Specification and measures of maximal entropy

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- First steps
- Uniqueness via uniformity
- Beyond uniform specification
- Beyond symbolic dynamics
- Applications

Big Ideas in Dynamics
February 3, 2023

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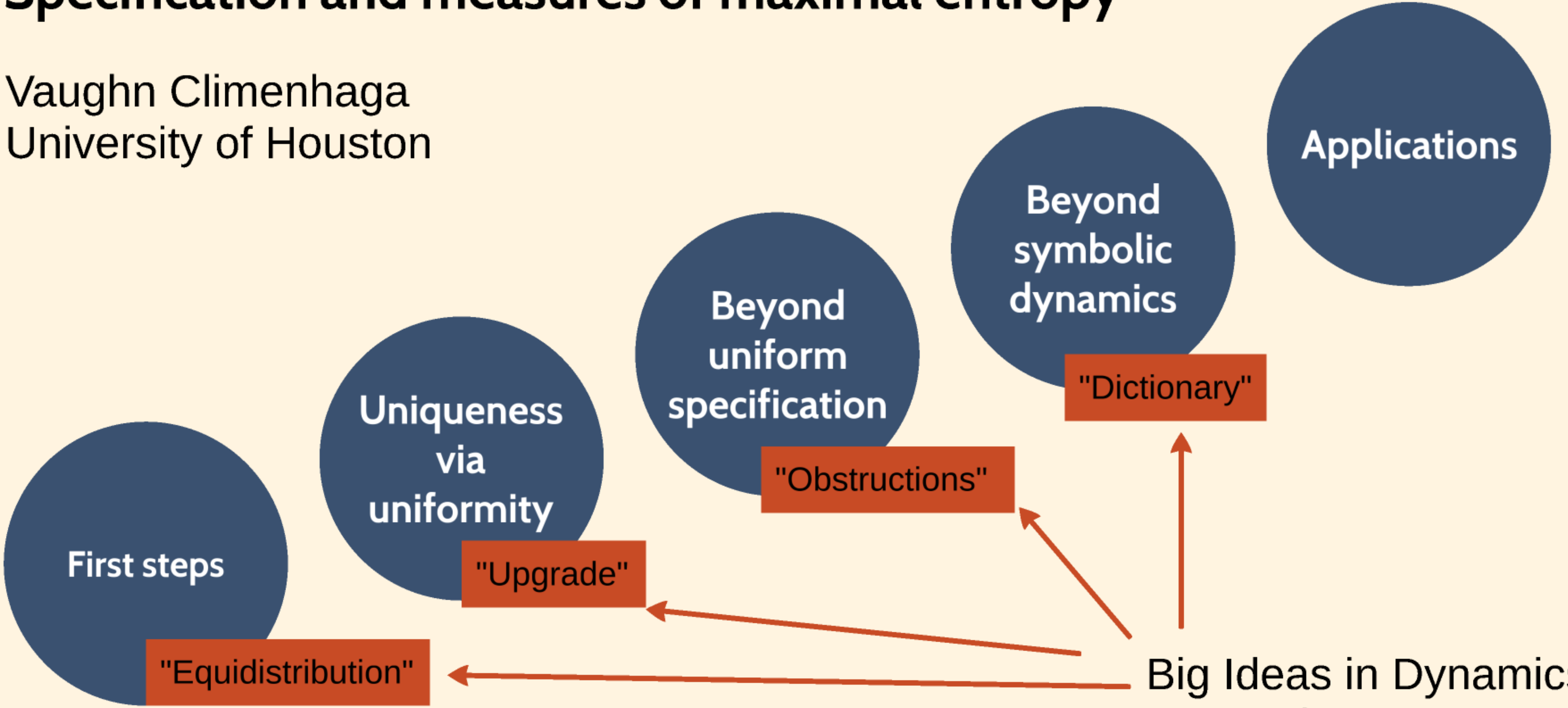

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Measures of maximal entropy

Consider an experiment with d possible outcomes, and a probability vector $\mathbf{p}$ giving their likelihoods

Entropy $H(\mathbf{p})$ = expected information gain

Fundamental inequality for probability vectors: *entropy maximized when all outcomes equally likely*

Entropy is maximized at a unique probability vector

Anything to say about dynamical systems?

Big idea: Equidistribution maximizes entropy

$H(\mathbf{p}) = \sum_{i=1}^{d} -p_i \log p_i \leq \log d$

"=" iff equidistributed ($p_i = 1/d$)

Assumption of maximum ignorance

The basic questions

Hyperbolic dynamics

Beyond uniform hyperbolicity

The basic questions

X a compact metric space, $f : X \to X$ continuous

$\mathcal{M}$ = {invariant probability measures}

Each $\nu \in \mathcal{M}$ has an **entropy** $h_\nu \in [0, \infty]$

Topological entropy $h = \sup\{h_\nu : \nu \in \mathcal{M}\}$

Measure of maximal entropy (MME):
$\nu \in \mathcal{M}$ such that $h_\nu = h$

Existence?
Uniqueness?
Properties?

Rate of expected information gain
(= expected rate of information gain)

Connections and motivations:
- Asymptotic behavior of system
- Periodic orbit estimates (Margulis)
- Other equilibrium states, thermodynamic formalism, multifractal analysis
- Physically relevant SRB measure

Uniform hyperbolicity

(Anosov diffeo, subshift of finite type, etc.)

- There is a unique MME
- It is mixing, K, Bernoulli
- It has exponential decay of correlations
- Similar results hold for equilibrium states for Hölder potentials

How to prove it?

Expansivity and specification

(Rufus Bowen, *Math. Syst. Theory*, 1974/5)

Nearby trajectories diverge

Can join any past to any future
(approximately) (and uniformly) (and repeatedly)

Ruelle-Perron-Frobenius operator

Markov partitions

Anisotropic Banach spaces

Looking ahead

Can we study existence and uniqueness of MMEs for systems that are:

- non-uniformly hyperbolic?
  (logistic map, Hénon, Lorenz)
- partially hyperbolic?
- hyperbolic with singularities?
  (billiards)

We will discuss non-uniform specification (Climenhaga-Thompson), but other approaches can be extended too.

Many open questions

References:

C.-T., *Israel Journal*, 2012
C.-T., *JLMS*, 2013
C.-T., *ETDS*, 2014
C.-T., *Advances*, 2016
C.-T., *Thermodynamic Formalism (Springer LNM 2290)*, 2021

Related:

C.-Pesin-Zelerowicz, *BAMS*, 2018

Some counter-examples

Some counterexamples

when we go beyond uniform hyperbolicity

Existence: can fail for some diffeos with only finitely many derivatives (Buzzi)

Uniqueness: can fail for some shift spaces

- Disjoint union of two shifts with the same entropy (this feels like cheating)
- Same thing, but "glue them together" without creating entropy (Haydn)
- The Dyck shift: symbols are ( ) [ ] and they must pair correctly (Krieger)

Alphabet = { 0, 1, 2, 3, 4 }
Rule: every sequence is either

- all red
- all blue
- ... red - 0s - blue - 0s - red - 0s... with # zeroes at least # adjacent reds and blues

Entropy is log(2), two ergodic MMEs

([ ] ( )) is legal, but ([ ) is not

Need not all close: ...(((((... is legal

Entropy is log(3), two ergodic MMEs

- One MME: every left bracket has a corresponding right bracket
- The other MME: vice versa

Existence and uniqueness

in shift spaces with specification

Shift space: a closed σ-invariant subset X of $\{1,\ldots,d\}^{\mathbb{N}}$, where σ is the left shift map.

Language: write $\mathcal{L}_n \subset \{1,2,\ldots,d\}^n$ for the set of words of length n that appear in some $x \in X$

Cylinders: given $w \in \mathcal{L}_n$, write $[w] = \{x \in X : x_1 \cdots x_n = w\}$

Topological entropy: $h = \lim \frac{1}{n} \log \#\mathcal{L}_n$

$\#\mathcal{L}_n \approx e^{nh}$: more precisely, $c_n = \#\mathcal{L}_n e^{-nh}$ is subexponential ($\lim \frac{1}{n} \log c_n = 0$)

Why does this limit exist? ....

Big idea:
Need to upgrade "subexponential" to "uniform" (in multiple places)

Shannon–McMillan–Breiman: If ν is an MME then $\nu[w] \approx e^{-nh}$ for "most" $w \in \mathcal{L}_n$.

Katok estimate: If ν is an MME and $\nu(Z) > 0$, then $\#\{n\text{-cylinders intersecting } Z\} \approx e^{nh}$.

Initial upgrades

Existence (in general)

Uniqueness for SFTs

Specification

Initial upgrades

$h = \lim \frac{1}{n} \log \#\mathcal{L}_n$ exists by Fekete's lemma:

- $\#\mathcal{L}_{n+k} \leq (\#\mathcal{L}_n)(\#\mathcal{L}_k)$
- $a_n := \log \#\mathcal{L}_n$ is subadditive ($a_{n+k} \leq a_n + a_k$)
- Fekete: $\lim \frac{1}{n} a_n$ exists and $= \inf \frac{1}{n} a_n$

Get $h \leq \frac{1}{n} a_n$ for all n, so $\log \#\mathcal{L}_n = a_n \geq nh$, so $\#\mathcal{L}_n \geq e^{nh}$.

Can also prove this directly:

$$\#\mathcal{L}_k \leq d^k \quad \Rightarrow \quad h = \lim_k \tfrac{1}{k} \log \#\mathcal{L}_k \leq \log d$$

$$\#\mathcal{L}_{nk} \leq (\#\mathcal{L}_n)^k \quad \Rightarrow \quad h = \lim_k \tfrac{1}{nk} \log \#\mathcal{L}_{nk} \leq \tfrac{1}{n} \log \#\mathcal{L}_n$$

Uniform lower counting bound for free. A uniform upper counting bound will require more hypotheses (later).

Fekete's lemma also guarantees existence of $h_\nu = \lim \frac{1}{n} H_\nu(\beta_n)$, since $n \mapsto H_\nu(\beta_n)$ is subadditive. Again, it *also* gives $h_\nu \leq \frac{1}{n} H_\nu(\beta_n)$ for all n, which will be important in the proof of uniqueness (via the "uniform Katok estimate").

Measure-theoretic entropy:

$$\beta_n = \{[w] : w \in \mathcal{L}_n\} \text{ (partition into } n\text{-cylinders)}$$

$$H_\nu(\beta_n) = \sum_{w \in \mathcal{L}_n} -\nu[w] \log \nu[w] \leq \log \#\mathcal{L}_n$$

$$h_\nu(\sigma) = \lim \frac{1}{n} H_\nu(\beta_n) \leq h$$

Constructing an MME

Misiurewicz's proof of the variational principle contains a construction that produces an MME for every shift space.

Equidistribution maximizes entropy:
Let m_n be any measure with $m_n[w] = 1/\#\mathcal{L}_n$ for all $w \in \mathcal{L}_n$.

Push forward and average to get invariance:
Let $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \sigma_*^k m_n$.

Then any limit point $\mu = \lim_k \mu_{n_k}$ is an MME.

Measure-theoretic entropy:

$$\beta_n = \{[w] : w \in \mathcal{L}_n\} \text{ (partition into } n\text{-cylinders)}$$

$$H_\nu(\beta_n) = \sum_{w\in\mathcal{L}_n} -\nu[w] \log \nu[w] \leq \log \#\mathcal{L}_n$$

$$h_\nu(\sigma) = \lim \frac{1}{n} H_\nu(\beta_n) \leq h$$

Upgrade SMB to uniform Gibbs bound

For SFTs, can use eigendata of transition matrix to build an MME (Parry measure) with the property that there is $c > 0$ such that $\mu[w] \geq ce^{-nh}$ for all $w \in \mathcal{L}_n$.

Uniqueness

Adler-Weiss argument: for SFTs and more

Idea: any two MMEs should be equidistributed, hence equivalent (absolutely continuous). But if μ is ergodic and $\nu \ll \mu$ is invariant, then $\nu = \mu$, giving uniqueness.

Suppose we have an ergodic MME satisfying the **Uniform Gibbs bound:** $\mu[w] \geq ce^{-nh}$ for all $w \in \mathcal{L}_n$

Then we immediately get a **Uniform counting bound:** $\#\mathcal{L}_n \leq Qe^{nh}$ with $Q = 1/c$.

This in turn leads to a **Uniform Katok estimate:** If ν is any MME and Z is covered by s_n n-cylinders, then $s_n \geq Q(2Q)^{-1/\nu(Z)}e^{nh}$.

Using the Gibbs bound gives $\mu(Z) \geq (c/2)^{1/\nu(Z)} > 0$ whenever $\nu(Z) > 0$, so $\nu \ll \mu$.

Proof of uniform Katok estimate

Theorem: If the shift space has an ergodic MME μ with the lower Gibbs bound, then μ is the *unique* MME.

Uniform Katok estimate

Upgrade "subexponential" to "uniform"

Theorem: If ν is any MME for a shift satisfying $\#\mathcal{L}_n \leq Qe^{nh}$, and if Z_n is a union of s_n n-cylinders, then $s_n \geq Q(2Q)^{-1/\nu(Z_n)}e^{nh}$.

Proof:

$$nh = h_\nu(\sigma^n) \leq H_\nu(\beta_n) = H_\nu(\zeta_n) + H_\nu(\beta_n \mid \zeta_n)$$

$$\begin{aligned} H_\nu(\beta_n \mid \zeta_n) &= \nu(Z_n)H_\nu(\beta_n|_{Z_n}) + \nu(Z_n^c)H_\nu(\beta_n|_{Z_n^c}) \\ &\leq \nu(Z_n)\log s_n + \nu(Z_n^c)\log \#\mathcal{L}_n \end{aligned}$$

$$\begin{aligned} nh &\leq \log 2 + \nu(Z_n)\log s_n + \nu(Z_n^c)\log \#\mathcal{L}_n \\ 0 &\leq \log 2 + \nu(Z_n)\log(s_n e^{-nh}) + \nu(Z_n^c)\log(\#\mathcal{L}_n e^{-nh}) \\ &\leq \log 2 + \nu(Z_n)\log(s_n e^{-nh}) + (1-\nu(Z_n))\log Q \\ &= \log(2Q) + \nu(Z_n)\log(s_n e^{-nh}Q^{-1}) \end{aligned}$$

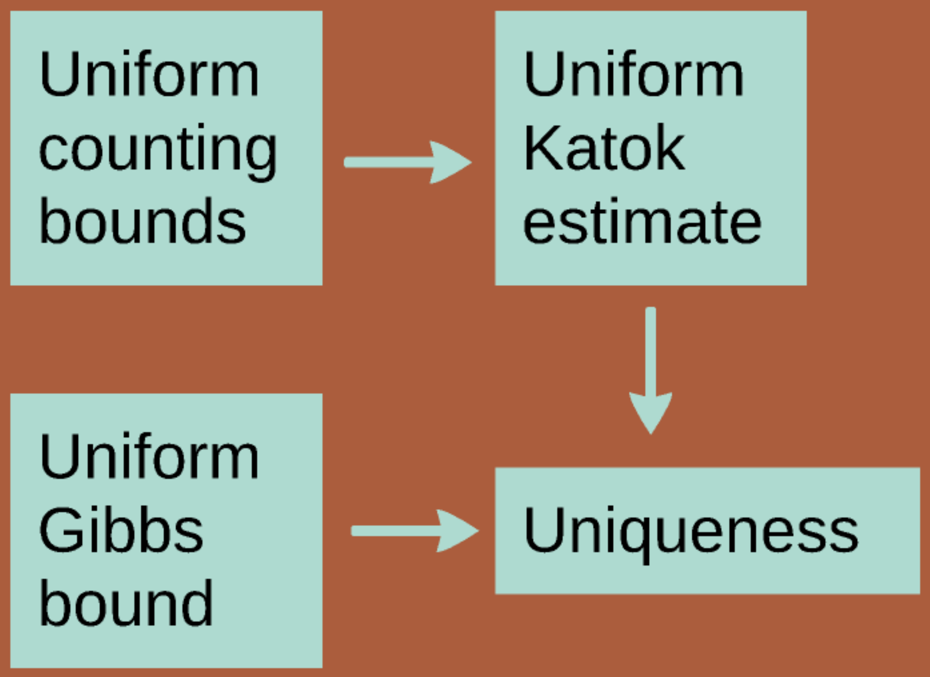
β_n is the partition into n-cylinders

$\zeta_n = \{Z_n, Z_n^c\}$ has $H_\nu(\zeta_n) \leq \log 2$

$H_\nu(\cdot \mid \cdot)$ is conditional entropy

$H_\nu(\beta_n|_{Z_n})$ is entropy of ν restricted to Z_n and nor ali ed

How to get uniformity?



Uniform counting for SFTs

Specification

Lower counting bound:

Natural map $\mathcal{L}_{n+m} \to \mathcal{L}_n \times \mathcal{L}_m$ is injective, so $\#\mathcal{L}_{n+m} \leq (\#\mathcal{L}_n)(\#\mathcal{L}_m)$.

$a_n = \log \#\mathcal{L}_n$ is subadditive: $a_{n+m} \leq a_n + a_m$

$a_{nk} \leq k a_n \Rightarrow \frac{1}{nk} a_{nk} \leq \frac{1}{n} a_n$

Sending $k \to \infty$ gives $a_n \geq nh$, so $\#\mathcal{L}_n \geq e^{nh}$.

Upper counting bound:

Cannot expect $\#\mathcal{L}_{n+m} \geq (\#\mathcal{L}_n)(\#\mathcal{L}_m)$.

Use mixing property to get $\#\mathcal{L}_{n+\tau+m} \geq (\#\mathcal{L}_n)(\#\mathcal{L}_m)$.

Proceed as above to get $\#\mathcal{L}_n \leq Q e^{nh}$.

Fekete's lemma: by subadditivity, $h = \lim \frac{1}{n} a_n$ exists (and = $\inf \frac{1}{n} a_n$)

Mixing SFT: $\tau \in \mathbb{N}$ such that in τ steps, we can get from any symbol to any other symbol

Given any $v \in \mathcal{L}_n$ and $w \in \mathcal{L}_m$, we can find $u \in \mathcal{L}_\tau$ such that $vuw \in \mathcal{L}_{n+\tau+m}$

Obtain injective map $\mathcal{L}_n \times \mathcal{L}_m \to \mathcal{L}_{n+\tau+m}$

Specification

A shift space has the **specification property** if there is $\tau \in \mathbb{N}$ such that for every $v, w \in \mathcal{L}$, there is $u \in \mathcal{L}_\tau$ such that $vuw \in \mathcal{L}$.

True for mixing SFTs.
Gives uniform counting bounds.

Proposition: Uniform counting bounds and specification give uniform Gibbs bounds via the Misiurewicz construction.

Idea of proof: Control $\sigma_k^* m_n[w]$ by estimating the number of words of length n that see the word w starting in position k.

$$\mathcal{L} = \bigcup_n \mathcal{L}_n$$

Equivalently, for every $w^1, \ldots, w^k \in \mathcal{L}$, there are $u^i \in \mathcal{L}_\tau$ such that $w^1 u^1 w^2 u^2 \cdots u^{k-1} w^k \in \mathcal{L}$.

Theorem (Bowen): Every shift space with specification has a unique MME.

Non-uniform specification

Climenhaga-Thompson:
Can use a weaker version of specification and still get uniqueness

First applied to beta-shifts, S-gap shifts

Also geodesic flow in nonpositive curvature, Lorenz attractor, and more

Big idea:
If "obstructions" have small entropy, uniform bounds still hold

Decompositions

Uniform counting bounds (still)

Uniqueness

Decomposing the language

Let X be a shift space with language $\mathcal{L}$.

A **decomposition** of $\mathcal{L}$ consists of $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ such that given any $w \in \mathcal{L}$, there are $u^{p,s} \in \mathcal{C}^{p,s}$ and $v \in \mathcal{G}$ satisfying $w = u^p v u^s$.

Say that $\mathcal{G}$ has **specification** if there is $\tau \in \mathbb{N}$ such that given any $w^1, \dots, w^k \in \mathcal{G}$, there are $u^i \in \mathcal{L}_\tau$ such that $w^1 u^1 w^2 u^2 \cdots u^{k-1} w^k \in \mathcal{L}$.

Define $h(\mathcal{C}^p \cup \mathcal{C}^s) = \overline{\lim}\, \frac{1}{n} \log \#(\mathcal{C}^p_n \cup \mathcal{C}^s_n)$, think of this as "entropy of obstructions to specification".

Suppose we can get $h(\mathcal{C}^p \cup \mathcal{C}^s) < h$...

Every word in $\mathcal{L}$ can be transformed into a "good" word (in $\mathcal{G}$) by removing a prefix from $\mathcal{C}^p$ and a suffix from $\mathcal{C}^s$.

Example: Given $S \subset \mathbb{N}$ infinite, the S-gap shift is $X \subset \{0,1\}^{\mathbb{Z}}$ defined by forbidding all words 10^n1 with $n \notin S$. One decomposition is

$$\mathcal{C}^p = \{0^n : n \geq 0\}$$

$$\mathcal{G} = \{10^{n_1}10^{n_2} \cdots 10^{n_k} : n_i \in S\}$$

$$\mathcal{C}^s = \{10^n : n \geq 0\}$$

Here $\mathcal{G}$ has specification (with $\tau = 0$) and $h(\mathcal{C}^p \cup \mathcal{C}^s) = 0$.

Uniform counting

Assume: decomposition such that $\mathcal{G}$ has specification and $h(\mathcal{C}^p \cup \mathcal{C}^s) < h$.

Earlier proofs give $\#\mathcal{L}_n \geq e^{nh}$ and $\#\mathcal{G}_n \leq Qe^{nh}$

Let $c_n = \#(\mathcal{C}_n^p \cup \mathcal{C}_n^s)e^{-nh}$, then $\sum c_n < \infty$, and

$$\#\mathcal{L}_n \leq \sum_{i+j+k=n} (\#\mathcal{C}_i^p)(\#\mathcal{G}_j)(\#\mathcal{C}_k^s)$$

$$\leq \sum_{i+j+k=n} (c_i e^{ih})(Qe^{jh})(c_k e^{kh})$$

$$= Qe^{nh} \sum_{i+j+k=n} c_i c_k \leq Qe^{nh} \sum_{i=0}^{\infty} c_i \sum_{k=0}^{\infty} c_k$$

We conclude that $\#\mathcal{L}_n \leq Q\Sigma^2 e^{nh}$.

$\overline{\lim}\, \frac{1}{n} \log c_n = h(\mathcal{C}^p \cup \mathcal{C}^s) - h < 0$
so c_n decays exponentially fast

Decomposition map $\mathcal{L}_n \to \bigcup_{i+j+k=n} \mathcal{C}_i^p \times \mathcal{G}_j \times \mathcal{C}^s$

What about uniform Gibbs?
This gives a Gibbs bound for the constructed MME, but only for "good" words

Uniqueness

Theorem: Let X be a shift space whose language $\mathcal{L}$ has a decomposition $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s$ such that

(1) $\mathcal{G}$ has specification

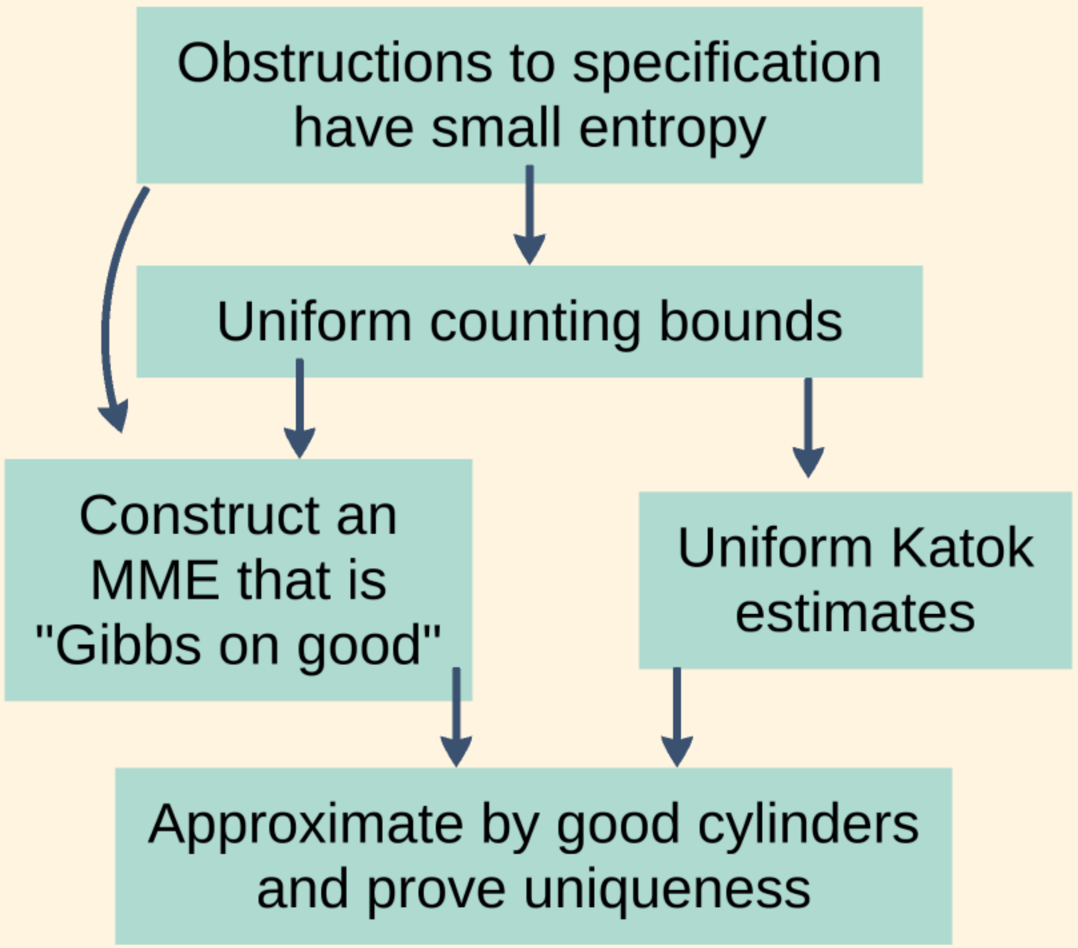
(2) $h(\mathcal{C}^p \cup \mathcal{C}^s) < h$

Then X has a unique MME.

Original proof by Climenhaga–Thompson required an extra condition. Recently Pacifico–Yang–Yang showed that this can be removed.

In the proof of uniqueness, an important step was “approximate Z by Z_n, a union of n-cylinders, and use the Gibbs property”. This must be done more carefully here because the Gibbs property only applies to **some** n-cylinders.

“The collection $\mathcal{G}^M := \{u^p v u^s : |u^p|, |u^s| \le M\}$ has specification for every $M \in \mathbb{N}$”



Topological/smooth dynamics

X a compact metric space, $f : X \to X$ continuous

Fix $\epsilon > 0$, replace cylinder $[x_1 \cdots x_n]$ with **Bowen ball** $B_n(x, \epsilon) = \{y \in X : d(f^k x, f^k y) < \epsilon \text{ for all } 0 \leq k < n\}$

$E \subset X$ is (n, ϵ)-separated if $B_n(x, \epsilon) \cap E = \{x\}$ for all $x \in E$

Replace $\#\mathcal{L}_n$ with $\Lambda_n^\epsilon := \max\{\#E : E \text{ is } (n, \epsilon)\text{-separated}\}$

Topological entropy: $h = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log \Lambda_n^\epsilon$

Need to remove this limit

Expansivity

Specification

The guts of the proof

A non-uniform result

Big idea:
Dictionary between symbolic and non-symbolic settings **if** we can work at a fixed scale

Expansivity

Let $h^\epsilon := \overline{\lim_{n\to\infty}} \frac{1}{n} \log \Lambda_n^\epsilon$, so $h = \lim_{\epsilon\to 0} h^\epsilon$.

Misiurewicz construction: let m_n be equidistributed on a maximal (n, ϵ)-separated set, and proceed as before. Limit measure μ has $h_\mu \geq h^\epsilon$.

Definition: $f: X \to X$ is **expansive** up to scale $\epsilon > 0$ if for every $x \neq y$ there is n such that $d(f^n x, f^n y) \geq \epsilon$.

Roughly speaking, all information makes it to scale ϵ, and we have $h^\delta = h^\epsilon$ for all $\delta \in (0, \epsilon]$, so $h = h^\epsilon$

In particular, Misiurewicz construction gives an MME.

Entropy at scale ϵ

Symbolic case:
equidistributed on n-cylinders

One-sided: $n \geq 0$ (appropriate if non-invertible)
Two-sided: $n \in \mathbb{Z}$ (appropriate if invertible)

We consider one-sided case for simplicity. Then expansive iff $\bigcap_{n\geq 0} B_n(x, \epsilon) = \{x\}$ for all $x \in X$.

Arguments and estimates are done with finite values of ϵ and n. Expansivity and uniform counting estimates guarantee that working with these finite values gives us a complete enough picture; sending $\epsilon \to 0$ and $n \to \infty$ doesn't override what we find out for fixed ϵ, n.

Specification

Dictionary: “replace cylinders by Bowen balls”

How to write specification in terms of cylinders?

Given $v \in \mathcal{L}_n$ and $w \in \mathcal{L}_m$, TFAE:

- $\exists u \in \mathcal{L}_\tau$ such that $vuw \in \mathcal{L}$
- $\exists u \in \mathcal{L}_\tau$ and $x \in X$ such that $x \in [vuw]$
- $\exists x \in X$ such that $x \in [v]$ and $\sigma^{n+\tau}(x) \in [w]$

Writing $[v] = B_n(y, \delta)$ and $[w] = B_m(z, \delta)$, can rewrite:

- $\exists x \in X$ s.t. $x \in B_n(y, \delta)$ and $\sigma^{n+\tau}(x) \in B_m(z, \delta)$

In non-symbolic systems, going from 1-step to multistep requires some expansion/contraction

"space of orbit segments"

$f\colon X \to X$ has **specification** down to scale $\delta > 0$ if there is $\tau \in \mathbb{N}$ such that: for every $(x_1, n_1), \ldots, (x_k, n_k) \in X \times \mathbb{N}$, there is $y \in X$ such that writing $s_j = \sum_{i=1}^{j-1}(n_i + \tau)$, we have $f^{s_j}(y) \in B_{n_j}(x, \delta)$ for each j.

Can δ-shadow anything using gaps of length τ

Theorem (Bowen): Let X be a compact metric space and $f\colon X \to X$ a continuous map with expansivity and specification. Then (X, f) has a unique MME.

up to scale $\epsilon > 40\delta$

down to scale δ

Technical irritants

Try to run the symbolic arguments through the dictionary

Proof of lower counting bound $\#\mathcal{L}_n \geq e^{nh}$ relied on submultiplicativity: use the injective map $\mathcal{L}_{n+k} \to \mathcal{L}_n \times \mathcal{L}_k$ to deduce that $\#\mathcal{L}_{n+k} \leq (\#\mathcal{L}_n)(\#\mathcal{L}_k)$

Let E_n^ϵ be a maximal (n, ϵ)-separated set, with $\Lambda_n^\epsilon = \#E_n$.

Direct analogue of symbolic argument: $E_{n+k}^\epsilon \to E_n^\epsilon \times E_k^\epsilon$ taking x to (y, z) such that $x \in B_n(y, \epsilon)$ and $f^n(x) \in B_k(z, \epsilon)$.

Might not be injective! To guarantee injectivity, we need to instead consider $E_{n+k}^{2\epsilon} \to E_n^\epsilon \times E_k^\epsilon$.

This cannot be iterated, so do it all at once: $E_{nk}^{2\epsilon} \to (E_n^\epsilon)^k$

Similar "scale-changing" is necessary for specification-based arguments

Proof of uniqueness relied on approximating Z by Z_n, a union of n-cylinders, and using Gibbs bound on each cylinder.

Approximation relies on cylinders forming a partition.

Bowen balls do not form a partition.

Construct and use a partition α_n such that

- $\alpha_n = \{A_1, A_2, \ldots, A_L\}$
- $E_n^{2\epsilon} = \{x_1, x_2, \ldots, x_L\}$ (in fact $L = \Lambda_n^{2\epsilon}$)
- $B_n(x_i, \epsilon) \subset A_i \subset B_n(x_i, 2\epsilon)$ for each i

Such a partition is called **adapted**.

This leads to yet more scale-changing

Obstructions to expansivity

The **non-expansive set** at scale $\epsilon > 0$ is
$\mathrm{NE}(\epsilon) = \{x \in X : \bigcap_{n\geq 0} B_n(x,\epsilon) \neq \{x\}\}$.

(X, f) is expansive up to scale ϵ iff $\mathrm{NE}(\epsilon) = \emptyset$.

Entropy of obstructions to expansivity at scale ϵ:
$h^\perp(\epsilon) = \sup\{h_\nu : \nu \text{ an inv. prob. meas., } \nu(\mathrm{NE}(\epsilon)) = 1\}$.

Proposition: If $h^\perp(\epsilon) < h$, then $h^\epsilon = h$.

A **decomposition** consists of $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset X \times \mathbb{N}$ such that given any $(x, n) \in X \times \mathbb{N}$, there are $p, g, s \in \mathbb{N}$ with $p + g + s = n$ and

$$(x, p) \in \mathcal{C}^p, \qquad (f^p x, g) \in \mathcal{G}, \qquad (f^{p+g} x, s) \in \mathcal{C}^s.$$

Thus Misiurewicz construction gives an MME

(With Pacifico-Yang-Yang improvement)

Theorem (Climenhaga–Thompson): Let X be a compact metric space and $f : X \to X$ continuous. Suppose $\epsilon > 40\delta > 0$ are such that $h^\perp(\epsilon) < h$ and that there is a decomposition satisfying

(1) $\mathcal{G}$ has specification at scale δ, and

(2) $h^\delta(\mathcal{C}^p \cup \mathcal{C}^s) < h$.

Then (X, f) has a unique MME.

Applications

The strategy is always to identify the obstructions to expansivity and specification, and then find a way to control their entropy

Symbolic examples

Partial hyperbolicity

Non-uniform hyperbolicity

Symbolic examples

Beta-shifts, S-gap shifts, and factors
(C.-T., *Israel Journal*, 2012)

Many shifts of quasi-finite type
(C., *Comm. Math. Phys.*, 2018)

S-limited shifts
(Matson-Sattler, *Real An. Exch.*, 2018)

1-sided almost specification
(C.-Pavlov, *ETDS*, 2019)

Negative beta shifts
(Shinoda-Yamamoto, *Nonlin*, 2020)

S-graph shifts
(Dillon, *DCDS*, 2022)

Partial hyperbolicity and dominated splittings

Bonatti-Viana examples
(C.-Fisher-T., *Nonlinearity*, 2018)

Mañe examples
(C.-Fisher-T., *ETDS*, 2019)

Certain partially hyperbolic attractors
(Fisher-Oliveira, *Nonlinearity*, 2020)

Katok example
(Tianyu Wang, *ETDS*, 2021)

Lorenz attractor, sectional-hyperbolic flows
(Pacifico, Fan Yang, Jiagang Yang,
Nonlinearity 2022 and arXiv:2209.10784)

Non-uniform hyperbolicity

Geodesic flows

Non-uniformly hyperbolic geodesic flows

Non-positive curvature
(Burns-C.-Fisher-T., GAFA, 2018)

No focal points
(Chen-Kao-Park, *Nonlinearity* 2020 and *Advances* 2021)

Surfaces with no conjugate points
(C.-Knieper-War, *Advances*, 2021)

CAT(-1) spaces
(Constantine-Lafont-T., *Groups Geom. Dyn.*, 2020)

Flat surfaces with singularities
(Call-Constantine-Erchenko-Sawyer-Work, *IMRN*, 2022)

Specification and measures of maximal entropy

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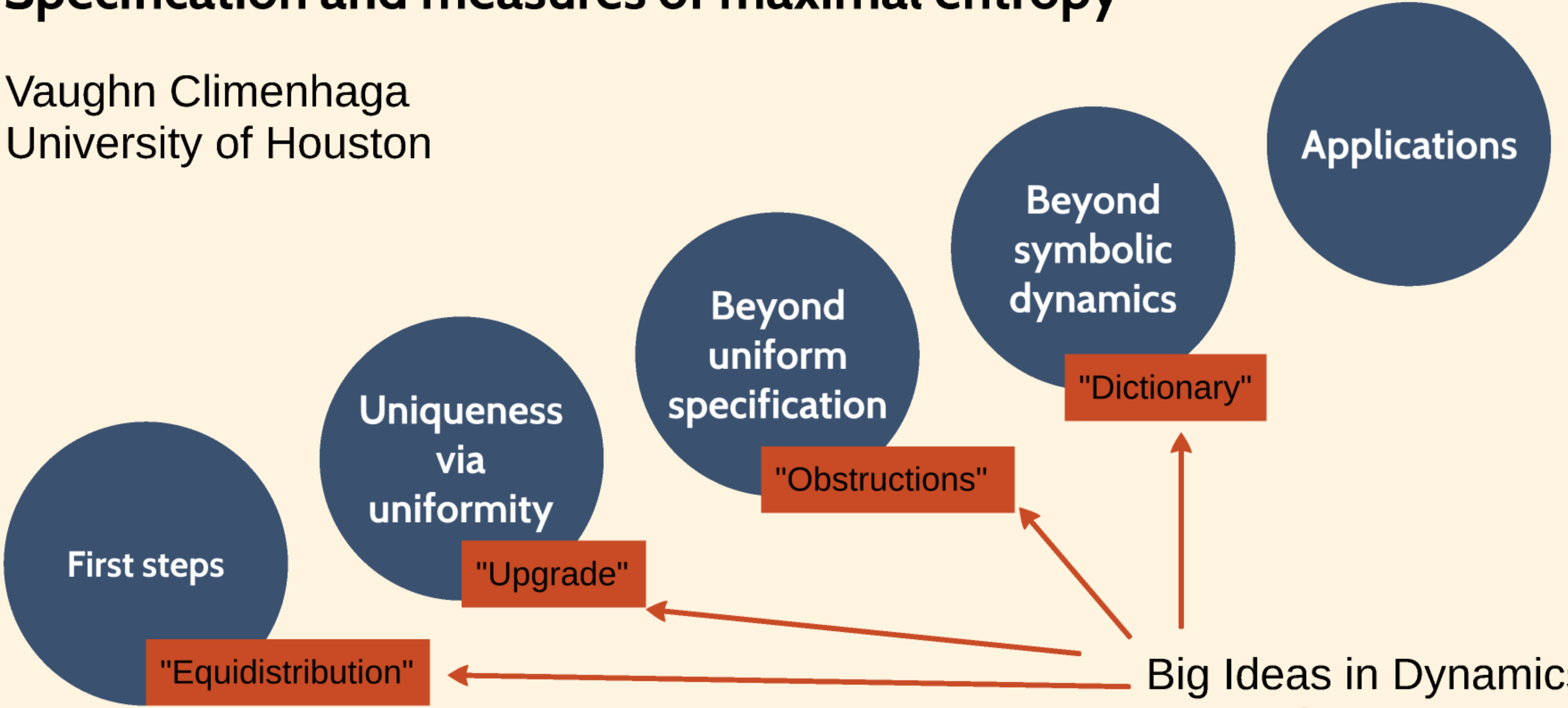

Big Ideas in Dynamics
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