ without invoking the tail entropy $h_g^*(5\eta)$. However, the constant L depends on η, and when $\delta < \rho$ there is also trouble with the term $\Lambda_n(\mathcal{Q},0,\delta;g)$. Thus we make the above estimate at exactly one scale, and deal with smaller scales via tail entropy.*

10.2. **Proof of Theorem 3.4.** Let g, r, q, q' be as in §3.5, and write $\chi = \chi_q, \chi' = \chi_{q'}$, $\mathcal{C} = \mathcal{C}(q,r)$, $\mathcal{C}' = \mathcal{C}(q',r)$. Consider the sets

$$A^+ = \{x : \text{there exists } K(x) \text{ so } \tfrac{1}{n} S_n^g \chi(x) < r \text{ for all } n > K(x)\},$$

$$A^- = \{x : \text{there exists } K(x) \text{ so } \tfrac{1}{n} S_n^{g^{-1}} \chi'(x) < r \text{ for all } n > K(x)\}.$$

Theorem 3.4 is an application of the following theorem, whose proof is based on the Katok pressure formula [41, 56].

Theorem 10.3. *Let $\mu \in \mathcal{M}_e(g)$. If either $\mu(A^+) > 0$ or $\mu(A^-) > 0$, then $h_\mu(g) + \int \varphi\, d\mu \le P(\mathcal{C} \cup \mathcal{C}', \varphi)$.*

Proof. Start with the case where $\mu(A^+) > 0$; we show that $h_\mu(g) + \int \varphi\, d\mu \le P(\mathcal{C},\varphi)$. Given $k \in \mathbb{N}$, let $A_k^+ = \{x \in A^+ : K(x) \le k\}$, and observe that $\mu(\bigcup_k A_k^+) > 0$, so there is some k such that $\mu(A_k^+) > 0$.

Note that for every $n > k$ and $x \in A_k^+$, we have $(x, n) \in \mathcal{C}$. It follows that for every $\delta > 0$ we have

$$\Lambda_n(A_k^+, \varphi, \delta; g) \leq \Lambda_n(\mathcal{C}, \varphi, \delta; g). \tag{10.5}$$

Fix $\alpha \in (0, \mu(A_k^+))$ and consider the quantity

$$s_n(\varphi, \delta, \mu; g) = \min \left\{ \sum_{x \in E} \exp\{S_n^g \varphi(x)\} : \mu \left(\bigcup_{x \in E} B_n(x, \delta) \right) \geq \alpha \right\}.$$

By Katok's pressure formula [56, Proposition 4], we have

$$h_\mu(g) + \int \varphi \, d\mu = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varphi, \delta, \mu; g).$$

Note that $s_n(\varphi, 2\delta, \mu; g) \leq \Lambda_n(A_k^+, \varphi, \delta; g) \leq \Lambda_n(\mathcal{C}, \varphi, \delta; g)$, where the second inequality uses (10.5). It follows that

$$h_\mu(g) + \int \varphi \, d\mu \leq P(\mathcal{C}, \varphi) = \lim_{\delta \to 0} P(\mathcal{C}, \varphi, \delta).$$

The case where $\mu(A^-) > 0$ is similar: obtain $A_k^- \subset A^-$ such that $K(x) \leq k$ for all $x \in A_k^-$ and $\mu(A_k^-) > 0$. Then observe that for $x \in A_k^-$, we have $(g^{-n}x, n) \in \mathcal{C}'$ for any $n \geq k$. Moreover, (n, ε) separated sets for g are in one to one correspondence with (n, ε) separated sets for g^{-1}, and $S_n^{g^{-1}} \varphi(x) = S_n^g \varphi(g^{-n+1}x)$. Then a simple argument shows that $P(A_k^-, \varphi, \varepsilon; g^{-1}) \leq P(\mathcal{C}', \varphi, \varepsilon; g)$.

Finally, Katok's pressure formula applied to g^{-1} tells us that

$$h_\mu(g) + \int \varphi \, d\mu = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varphi, \delta, \mu; g^{-1}).$$

Thus $h_\mu(g) + \int \varphi \, d\mu \leq \lim_{\delta \to 0} P(A_k^-, \varphi, \varepsilon; g^{-1}) \leq \lim_{\delta \to 0} P(\mathcal{C}', \varphi, \delta)$. $\square$

Now, to prove Theorem 3.4, by the hypothesis [**E**], if $\Gamma_\varepsilon(x) \neq \{x\}$, then either there are only finitely many n so that $\frac{1}{n} S_n^g \chi(x) \geq r$, or there are only finitely many n so that $\frac{1}{n} S_n^{g^{-1}} \chi'(x) \geq r$. Thus, if $x \in \mathrm{NE}(\varepsilon)$, then either $x \in A^+$ or $x \in A^-$. Thus, if μ is an ergodic measure satisfying $\mu(\mathrm{NE}(\varepsilon)) > 0$; then at least one of A^+ or A^- has positive μ-measure. Thus, Theorem 10.3 applies, and we conclude that

$$h_\mu(g) + \int \varphi \, d\mu \leq P(\mathcal{C} \cup \mathcal{C}', \varphi) = \max(P(\mathcal{C}, \varphi), P(\mathcal{C}', \varphi)).$$

11. Proofs of Lemmas

11.1. General estimates on partition sums.

Proof of Lemma 2.1. It suffices to consider (n,δ)-separated sets of maximum cardinality in the supremum for the partition sum. Otherwise, we could increase the partition sum by adding in another point. An (n,δ)-separated set of maximum cardinality must be (n,δ)-spanning, or else we could add in another point and still be (n,δ)-separated. The first inequality follows.

The second inequality holds by observing that if E_n is any $(n,2\delta)$-separated set and F_n is any (n,δ)-spanning set, then the map $\pi\colon E_n \to F_n$ taking x to the nearest point in F_n has the property that $d(x,\pi(x)) \leq \delta$ and hence is injective. Thus, for any E which is $(n,2\delta)$ separated,

$$\sum_{y\in F_n} e^{S_n\varphi(y)} \geq \sum_{x\in E_n} e^{S_n\varphi(\pi(x))} \geq \sum_{x\in E_n} e^{S_n\varphi(x)-n\operatorname{Var}(\varphi,\delta)},$$

and thus $\sum_{y\in F_n} e^{S_n\varphi(y)} \geq e^{-n\operatorname{Var}(\varphi,\delta)}\Lambda_n(\mathcal{D},\varphi,2\delta)$. □

Proof of Lemma 2.2. It is shown in [8, Proposition 2.2] that given any $\delta > 0$ and $\alpha > h_f^*(\varepsilon)$, there is a constant K such that

$$\Lambda^{\mathrm{span}}(B_n(x,\varepsilon),0,\delta;f) \leq Ke^{\alpha n}$$

for every $x \in X$ and $n \in \mathbb{N}$; that is, every Bowen ball $B_n(x,\varepsilon)$ has an (n,δ)-spanning subset with cardinality at most $Ke^{\alpha n}$. Let $E_n \subset \mathcal{D}_n$ be a (n,ε) separated set of maximum cardinality. Then

$$\mathcal{D}_n \subset \bigcup_{x\in E_n} B(x,n,\varepsilon).$$

For each $x \in E_n$ let F_x be an (n,δ)-spanning set for $B_n(x,\varepsilon)$ with

$$\#F_x \leq Ke^{\alpha n}$$

Then the set $G_n = \bigcup_{x\in E_n} F_x$ is (n,δ)-spanning for $\mathcal{D}_n$ and has

$$\#G_n \leq (\#E_n)Ke^{\alpha n}$$

We conclude that

$$\Lambda_n^{\mathrm{span}}(\mathcal{D},0,\delta) \leq \Lambda_n(\mathcal{D},0,\varepsilon)Ke^{\alpha n}.$$

The result follows by using the second inequality in Lemma 2.1 and sending $\delta \to 0$. □

11.2. **Pressure and partition sums of perturbations.**

Proof of Lemma 3.2. With η and C as in the statement of the lemma, put $\alpha = \eta/C$. By the Anosov shadowing lemma if $\{x_n\}$ is an α-pseudo orbit for f, then there exists an f-orbit that η-shadows $\{x_n\}$.

Now fix $g \in \mathrm{Diff}(M)$ with $d_{C^0}(f,g) < \alpha$. Then every g-orbit is an α-pseudo orbit for f, and hence for every $x \in M$, we can find a point $\pi(x) \in M$ such that

$$d(f^n(\pi x), g^n x) < \eta \text{ for all } n \in \mathbb{Z}. \tag{11.1}$$

We prove (i). By expansivity of f, we have

$$P(f,\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n^{\mathrm{span}}(\varphi, 3\eta; f). \tag{11.2}$$

Let E_n be a (n,η)-spanning set for g. Then from (11.1) we see that $\pi(E_n)$ is $(n,3\eta)$-spanning for f; indeed, given $x \in M$ we choose $y \in \pi^{-1}(x)$ and observe that there is $z \in E_n$ such that $d(g^k y, g^k z) < \eta$ for all $0 \le k < n$. Then we have

$$\begin{aligned} d(f^k x, f^k(\pi z)) &= d(f^k(\pi y), f^k(\pi z)) \\ &\le d(f^k(\pi y), g^k y) + d(g^k y, g^k z) + d(g^k z, f^k(\pi z)) < 3\eta \end{aligned}$$

for every $0 \le k < n$, showing that $\pi(E_n)$ is $(n,3\eta)$-spanning for f. It follows that

$$\Lambda_n^{\mathrm{span}}(\varphi, 2\eta; f) \le \sum_{x\in\pi(E_n)} e^{S_n^f\varphi(x)} = \sum_{x\in E_n} e^{S_n^f\varphi(\pi x)}. \tag{11.3}$$

Note that

$$S_n^f\varphi(\pi x) = \sum_{k=0}^{n-1} \varphi(f^k(\pi x)) \le \sum_{k=0}^{n-1} (\varphi(g^k x) + \mathrm{Var}(\varphi,\eta))$$

and together with (11.2) and (11.3) this gives

$$P(f,\varphi) \le \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in E_n} e^{n\,\mathrm{Var}(\varphi,\eta) + S_n^g\varphi(x)}.$$

Taking an infimum over all (n,η)-spanning sets for g gives

$$P(f,\varphi) \le \mathrm{Var}(\varphi,\eta) + P(g,\varphi,\eta)$$

by the first inequality in Lemma 2.1. This completes the proof of (i) since $P(g,\varphi) \ge P(g,\varphi,\eta)$.

Now we prove (ii). Let E_n be a maximal $(n,3\eta)$ separated set for g. As in the previous argument, we see from (11.1) that $\pi(E_n)$ is (n,η)-separated for f: indeed, for every $x, y \in E_n$ there is $0 \le k < n$ such that $d(g^k x, g^k y) \geq 3\eta$, and hence

$$d(f^k(\pi x), f^k(\pi y)) \geq d(g^k x, g^k y) - d(g^k x, f^k \pi x) - d(g^k y, f^k \pi y) > \eta.$$

In particular, we have

$$\begin{aligned} \Lambda_n(\varphi, \eta; f) &\geq \sum_{x \in \pi(E_n)} e^{S_n^f \varphi(x)} = \sum_{x \in E_n} e^{S_n^f \varphi(\pi x)} \\ &\geq \sum_{x \in E_n} e^{S_n^g \varphi(x) - n \operatorname{Var}(\phi, \eta)} \geq \Lambda_n(\varphi, 3\eta; g) e^{-n \operatorname{Var}(\varphi, \eta)}, \end{aligned} \tag{11.4}$$

as required. $\square$

11.3. **Local product structure.** We prove Lemma 3.5 following the usual proof of local product structure: obtain both leaves as graphs of functions ϕ_1, ϕ_2 and observe that the intersection of the leaves is the unique fixed point of $\phi_1 \circ \phi_2$, which is a contraction.

Given $x, y \in F^1 \oplus F^2$, let z' be the unique point of intersection of $(x + F^1) \cap (y + F^2)$. Translating the coordinate system so that z' becomes the origin, we assume without loss of generality that $x \in F^1$ and $y \in F^2$. Then $W^1(x)$ and $W^2(y)$ are graphs of C^1 functions $\phi_1 \colon F^1 \to F^2$ and $\phi_2 \colon F^2 \to F^1$ with $\|D\phi_i\| < \beta$. That is, $W_1(x) = \{a + \phi_1(a) : a \in F^1\}$ and $W_2(y) = \{\phi_2(b) + b : b \in F^2\}$. Thus $z \in W_1 \cap W_2$ if and only if $z = a + \phi_1(a) = \phi_2(b) + b$ for some $a \in F^1$ and $b \in F^2$. This occurs if and only if $b = \phi_1(a)$ and $a = \phi_2(b)$; that is, if and only if $a = \phi_2 \circ \phi_1(a)$ and $b = \phi_1(a)$. Because $\phi_2 \circ \phi_1$ is a contraction on the complete metric space F_1 it has a unique fixed point a.

For the estimate on the distances from z to x, y we observe that

$$\begin{aligned} \|a\| &= d(a, 0) = d(\phi_2 b, \phi_2 y) \leq \beta d(b, y) \leq \beta(\|b\| + \|y\|), \\ \|b\| &= d(b, 0) = d(\phi_1 a, \phi_1 x) \leq \beta d(a, x) \leq \beta(\|a\| + \|x\|). \end{aligned}$$

Recall that by the definition of $\bar{\kappa}$ we have $\|x\|, \|y\| \leq \bar{\kappa}\|x - y\|$. Thus we have

$$\|a\| \leq \beta(\beta(\|a\| + \|x\|) + \|y\|) \leq \beta^2 \|a\| + \beta(1 + \beta)\bar{\kappa} d(x, y),$$

which gives $\|a\| \leq \frac{\beta}{1-\beta} \bar{\kappa} d(x, y)$, and similarly for $\|b\|$. Thus

$$d(a, x) \leq \|a\| + \|x\| \leq \left(\frac{\beta}{1 - \beta} + 1 \right) \bar{\kappa} d(x, y) = \frac{\bar{\kappa} d(x, y)}{1 - \beta}.$$

The bound on $d_{W^1}(z, x)$ follows since $\|D\phi_1\| < \beta$. The other distance bound is similar.

11.4. **Density of leaves.** Before proving Lemma 5.4, we prove the following general result.

Lemma 11.1. *Let W be a foliation of a compact manifold M such that $W(x)$ is dense in M for every $x \in M$. Then for every $\alpha > 0$ there is $R > 0$ such that $W_R(x)$ is α-dense in M for every $x \in M$.*

Proof of Lemma 11.1. Given $R > 0$, define a function $\psi_R \colon M \times M \to [0, \infty)$ by $\psi_R(x, y) = \operatorname{dist}(y, W_R(x))$. Note that for each R, the map $x \mapsto W_R(x)$ is continuous (in the Hausdorff metric) and hence ψ_R is continuous. Moreover, since $W(x) = \bigcup_{R>0} W_R(x)$ is dense in M for each $x \in M$, we have $\lim_{R\to\infty} \psi_R(x, y) = 0$ for each $x, y \in M$. Finally, when $R \geq R'$ we see that $W_R(x) \supset W_{R'}(x)$ and so $\psi_R(x, y) \leq \psi_{R'}(x, y)$. Thus $\{\psi_R : R > 0\}$ is a family of continuous functions that converge monotonically to 0 pointwise. By compactness of $M \times M$, the convergence is uniform, hence for every $\alpha > 0$ there is R such that $\psi_R(x, y) < \alpha$ for all $x, y \in M$. □

Proof of Lemma 5.4. Put $\delta = \rho'$. By the local product structure for W^{cs}, W^u we can put $\alpha = \delta/\kappa$ and observe that if $d(y, z) < \alpha$, then

$$W^u_\delta(z) \cap W^{cs}_\delta(y) \neq \emptyset. \tag{11.5}$$

By Lemma 11.1 there is $R > 0$ such that $W^u_R(x)$ is α-dense in $\mathbb{T}^d$ for every $x \in \mathbb{T}^d$. Thus for every $x \in \mathbb{T}^d$ there is $z \in W^u_R(x)$ such that $d(y, z) < \alpha$, and in particular (11.5) holds. The result follows by observing that $W^u_{R+\delta}(x) \supset W^u_\delta(z)$. □

11.5. **Contraction in the Grassmannian.**

11.5.1. *Proof of Lemma 8.7.* Given $A \colon \mathbb{R}^4 \to \mathbb{R}^4$ such that $AC_\beta \subset C_\beta$ we first find the map $\hat{A} \colon U_\beta \to U_\beta$ that it induces.

Given $v \in F^u$ we want to find $(\hat{A}L)(v) \in F^s$ such that $(\hat{A}L)v + v \in A\Gamma(L)$; that is, such that there is $w \in F^s$ with

$$v + (\hat{A}L)v = Aw + ALw = A^{uu}w + A^{su}w + A^{us}Lw + A^{ss}Lw,$$

from which we deduce that

$$\begin{aligned} v &= A^{uu}w + A^{us}Lw, \\ (\hat{A}L)v &= A^{su}w + A^{ss}Lw = (A^{su} + A^{ss}L)(A^{uu} + A^{us}L)^{-1}v. \end{aligned}$$

We conclude that the map $\hat{A} \colon U_\beta \to U_\beta$ is given by

$$\hat{A}(L) = (A^{su} + A^{ss}L)(A^{uu} + A^{us}L)^{-1}. \tag{11.6}$$

Fix $L, L' \in U_\beta$. For convenience we write

$$\begin{aligned} A_u &= A^{uu} + A^{us}L \colon F^u \to F^u, \\ A_s &= A^{su} + A^{ss}L \colon F^u \to F^s, \\ A'_u &= A^{uu} + A^{us}L' \colon F^u \to F^u, \\ A'_s &= A^{su} + A^{ss}L' \colon F^u \to F^s. \end{aligned}$$

Now we have

$$\begin{aligned} \hat{A}L - \hat{A}L' &= A_s A_u^{-1} - A'_s (A'_u)^{-1} \\ &= (A_s A_u^{-1} A'_u - A'_s)(A'_u)^{-1} \\ &= A_s A_u^{-1}(A'_u - A_u)(A'_u)^{-1} + (A_s - A'_s)(A'_u)^{-1} \\ &= A_s A_u^{-1} A^{us}(L' - L)(A'_u)^{-1} + A^{ss}(L - L')(A'_u)^{-1}, \end{aligned}$$

together with the following bounds:

$$\begin{aligned} \|A_s\| &\le \|A^{su}\| + \|A^{ss}L\| \le \xi + K\beta, \\ \|A_u^{-1}\| &\le \|(A^{uu})^{-1}\| \|(I + A^{us}L(A^{uu})^{-1})^{-1}\| \le \lambda \sum_{n \ge 0} (\xi\beta\lambda)^n = \frac{\lambda}{1 - \xi\beta\lambda}, \\ \|(A'_u)^{-1}\| &\le \frac{\lambda}{1 - \xi\beta\lambda}. \end{aligned}$$

Putting it all together we get

$$\|\hat{A}L - \hat{A}L'\| \le \left((\xi + K\beta) \frac{\lambda^2}{(1 - \xi\beta\lambda)^2} \xi + \frac{\lambda^2}{1 - \xi\beta\lambda} \right) \|L - L'\|,$$

which proves the lemma.

11.5.2. *Proof of Lemma 8.8.* Let $\eta_k^j = F_{k-1} \circ \cdots \circ F_j \circ G_{j-1} \circ \cdots \circ G_0(\eta)$. Then $\eta_k^k = \eta_k$ and $\eta_k^0 = F_{k-1} \circ \cdots \circ F_0(\eta)$. Hence

$$d(\xi_k, \eta_k) = d(\xi_k, \eta_k^k) \le d(\xi_k, \eta_k^0) + \sum_{j=0}^{k-1} d(\eta_k^j, \eta_k^{j+1}).$$

Property (2) gives

$$d(\xi_k, \eta_k^0) = d(F_{k-1} \circ \cdots \circ F_0(\xi), F_{k-1} \circ \cdots \circ F_0(\eta)) \le \prod_{j=0}^{k-1} \lambda_j \cdot d(\xi, \eta).$$

We also have

$$\begin{aligned}
d(\eta_k^j, \eta_k^{j+1}) &= d(F_{k-1} \circ \cdots \circ F_{j+1} \circ F_j \circ G_{j-1} \circ \cdots \circ G_0(\eta), \\
&\qquad F_{k-1} \circ \cdots \circ F_{j+1} \circ G_j \circ G_{j-1} \circ \cdots \circ G_0(\eta)) \\
&= d(F_{k-1} \circ \cdots \circ F_{j+1} \circ F_j(\eta_j^j), F_{k-1} \circ \cdots \circ F_{j+1} \circ G_j(\eta_j^j)) \\
&\leq \prod_{i=j+1}^{k-1} \lambda_i \cdot d(F_j(\eta_j^j), G_j(\eta_j^j)) \qquad \text{by (2)} \\
&\leq \prod_{i=j+1}^{k-1} \lambda_i \cdot C \prod_{i=0}^{j-1} \lambda_i = C \prod_{i \neq j} \lambda_i \qquad \text{by (1).}
\end{aligned}$$

Thus

$$d(\xi_k, \eta_k) \leq \prod_{i=j}^{k-1} \lambda_i \cdot d(\xi, \eta) + C \sum_{j=0}^{k-1} \prod_{i \neq j} \lambda_i = \prod_{j=0}^{k-1} \lambda_j \cdot \left[d(\xi, \eta) + C \sum_{j=0}^{k-1} \frac{1}{\lambda_j} \right].$$

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