Specification and Markov properties in shift spaces

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Thermodynamic formalism

Let (X, σ) be a shift space on a finite alphabet. Then it has a measure of maximal entropy (MME). (Maximizes $h_\mu(\sigma)$)

1. For which classes of shifts is the MME unique?
2. Does the MME have exponential decay of correlations (EDC)?
3. What about equilibrium states for non-zero potentials?
(Maximize $h_\mu(\sigma) + \int \varphi \, d\mu$)

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Connections to smooth dynamics: for uniformly hyperbolic diffeomorphisms, **physically relevant** invariant measures arise as equilibrium states for the "geometric potential", and display strong stochastic properties.

- Goal is to develop techniques that extend this theory for non-uniformly hyperbolic systems.

Subshifts of finite type / Markov shifts

- A (finite alphabet) $\rightsquigarrow$ $A^* = \bigcup_{n\geq 0} A^n = \{\text{finite words over } A\}$
- $X \subset A^{\mathbb{N}}$ a shift space if closed and σ-invariant
- Language is $\mathcal{L} = \{x_{[i,j)} = x_i x_{i+1} \cdots x_{j-1} \mid x \in X, i \leq j\} \subset A^*$
- $x \in X \Leftrightarrow x_{[i,j)} \in \mathcal{L}$ for all $i \leq j$

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X is Markov if there is n s.t. $x \in X$ iff $x_{[i,j)} \in \mathcal{L}$ whenever $j \leq i + n$

- When $n = 2$, present X via transition matrix or graph

Theorem (Parry, Ruelle, Sinai, Bowen – 60s and 70s)

If X is a transitive Markov shift (SFT), then

1. *there is a unique MME μ;*
2. *μ has EDC (up to a period);*
3. *same is true for every Hölder potential.*

Non-Markov shifts

Even if X is **not** Markov, it may still admit a tower:

- Σ a countable-state Markov shift; (need **recurrence**)
- $\pi \colon \Sigma \to X$ a shift-commuting map. (need eq. states to **lift**)

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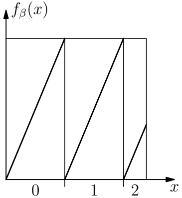
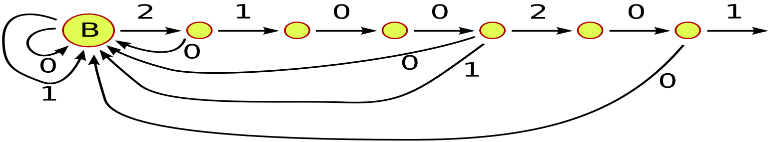


Example

Fix $\beta > 1$. Let X be the coding space for $f_\beta \colon x \mapsto \beta x \pmod 1$. This β-shift is typically not Markov, but admits a tower.



Every Lipschitz potential on a β-shift has a unique ES, with EDC.

(Hofbauer 1978, Walters 1978)

Thermodynamics and towers

Let Σ be ctbl-state Markov and $\pi\colon \Sigma \to X$ shift-commuting, 1-1.

- Inducing on a state B in Σ gives Σ as a suspension over a countable-state full shift – this is the ‘tower’ referred to.

Theorem (Sarig, Young – 1990s)

If φ is Hölder and Σ is strongly positive recurrent (SPR) w.r.t. $\varphi \circ \pi$, then it has a unique equilibrium state μ. Moreover, the tower has exponential tails w.r.t. μ, and thus μ has EDC.

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Warning: π need not be onto.

Definition

(X, φ) has an SPR model if there are Σ, π as above s.t. $(\Sigma, \varphi \circ \pi)$ is SPR and every ES μ for (X, φ) has $\mu(\pi\Sigma) = 1$.

- SPR model for a Hölder $\varphi \Rightarrow$ uniqueness and EDC.

Specification

Alternate approach to uniqueness given by specification property.

Definition

A language $\mathcal{L}$ has specification if there is $\tau \in \mathbb{N}$ such that for every $u, v \in \mathcal{L}$, there is $w \in \mathcal{L}$ with $|w| \leq \tau$ such that $uwv \in \mathcal{L}$.

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Theorem (Bowen - 1974)

If the language of a shift X has specification, then every Hölder φ has a unique equilibrium state μ_φ.

Bowen's result does not guarantee correlations decay exponentially.

Theorem (C., following Bertrand & Thomsen)

If the language of a shift X has specification, then (X, φ) has an SPR model for every Hölder φ. Thus μ_φ has EDC.

Specification to synchronisation to a tower

Most of the work for this theorem done previously:

1. *A. Bertrand 1988:* if $\mathcal{L}$ has specification then it has a synchronising word w (if $uw \in \mathcal{L}$ and $wv \in \mathcal{L}$ then $uwv \in \mathcal{L}$)
2. *K. Thomsen 2006:* if $\mathcal{L}$ has a synchronising word w, and if omitting all appearances of w gives a language $\mathcal{L}'$ with smaller entropy, then there is an SPR model

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Key idea: study entropy of **part** of the language, compare to whole

Definition

Given $\mathcal{D} \subset \mathcal{L}$, let $\mathcal{D}_n = \{w \in \mathcal{D} : |w| = n\}$. The entropy of $\mathcal{D}$ is

$$h(\mathcal{D}) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathcal{D}_n$$

Shifts of quasi-finite type

Buzzi (2005) introduced the following generalization of SFTs. Let X be a shift and $\mathcal{L}$ its language. The left and right constraints are

$$\mathcal{C}^\ell := \{aw \in \mathcal{L} \mid a \in A, w \in A^*, \text{ and } \exists v \in \mathcal{L} \text{ s.t. } wv \in \mathcal{L}, awv \notin \mathcal{L}\}$$
$$\mathcal{C}^r := \{wa \in \mathcal{L} \mid w \in A^*, a \in A, \text{ and } \exists v \in \mathcal{L} \text{ s.t. } vw \in \mathcal{L}, vwa \notin \mathcal{L}\}$$

X is Markov iff there is n such that $\mathcal{C}_n^\ell = \mathcal{C}_n^r = \emptyset$.

Definition

X is of quasi-finite type (QFT) if $\min\{h(\mathcal{C}^\ell), h(\mathcal{C}^r)\} < h(\mathcal{L})$.

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X is of quasi-finite type (QFT) if $\min\{h(\mathcal{C}^\ell), h(\mathcal{C}^r)\} < h(\mathcal{L})$.

Theorem (Buzzi 2005)

QFTs have countable-state Markov models with SPR components. Transitive QFTs can have multiple MMEs.

Non-uniform specification

QFTs generalise SFTs: constraints may be non-empty, but must be thermodynamically small. Similar idea for specification...

Definition

A decomposition of $\mathcal{L}$ is a choice of $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ s.t. $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$.

Then every word in $\mathcal{L}$ can be written as *uvw* for some choice of $u \in \mathcal{C}^p$, $v \in \mathcal{G}$, $w \in \mathcal{C}^s$. In particular, $\mathcal{L} = \bigcup_M \mathcal{G}^M$, where

$$\mathcal{G}^M = \{uvw \in \mathcal{L} \mid u \in \mathcal{C}^p, v \in \mathcal{G}, w \in \mathcal{C}^s, |u|, |w| \leq M\}$$

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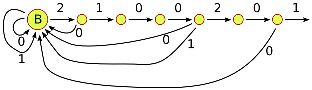
Theorem (C.–Thompson 2012)

Suppose $\mathcal{L}(X)$ has a decomposition such that

1. *$\mathcal{G}^M$ has specification for every M*
2. $h(\mathcal{C}^p \cup \mathcal{C}^s) = \max\{h(\mathcal{C}^p), h(\mathcal{C}^s)\} < h(\mathcal{L})$

Then X has a unique MME μ.

Application to β-shifts and factors



- $\mathcal{C}^p = \emptyset$
- $\mathcal{G}$: paths starting and ending at B
- $\mathcal{C}^s$: paths that never return to B

Then $h(\mathcal{C}^p \cup \mathcal{C}^s) = 0$; same holds for all factors.

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Every subshift factor of a β-shift has a unique MME.

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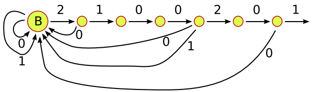

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Theorem (Walters 1978, C.–Thompson 2013)

Every Hölder potential on a β-shift has a unique ES, with EDC.

1. Does the unique MME of a β-shift **factor** have EDC?
2. What about non-zero potentials?

Getting a tower

Theorem (C. 2016)

Suppose $\mathcal{L}(X)$ has a decomposition $\mathcal{C}^p\mathcal{G}\mathcal{C}^s$ such that

1. *$\mathcal{G}$ has specification*
2. $h(\mathcal{C}^p \cup \mathcal{C}^s) = \max\{h(\mathcal{C}^p), h(\mathcal{C}^s)\} < h(\mathcal{L})$
3. *if $uvw \in \mathcal{L}$ and $uv, vw \in \mathcal{G}$, then $v, uvw \in \mathcal{G}$* (if v is long)

Then X has an SPR model. In particular, it has a unique MME, and this MME has EDC.

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Examples:

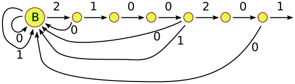
- Every subshift factor of a β-shift has such a decomposition.
- If X is a transitive QFT for which **both** $\mathcal{C}^\ell$ and $\mathcal{C}^r$ have small entropy, then it admits such a decomposition.
- Same true for topologically exact QFTs with $h(\mathcal{C}^\ell) < h(\mathcal{L})$.

Almost specification

C.–Thompson approach was motivated by M. Boyle's open problem list: K. Thomsen asked if factors of β-shifts have unique MMEs.

Original (failed) attempt used almost specification:

$\exists\ g(n) = o(n)$ s.t. $\forall\ u_1, u_2 \in \mathcal{L}\ \exists\ u_1', u_2'$ s.t. $u_1'u_2' \in \mathcal{L}$ and $d_H(u_i, u_i') \le g(|u_i|)$.



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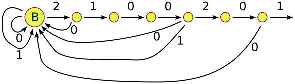

Theorem (Kulczycki–Kwietniak–Oprocha 2014, Pavlov 2016)

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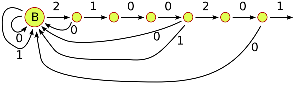

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Theorem (C.–Pavlov 2016)

If X has almost specification with $g \equiv 1$, or one-sided almost specification with g bounded, then it has an SPR model.

Non-zero potentials

SFTs, β-shifts, and S-gap shifts have a curious property.

Theorem

If X is an SFT, a β-shift, or an S-gap shift, then every Hölder potential is hyperbolic: all equilibrium states have $h(\mu) > 0$.

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This property does not hold universally.

Example (Conrad 2013)

Let $X = \overline{\{0^n 1^n \mid n \in \mathbb{N}\}^{\mathbb{Z}}}$ and $\varphi = t\chi_{[1]}$. Then

- $\mathcal{L}(X)$ has a decomposition with $h(\mathcal{C}^p \cup \mathcal{C}^s) < h(\mathcal{L})$
- for large t, δ_1 is the unique ES for $t\varphi$
- there is t_0 such that $t_0\varphi$ has multiple equilibrium states

Hölder (sometimes) implies hyperbolic

Given $g \colon \mathbb{N} \to \mathbb{N}$, say that $\mathcal{L}$ is g-Hamming approachable by $\mathcal{G}$ if every $w \in \mathcal{L}$ has $w' \in \mathcal{G}$ with $d_H(w, w') \leq g(|w|)$.

Theorem (C.–Cyr)

If g satisfies $\frac{g(n)}{\log n} \to 0$, and $\mathcal{L}$ is g-Hamming approachable by some $\mathcal{G}$ with specification, then every Hölder potential is hyperbolic.

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Application: if X is a subshift factor of a β-shift, then every Hölder potential on X has a unique equilibrium state, which has EDC.

Open question: what about the coding spaces for $x \mapsto \alpha + \beta x$?

The non-symbolic setting

Similar results hold for non-symbolic systems: X a compact metric space, $f\colon X \to X$ continuous, $\varphi\colon X \to \mathbb{R}$ continuous.

Replace $\mathcal{L}$ with $X \times \mathbb{N}$ (space of finite orbit segments)

$$(x, n) \leftrightsquigarrow x, f(x), f^2(x), \dots, f^{n-1}(x)$$

Ask for $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset X \times \mathbb{N}$ such that

- every (x, n) has $p, g, s \in \mathbb{N}_0$ such that $p + g + s = n$, $(x, p) \in \mathcal{C}^p$, $(f^p x, g) \in \mathcal{G}$, and $(f^{p+g} x, s) \in \mathcal{C}^s$
- every $\mathcal{G}^M$ has specification
- φ has the Bowen property (bounded distortion) on $\mathcal{G}$
- $P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(X, \varphi)$

Together with weak expansivity condition, this gives uniqueness.

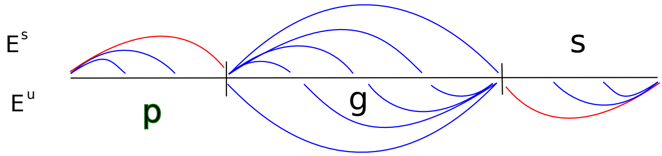
Other applications

Theorem (C.–Fisher–Thompson 2015)

For every Hölder continuous $\varphi\colon \mathbb{T}^4 \to \mathbb{R}$ there is a C^1-open set of diffeos $f : \mathbb{T}^4 \to \mathbb{T}^4$ (given by Bonatti and Viana) such that

- *f has a dominated splitting but is not partially hyperbolic*
- *$(\mathbb{T}^4, f, \varphi)$ has a unique equilibrium state*

$T_x\mathbb{T}^4$ splits into non-uniformly expanding and contracting E^u, E^s.



Similar approach works for geodesic flow on rank one manifolds of non-positive curvature (Burns–C.–Fisher–Thompson)