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THERMODYNAMIC FORMALISM AND MULTIFRACTAL ANALYSIS FOR GENERAL TOPOLOGICAL DYNAMICAL SYSTEMS

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Abstract

We investigate to what degree results in dimension theory and multifractal formalism can be derived as a direct consequence of thermodynamic properties of a dynamical system. We show that under quite general conditions, various multifractal spectra (the entropy spectrum for Birkhoff averages and the dimension spectrum for pointwise dimensions, among others) may be obtained as Legendre transforms of functions $T\colon \mathbb{R} \to \mathbb{R}$ arising in the thermodynamic formalism. We impose minimal requirements on the maps we consider, and obtain partial results for any continuous map f on a compact metric space. In order to obtain complete results, the primary hypothesis we require is that the functions T be continuously differentiable. This makes rigorous the general paradigm of reducing questions regarding the multifractal formalism to questions regarding the thermodynamic formalism. These results hold for a broad class of measurable potentials, which includes (but is not limited to) continuous functions. We give applications that include most previously known results, as well as some new ones.

Along the way, we show that Bowen's equation, which characterises the Hausdorff dimension of certain sets in terms of the topological pressure of an expanding conformal map, applies in greater generality than has been heretofore established. In particular, we consider an arbitrary subset Z of a compact metric space and require only that the lower Lyapunov exponents be positive on Z, together with a tempered contraction condition. Among other things, this allows us to compute the dimension spectrum for Lyapunov exponents in terms of the entropy spectrum for Lyapunov exponents, and is also a crucial tool in the aforementioned results on the dimension spectrum for local dimensions.

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List of Symbols

$\wedge$	Common prefix of two points in symbolic space, p. 69
$\alpha_{\min}, \alpha_{\max}$	Minimum and maximum values of local quantities, p. 31, 86, 89
A	A map associated with the Legendre transform, p. 30
$\mathcal{A}(E)$	Set of points with Lyapunov exponents in the set $E \subset \mathbb{R}$, p. 66
$\mathcal{A}_f$	Set of “nearly continuous” potential functions, p. 31
$a(x)$	Conformal derivative of f at x, p. 65
$\mathbf{B}$	Set of points with tempered contraction, p. 66
$\mathcal{B}(\alpha)$	Birkhoff spectrum, p. 11
$B(x, n, \delta)$	The Bowen ball centred at x of order n and radius δ, p. 19
$\mathcal{C}(\varphi)$	Set of points at which φ is discontinuous, p. 31
$\underline{Ch}_{\text{top}}(Z), \overline{Ch}_{\text{top}}(Z)$	Lower and upper capacity topological entropies of Z, p. 19
$\underline{CP}_Z(\varphi), \overline{CP}_Z(\varphi)$	Lower and upper capacity topological pressures of φ on Z, p. 20
D^-, D^+	One-sided derivatives, p. 30
$\mathcal{D}(\alpha)$	Dimension spectrum, p. 14, 83
$\underline{d}_\mu(x), \overline{d}_\mu(x)$	Lower and upper pointwise dimensions of μ at x, p. 23

$\mathcal{D}(Z,\varepsilon)$	Open covers of Z by sets of diameter $\leq \varepsilon$, p. 17
$\mathcal{D}^b(Z,\varepsilon)$	Open covers of Z by balls of radius $\leq \varepsilon$, p. 18
$\dim(\mu)$	The dimension of the measure μ, p. 4
$\underline{\dim}_B(Z), \overline{\dim}_B(Z)$	Lower and upper box dimensions of Z, p. 19
$\dim_H(Z)$	Hausdorff dimension of Z, p. 18
$\dim_H^b(Z)$	Hausdorff dimension of Z defined using covers by balls with weights according to diameter, p. 18
$\dim_H^{b'}(Z)$	Hausdorff dimension of Z defined using covers by balls with weights according to radius, p. 18
$\dim_H(\mu)$	Hausdorff dimension of a measure μ, p. 22
$\underline{\dim}_\mu(x), \overline{\dim}_\mu(x)$	Local and upper pointwise quantities for an arbitrary dimensional quantity, p. 24
$\mathcal{E}(\alpha)$	Entropy spectrum, p. 14, 83
E_n	A maximal (n,δ)-separated set, p. 26
$F_\alpha^{\varepsilon,N}(\varphi), F_\alpha^\varepsilon(\varphi)$	"Approximate level sets" for the Birkhoff averages of φ, p. 48
$G_\alpha^{\varepsilon,N}, G_\alpha^\varepsilon$	"Approximate level sets" for the pointwise dimensions of μ, p. 101
$h(\mu)$	Entropy of a measure μ (defined as a dimension), p. 23
$\underline{h}_\mu(x), \overline{h}_\mu(x)$	Lower and upper local entropies of μ at x, p. 23
$h_{\text{top}}(Z)$	Carathéodory topological entropy of Z, p. 20
$I_A(h)$	Set of values of α for which the Legendre transform of T is larger than h, p. 35
$I_Q(h)$	Set of values of q that correspond to a value of the Legendre transform larger than h, p. 36
$K_\alpha^{\mathcal{B}}$	Level sets for Birkhoff averages, p. 11
$K_\alpha^{\mathcal{D}}$	Level sets for pointwise dimensions, p. 14

$K_\alpha^{\mathcal{E}}$	Level sets for local entropies, p. 14
$K_\alpha^{\mathcal{L}}$	Level sets for Lyapunov exponents, p. 13
$\underline{\lambda}(x), \overline{\lambda}(x), \lambda(x)$	(Lower and upper) Lyapunov exponent at x, p. 65
$\lambda_n(x)$	Approximate Lyapunov exponents at x, p. 65
L_1, L_2, L_3, L_4	Various versions of the Legendre transform, p. 27, 29, 86
$\mathcal{L}_D(\alpha), \mathcal{L}_E(\alpha)$	Multifractal spectra for Lyapunov exponents, p. 14, 40
μ_n	An atomic measure supported on n-trajectories of an (n, δ)-separated set, p. 26, 49
$\mathcal{M}(X)$	The set of probability measures on X, p. 25
$\mathcal{M}^f(X)$	The set of invariant probability measures on X, p. 25
$\mathcal{M}_E^f(X)$	The set of ergodic invariant probability measures on X, p. 25
$m_h(\cdot, s, \delta)$	Outer measure at scale δ for Carathéodory topological entropy, p. 20
$m_H(\cdot, s)$	s-dimensional Hausdorff outer measure, p. 17
$m_H^b(\cdot, s)$	s-dimensional Hausdorff outer measure defined using covers by balls weighted according to diameter, p. 18
$m_H^{b'}(\cdot, s)$	s-dimensional Hausdorff outer measure defined using covers by balls weighted according to radius, p. 18
$m_P(\cdot, s, \varphi, \delta)$	Outer measure at scale δ for Carathéodory topological pressure of the potential φ, p. 21
$N(Z, r)$	Minimal cardinality of an r-dense subset of Z, p. 18
$P_\mu(\varphi)$	Pressure of a potential φ for a measure μ (defined as a dimension), p. 23
$\underline{P}_\mu(x), \overline{P}_\mu(x)$	Lower and upper local pressures for a measure μ at x, p. 23
P_n^δ	Maximum cardinality of an (n, δ)-separated set, p. 50
$P_Z(\varphi)$	Carathéodory topological pressure of φ on Z, p. 21

$P_Z^*(\varphi)$	Variational pressure of φ on Z, p. 25
$\mathcal{P}(Z,N,\delta)$	Set of covers of Z by Bowen balls of length $\geq N$ and radius δ, p. 19
φ_1	Centred potential function, p. 14
φ_q	One-parameter family of potential functions $\varphi_{q,T_{\mathcal{D}}(q)}$, p. 88
$\varphi_{q,t}$	Two-parameter family of potential functions for implicit definition of $T_{\mathcal{D}}$, p. 87
$R_\eta(I_Q)$	Domain in $\mathbb{R}^2$ lying just under the graph of $T_{\mathcal{D}}$, p. 88
Q	A map associated with the Legendre transform, p. 30
Q_n^δ	Cardinality of a minimal (n,δ)-spanning set, p. 19
$R_n^\delta(\varphi)$	Partition function for φ, p. 20
σ_n	An atomic measure supported on a maximal (n,δ)-separated set, p. 26, 49
Σ_n^+	One-sided shift space on n symbols, p. 33
$\mathcal{S}_D(\alpha),\mathcal{S}_E(\alpha)$	Arbitrary multifractal spectra, p. 8
$S_n\varphi$	nth Birkhoff sum of φ, p. 11, 20
$T_{\mathcal{B}}(q)$	One-dimensional cross-section of pressure function, p. 11
$T_{\mathcal{D}}(q)$	Implicitly defined cross-section of pressure function, p. 15, 88
$T_{\mathcal{E}}(q)$	One-dimensional cross-section of pressure for centred potential, p. 14
$\hat{X}$	The exceptional set, p. 7
X'	Set of "good" points for dimension results, p. 87
$\mathbf{Z}(\mu)$	Set of points with zero Lyapunov exponent and zero local entropy along some subsequence of times, p. 87

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Epigraph

We must learn to free ourselves from seeing things the way they are!

Per Bak

“Beware the Jabberwock, my son!
    The jaws that bite, the claws that catch!
Beware the Jubjub bird, and shun
    The frumious Bandersnatch!”

He took his vorpal sword in hand:
    Long time the manxome foe he sought—
So rested he by the Tumtum tree,
    And stood awhile in thought.

    . . .

“And hast thou slain the Jabberwock?
    Come to my arms, my beamish boy!
O frabjous day! Callooh! Callay!”
    He chortled in his joy.

Lewis Carroll, “Jabberwocky”

Chapter 1

Introduction and overview

1.1 Asymptotic quantities of dynamical systems

1.1.1 Ergodic theorems

Many of the most important characteristics of a dynamical system are asymptotic quantities that capture a certain aspect of the long-term behaviour of the system. These include (among others):

1. *Birkhoff averages*, the asymptotic means of sequences of observations;

2. *Lyapunov exponents*, the asymptotic growth rates of the magnitudes of small initial errors or uncertainties;

3. *measure-theoretic entropy*, the asymptotic growth rate of the information gained by coarse observation of the system for a fixed period of time;

4. *topological entropy*, the asymptotic growth rate of the number of orbit segments of a fixed length that are distinguishable at a coarse scale.

The first two properties are local, characterising a single trajectory of the system, while the latter two are global, characterising the system as a whole. Broadly speaking, we will be interested in the relationship between local quantities and global quantities in the setting where X is a compact metric space and the dynamical system is given by a map $f\colon X \to X$.

The story begins with Birkhoff's ergodic theorem, which shows that Birkhoff averages may be thought of as characteristics of a measure, not just a trajectory: if μ is an ergodic invariant probability measure on X and $\varphi \in L^1(X, \mu)$ is an observable, then for μ-a.e. point x, the average value of φ along a finite trajectory of x converges to $\int \varphi \, d\mu$ as the length of the trajectory goes to infinity.

For Lyapunov exponents, a similar result is provided by Oseledets' multiplicative ergodic theorem: if μ is an ergodic invariant measure for a smooth dynamical system on a manifold X, then for μ-a.e. point x, the tangent space admits a splitting into Lyapunov subspaces, on each of which the Lyapunov exponent exists and is independent of x. Thus the set of Lyapunov exponents of the system is a property of the measure μ, not just a pointwise property.

Both of these results show that certain locally defined quantities carry a global meaning. The story continues for measure-theoretic entropy: both the Shannon–McMillan–Breiman theorem and the Brin–Katok entropy formula [BK83] show that the measure-theoretic entropy of an ergodic invariant probability measure μ is equal to the almost-everywhere value of a certain *local entropy* at x. This local quantity is defined as the asymptotic rate of decay of the measure of the set of points whose trajectory is indistinguishable from the trajectory of x (at a coarse scale) for some finite length of time; it may be interpreted as the average rate at which information is gained if we observe a trajectory that shadows the trajectory of x.

Thus for each of the first three quantities listed above, there are results in ergodic theory that allow us to interpret the quantity at either a local level (as a property of a single trajectory) or a global level (as a property of a measure). What about the fourth quantity, the topological entropy?

1.1.2 Dimension theory

A deeper understanding of topological entropy pushes us in a different direction than the ergodic results in the previous section. Here, the key insight dates back to Bowen [Bow73], who introduced a definition of topological entropy for arbitrary sets $Z \subset X$. When Z is compact and invariant, this definition agrees with the usual one; as we will later see, it has many advantages over the classical definition in the case when Z is not necessarily compact or invariant.

Bowen's definition of topological entropy mirrors the definition of *Hausdorff dimension*, and accordingly invites us to think of entropy as a dimensional characteristic, writing $h_{\text{top}}(Z)$ and speaking of the entropy of the set Z (with respect to the underlying dynamics f) rather than the entropy of f (on the underlying phase space X).

This shift in focus is of profound importance for our purposes here; following the paradigm laid out by Pesin in [Pes98], we work with topological entropy and its more sophisticated sibling, *topological pressure*, as specific examples of a broad class of *Carathéodory dimension characteristics*. This equips us with a powerful set of tools to examine the relationship between the local and global quantities mentioned above.

Precise definitions of all these dimensional quantities will be given in Chapter 2. For the time being, we give a broad description of two types of dimensional quantities, of which Hausdorff dimension and box dimension are the canonical examples, and then we describe some of the ways in which they let us connect local and global quantities.

Quantities of the first type, which we will refer to as *Carathéodory dimensions*, are defined as follows. A one-parameter family of set functions $Z \mapsto m(Z,t)$ is constructed, having the property that for every $Z \subset X$ there is a critical value $t = t_c$ such that $m(Z,t) = \infty$ for $t < t_c$ and $m(Z,t) = 0$ for $t > t_c$. This critical value is the Carathéodory dimension of Z. By choosing the set functions $m(\cdot,t)$ appropriately, this construction can give the Hausdorff dimension, Bowen's topological entropy, topological pressure in the sense of Pesin and Pitskel' [PP84], and a variety of other dimensional characteristics.

Quantities of the second type, which we will refer to as *capacities*, are defined differently.[1] Given a characteristic scale $r > 0$, some quantity $E(Z,r)$ is introduced to characterise how "large" Z appears at that scale. The growth rate $\lim_{r\to 0} \log E(Z,r)/\log(1/r)$ is the capacity of Z. By choosing the quantity $E(Z,r)$ appropriately, this construction can give the box dimension, the classical definitions and topological entropy and pressure, and other quantities as well.

[1] In fact, the key difference between the two types of quantities is slightly different than what is stated here, but for the examples we consider, this description is accurate.

Broadly speaking, the relationship between the two types of quantities is as follows: Carathéodory dimensions are generally more well-suited to quantify the sets in which we will be interested (in particular, sets on which some local quantity converges asymptotically but non-uniformly), while capacities are easier to work with and can be used to obtain effective upper bounds for Carathéodory dimensions of certain sets (in particular, sets on which that convergence is uniform).

1.1.3 Dimensions of measures

Once topological entropy is understood as a dimensional quantity, the local and global measure-theoretic entropies can also be understood in this framework. Given an arbitrary Carathéodory dimensional quantity $\dim(Z)$ (which may be Hausdorff dimension, topological entropy, or something else entirely), a standard procedure for defining the dimension of a measure is given in [Pes98]:

$$\dim(\mu) = \inf\{\dim(Z) \mid \mu(Z) = 1\}.$$

When the dimension in question is topological entropy in Bowen's sense, it can be shown that this definition gives the measure-theoretic entropy of an ergodic measure μ. Furthermore, there is a standard way of defining a local dimensional quantity $\dim_\mu(x)$: for topological entropy, $\dim_\mu(x)$ is exactly the local entropy that appears in the Brin–Katok entropy formula.

The application of dimension theory to ergodic invariant measures gives us important insights into the relationships between various local and global asymptotic quantities. One key fact is that local information can give global results.

1. If $\dim_\mu(x)$ exists and is constant μ-almost everywhere, then the common value is equal to $\dim(\mu)$.
2. If $\dim_\mu(x)$ exists and is constant *everywhere* on a set $Z \subset X$ with $\mu(Z) > 0$, then the common value is equal to $\dim(Z)$.

Another key fact is that at the local level, the relationship between various asymptotic quantities becomes relatively transparent. In particular, we are occasionally interested in the *pointwise dimensions* of a measure, which give the rate of decay of $\mu(B(x,r))$ as $r \to 0$. For a conformal map f and a point x with positive Lyapunov exponent, it is not too difficult to prove that

$$\text{pointwise dimension} = \frac{\text{local entropy}}{\text{Lyapunov exponent}}. \tag{1.1}$$

More sophisticated versions of this relationship in the non-conformal case and for the corresponding global (measure-theoretic) quantities are at the heart of important results by Margulis, Ruelle, Pesin, and Ledrappier–Young. In a different direction, the use of the local quantities in (1.1) to relate global (setwise) dimensional properties is the key idea in results of Bowen [Bow79] and Ruelle [Rue82] on Hausdorff dimension of conformal repellers.

1.2 Multifractal analysis

1.2.1 Motivations: non-generic points

The various ergodic theorems in the opening section all have the same general form: given an ergodic invariant measure μ and a certain finite time local quantity $\psi_n(x)$, the corresponding asymptotic quantity $\lim_{n\to\infty} \frac{1}{n}\psi_n(x)$ exists and is constant on a set Z such that $\mu(Z) = 1$.

1. *Birkhoff averages*: Given an observable φ, the finite time local quantity is $\psi_n(x) = \varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{n-1}(x))$.

2. *Lyapunov exponents*: Given a vector $v \in T_xM$, the finite time local quantity is $\psi_n(x) = \log \|Df_x^n(v)\|$.

3. *Local entropies*: The finite time local quantity is $\psi_n(x) = -\log \mu(B_n(x))$, where $B_n(x)$ is either an n-cylinder relative to some partition (Shannon–McMillan–Breiman) or a Bowen ball of length n (Brin–Katok).

What these theorems do not tell us is the asymptotic behaviour of these quantities for points x that lie outside the set Z. It is the study of these *non-generic* points that lies at the heart of the multifractal analysis and of the present work.

As such points make up a set with zero measure (if indeed there are any of them at all), it is natural to dismiss them as inconsequential. Nevertheless, there are several reasons to be concerned with these points and to ask for results that hold *everywhere*, not just almost everywhere. One of these is that as mentioned in the previous section, knowledge of local dimensional quantities at every point in a set can give more precise information about the dimension of that set than can almost everywhere knowledge. To give a few other reasons, we temporarily restrict our attention to a specific example, the Birkhoff averages of a continuous function.

First and foremost, if we are given only a compact metric space X, a continuous map $f\colon X \to X$, and a continuous function $\varphi\colon X \to \mathbb{R}$, then it is not at all clear to which invariant measure Birkhoff's ergodic theorem ought to be applied. Certainly there is at least one ergodic invariant measure (thanks to Krylov–Bogolyubov), but for most systems of interest, this measure is not unique, and so before we can apply any ergodic theorems, we must select between all the available measures. If there is no clear way to make this selection, then other methods must be employed to study the asymptotic behaviour of the Birkhoff averages.

There are situations, though, in which X carries a natural measure (or measure class) that is of more interest to us than other measures. If X is a smooth manifold, for example, then we are primarily concerned with measures that are absolutely continuous with respect to Lebesgue measure. If some measure in this class is invariant, then we have a natural measure to which Birkhoff's ergodic theorem may be applied. Even if there is no absolutely continuous invariant measure, one can sometimes prove the existence of a *physical measure* for which Lebesgue-a.e. point x is generic (satisfies the result of Birkhoff's ergodic theorem).

Given the existence of a physical measure, what value is there in studying the asymptotics of the Birkhoff averages on the set of non-generic points, which has zero Lebesgue measure and is thus in some sense "unobservable"?

One answer is pragmatic. If we study a dynamical system as a model of some real-world behaviour, then asymptotic quantities are of interest only insofar as they are approached by finitely determined quantities, which are all we can actually measure. However, the *value* of an asymptotic quantity does not in and of itself reveal how the finitely determined quantity approaches that limit: what the asymptotic rate of convergence is, how long we must wait in order to observe convergence to within some specified error bound, the presence of large deviations, etc. It turns out that many aspects of this behaviour at large but finite times is related to the asymptotic behaviour at non-generic points, and so the study of such points does in fact yield useful information.

The second answer is aesthetic. Even if we disregard all thought of applications, adopt G.H. Hardy's distaste for "usefulness" as a desideratum, and only consider a dynamical system qua mathematical object, we will nevertheless find that there is a systematic theory to the behaviour of asymptotic quantities at non-generic points, and that it is elegant and beautiful. And in the end, what other motivation do we need?

1.2.2 Multifractal spectra

Having provided what must be hoped to be sufficient motivation, we now introduce the primary dramatis personae of the present narrative, beginning with the various *multifractal spectra*. Precise definitions will follow in the appropriate chapters: what we give here is an overview of the guiding principles.

Given a dynamical system $f\colon X \to X$, there are three components of a given multifractal spectrum:

1. the object to be studied, usually a function $\varphi\colon X \to \mathbb{R}$ or a measure μ on X;
2. a local asymptotic quantity associated to that object, such as the Birkhoff averages, the local entropies, or the pointwise dimensions;
3. a global dimensional quantity, such as the topological entropy (in Bowen's sense) or the more familiar Hausdorff dimension.

Given $\alpha \in \mathbb{R}$, we denote by K_α the *level set* of points x at which the local asymptotic quantity takes the value α. Writing $\hat{X}$ for the *exceptional set* of points at which the local asymptotic quantity does not exist (the finite time quantities do not converge), one obtains a *multifractal decomposition*

$$X = \left(\bigcup_{\alpha\in\mathbb{R}} K_\alpha \right) \cup \hat{X}. \tag{1.2}$$

In many examples of interest, the level sets K_α are dense in X, and so from the topological point of view, this decomposition is very intricate.

From the measure-theoretic point of view, on the other hand, this decomposition is very simple. For the Birkhoff averages and the local entropies, the various ergodic theorems cited above imply that every ergodic measure is supported on a single level set. The analogous result for pointwise dimensions of *hyperbolic* measures was proved by Barreira, Pesin, and Schmeling [BPS99].

From the dimensional point of view, this decomposition is surprisingly interesting. We define the *dimension spectrum* and the *entropy spectrum* for the local quantity in question by

$$\mathcal{S}_D(\alpha) = \dim_H(K_\alpha), \qquad \mathcal{S}_E(\alpha) = h_{\text{top}}(K_\alpha). \tag{1.3}$$

Here $h_{\text{top}}(K_\alpha)$ denotes topological entropy in Bowen's sense: because the level sets K_α are dense in many natural situations, box dimension and capacity entropy are of little use in analysing them, as these quantities assign the same value to a set and to its closure.

After the circuitous manner in which these multifractal spectra are defined—as dimensions of level sets of asymptotic local quantities—nothing suggests that their dependence on α should be anything but pathological. Nevertheless, here we encounter one of the great surprises of the theory, the so-called *multifractal miracle*: for a broad class of systems, the multifractal spectra are analytic functions of α! Moreover, they are concave and can be obtained as the *Legendre transform* of convex functions defined at the global level.

This is the central mystery of the subject, and the primary goals of the present work are to elucidate this phenomenon and investigate the generality in which it occurs.

1.3 Overview of results

1.3.1 The key tool: thermodynamic formalism

Direct computation of the various multifractal spectra, numerical or otherwise, is quite difficult. In the first place, in order to determine the level sets K_α explicitly, one needs to first compute the local asymptotic quantity at *every* point of X, which means that ergodic theorems cannot be used. Even if this is accomplished, it still remains to compute the (Bowen) topological entropy or Hausdorff dimension of K_α for every value of α. Because this quantity is a Carathéodory dimension, and hence relies on the computation of a critical point, rather than a growth rate, it is more difficult to compute than the (capacity) topological entropy or the box dimension.

The failure of the direct frontal assault leads us to introduce our second set of protagonists, the convex functions mentioned in the previous section, which sometimes also bear the name of multifractal spectra, but are of a different nature altogether. The spectra $\mathcal{S}_D(\alpha)$ and $\mathcal{S}_E(\alpha)$ are defined in terms of a one-parameter family of sets K_α and a fixed dimensional quantity that measures them. In contrast, these convex functions are defined in terms of a fixed set—the entire phase space X—and a one-parameter family of dimensional quantities $T(q)$.

Included in this list are the *Henschel–Procaccia spectrum*, the *Rényi spectrum*, and the *correlation entropies*, but the most important for our purposes will be the *topological pressure* and various thermodynamic functions derived from it.

The topological pressure is the common thread that binds together the disparate actors in our tale, the lynchpin on which the narrative hangs. It may be thought of as a generalisation of the topological entropy that incorporates information regarding the Birkhoff sums of a potential function φ. The entropy $h_{\text{top}}(Z)$ is defined in terms of orbit segments of finite length that are distinguishable at a fixed coarse scale. By counting each such segment with a weight that depends on the Birkhoff sum of φ along the orbit, one obtains the topological pressure $P_Z(\varphi)$.

As with the entropy, there are two flavours of pressure: the Carathéodory pressure, defined as a critical value of a set function, and the capacity pressure, defined as a growth rate of a certain *partition function*. On compact invariant sets, the two pressures agree; furthermore, on such sets they satisfy a crucial *variational principle*:

$$P_X(\varphi) = \sup_{\mu \in \mathcal{M}^f(X)} \left(h(\mu) + \int \varphi \, d\mu \right). \tag{1.4}$$

Here the supremum is over all f-invariant probability measures, and $h(\mu)$ is the measure-theoretic entropy. A measure μ that achieves the supremum in (1.4) is called an *equilibrium state*.

As the topological pressure will appear time and time again in various incarnations to fill various roles, we take a moment to list a few of its guises here. Because of the central role the pressure plays, this amounts to giving a capsule summary of everything that will come in the rest of the story.

1. If a potential φ is fixed, the set function $Z \mapsto P_Z(\varphi)$ is a dimensional quantity that characterises subsets of X. Conversely, if the set Z is fixed and φ is allowed to vary, the pressure defines a convex function $C(X) \to \mathbb{R}$, where $C(X)$ is the space of all continuous potential functions on X.

2. If $Z = X$ and φ varies, then (1.4) may be interpreted in terms of the *Legendre transform*, a standard tool in thermodynamics. The pressure function $P_X : C(X) \to \mathbb{R}$ is shown by the *variational principle* to be the Legendre transform of the entropy function $h : \mathcal{M}^f(X) \to [0, \infty)$. In Chapter 2, we will see that under certain conditions, the converse is also true: entropy is the Legendre transform of pressure. This well-known duality lies at the heart of our approach to multifractal analysis.

3. If we restrict our attention to the one-dimensional subspace of $C(X)$ spanned by a fixed potential φ, then we obtain a function $T : \mathbb{R} \to \mathbb{R}$ given by $T(q) = P_X(q\varphi)$. In Chapter 3, we will see that the equilibrium states ν_q for the potentials $q\varphi$ can be used to characterise the entropy spectrum for Birkhoff averages by establishing a function $q \mapsto \alpha(q)$ such that $\nu_q(K_{\alpha(q)}) = 1$ and $h_{\text{top}}(K_{\alpha(q)}) = h(\nu_q)$.

4. If we restrict our attention to this subspace and also fix a subset $Z \subset X$, then we obtain *Bowen's equation* $P_Z(q\varphi) = 0$. For a broad class of potentials φ, the root of this equation is equal to a dimensional quantity introduced by Barreira and Schmeling [BS00], denoted $\dim_\varphi(Z)$. In Chapter 4, we will see that if f is *conformal* and φ is the *geometric potential*, then $\dim_\varphi(Z)$ is nothing else but the Hausdorff dimension.

5. One can also consider level sets K_α on which the *ratio* of the Birkhoff sums of two functions φ and ψ converges to α. In this case, we will see in Chapter 5 that if we implicitly define a function T by $P_X(q\varphi + T(q)\psi) = 0$, then the equilibrium states ν_q for the potentials $q\varphi + T(q)\psi$ are supported on the

level sets $K_{\alpha(q)}$ for some function $q \mapsto \alpha(q)$. Using the results from Chapter 4, we will show that if ψ is the geometric potential for a conformal map f and μ is a *Gibbs measure* for φ—a measure for which the local entropies are determined by the Birkhoff averages of φ—then this allows us to characterise the dimension spectrum for pointwise dimensions of μ by showing that $\dim_H(K_{\alpha(q)}) = \dim_H(\nu_q)$.

1.3.2 Deriving multifractal results from thermodynamics

1.3.2.1 The Birkhoff spectrum

Given a compact metric space X, a continuous map $f\colon X \to X$, and a (Borel) measurable function $\varphi\colon X \to \mathbb{R}$, the level sets for Birkhoff averages are

$$K_\alpha^{\mathcal{B}} = K_\alpha^{\mathcal{B}}(\varphi) = \left\{ x \in X \;\middle|\; \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha \right\}, \tag{1.5}$$

where $S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k(x))$ is the nth Birkhoff sum. The *entropy spectrum for Birkhoff averages*, which we will simply refer to as the *Birkhoff spectrum*, is

$$\mathcal{B}(\alpha) = h_{\text{top}}\,(K_\alpha^{\mathcal{B}}), \tag{1.6}$$

where we reiterate that h_{top} is topological entropy in the sense of Bowen. Chapter 3 gives results for the multifractal analysis of the Birkhoff spectrum: these comprise the first half of the results in [Cli10b].

The strongest result is Theorem 3.1.1, which applies to bounded Borel functions $\varphi\colon X \to \mathbb{R}$ for which the closure of the set of discontinuities is given zero weight by every invariant measure; we denote this class of functions by $\mathcal{A}_f$. (Note that $C(X) \subset \mathcal{A}_f$.) For such maps and functions, we show that the function $T_{\mathcal{B}}\colon q \mapsto P_X(q\varphi)$ is the Legendre transform of $\mathcal{B}(\alpha)$, ***without any further restrictions on f and φ***. Furthermore, we show that $\mathcal{B}(\alpha)$ is the Legendre transform of $T_{\mathcal{B}}$, completing the multifractal formalism, ***provided $T_{\mathcal{B}}$ is continuously differentiable and equilibrium measures exist***. If the hypotheses on $T_{\mathcal{B}}$ only hold for certain values of q, we still obtain a partial result on $\mathcal{B}(\alpha)$ for the corresponding values of α.

As stated, Theorem 3.1.1 does not deal with phase transitions—that is, points at which the pressure function is non-differentiable. Such points correspond (via the Legendre transform) to intervals over which the Birkhoff spectrum is affine (if the multifractal formalism holds). In Theorem 3.1.3, we give slightly stronger conditions on the map f, which are still fundamentally thermodynamic in nature, under which we can establish the complete multifractal formalism even in the presence of phase transitions.

It is often the case that thermodynamic considerations demonstrate the existence of a *unique* equilibrium state for certain potentials. In Proposition 3.2.1, we show that if the entropy function is upper semi-continuous, then uniqueness of the equilibrium state implies differentiability of the pressure function and allows us to apply Theorem 3.1.1. However, Example 3.1.2 shows that there are systems for which the pressure function is differentiable, and hence Theorem 3.1.1 can be applied, even though the equilibrium state is non-unique.

One would like to understand for which classes of discontinuous potentials the multifractal formalism holds. Things work well for potentials in $\mathcal{A}_f$ because the weak* topology is the same at any f-invariant measure whether we consider continuous test functions or test functions in $\mathcal{A}_f$.

If there are invariant measures that give positive weight to the closure of the set of discontinuities of φ, then things are more delicate. One needs to establish conditions under which the invariant measures in which we are particularly interested still give zero weight to this set. This is achieved by comparing the topological entropy of this set with the Legendre transform of $T_{\mathcal{B}}$, which allows us to obtain in Theorem 3.1.4 partial results for any measurable potential that is bounded above and below.

Ideally, we would be able to include *unbounded* potentials in these results. In particular, we would like to be able to consider the geometric potential $\varphi(x) = -\log|f'(x)|$ for a multimodal map f; the presence of critical points leads to singularities of φ, and so φ is not bounded above. Theorem 3.1.5 shows that the results of Theorem 3.1.1 still hold for $q \leq 0$ (that is, values of q such that $q\varphi$ is bounded above) and for the corresponding values of α. The question of what happens for $q > 0$ is more delicate and remains open.

1.3.2.2 Bowen's equation

In order to use the techniques of thermodynamic formalism and equilibrium states to examine the Hausdorff dimensions of level sets in a given multifractal decomposition, we need a way of determining Hausdorff dimension using topological pressure. Chapter 4 gives the key result for our purposes, a version of Bowen's equation that appeared in [Cli10a] and is given here as Theorem 4.2.1.

The result is given in the setting where X is a compact metric space and $f\colon X \to X$ is a conformal map. Write $f'(x)$ for the conformal derivative of f—that is, the factor by which f expands or contracts nearby points around x. The precise definition is given in (4.4); two important cases of the definition are when X is an interval, in which case $f'(x)$ is the absolute value of the usual derivative, and when X is a Riemannian manifold, in which case $f'(x)$ is the positive real number such that $Df(x)/f'(x)$ is an isometry.

In this setting, Theorem 4.2.1 shows that for any $Z \subset X$, $\dim_H Z$ is the smallest value of t satisfying $P_Z(-t\log(f')) = 0$, provided the following conditions are met:

1. f has no critical points ($f'(x) = 0$) or singularities ($f'(x) = \infty$) in X;
2. every point $x \in Z$ has positive *lower Lyapunov exponent* ($(f^n)'(x)$ grows exponentially in n);
3. every point $x \in Z$ satisfies a *tempered contraction* condition.

Note that the first condition must hold on all of X, while the latter two only need to hold at points in Z.

One can develop the multifractal formalism for Lyapunov exponents; here the level sets are

$$K_\alpha^{\mathcal{L}} = \left\{ x \in X \,\middle|\, \lim_{n\to\infty} \frac{1}{n} \log(f^n)'(x) = \alpha \right\}. \tag{1.7}$$

It is shown in Theorem 4.2.1 that $P_{K_\alpha^{\mathcal{L}}}(-t\log(f')) = h_{\text{top}}(K_\alpha^{\mathcal{L}}) - t\alpha$, and hence

$$\dim_H(K_\alpha^{\mathcal{L}}) = \frac{h_{\text{top}}(K_\alpha^{\mathcal{L}})}{\alpha}. \tag{1.8}$$

(Note the similarity to (1.1).)

The Lyapunov exponent of a point x is nothing but the Birkhoff average of $\log(f')$ at x, and so $K_\alpha^{\mathcal{L}} = K_\alpha^{\mathcal{B}}(\log(f'))$. Then the entropy spectrum for Lyapunov exponents $\mathcal{L}_E(\alpha)$ is given as a special case of Theorem 3.1.1, and the form of the dimension spectrum for Lyapunov exponents $\mathcal{L}_D(\alpha)$ follows from (1.8).

1.3.2.3 Entropy and dimension spectra of Gibbs measures

There are two other multifractal spectra that are commonly studied: the *entropy spectrum for local entropies*, and the *dimension spectrum for pointwise dimensions*. Where there is no risk of confusion, we will refer to these as simply as the *entropy spectrum* and the *dimension spectrum*, respectively. We discuss them in Chapter 5, where we give results on their multifractal analysis that comprise the second half of the results from [Cli10b].

Both these spectra are characteristics of a fixed measure μ. The level sets for local entropies $h_\mu(x)$ and for pointwise dimensions $d_\mu(x)$ are

$$\begin{aligned} K_\alpha^{\mathcal{E}} &= \{x \in X \mid h_\mu(x) = \alpha\}, \\ K_\alpha^{\mathcal{D}} &= \{x \in X \mid d_\mu(x) = \alpha\}. \end{aligned}$$

The entropy spectrum is given by $\mathcal{E}(\alpha) = h_{\text{top}}\,(K_\alpha^{\mathcal{E}})$, and the dimension spectrum by $\mathcal{D}(\alpha) = \dim_H(K_\alpha^{\mathcal{D}})$.

In order to obtain results on these spectra, we need some way to obtain effective estimates on the small-scale features of μ. In particular, computing $h_\mu(x)$ requires estimates on $\mu(B_n(x))$, where $B_n(x)$ is either an n-cylinder or a Bowen ball, and computing $d_\mu(x)$ requires estimates on $\mu(B(x,r))$ for small values of r.

These estimates are given by the assumption that μ is a *weak Gibbs measure* for a potential function φ, which means that the value of $h_\mu(x)$ is determined by the Birkhoff average of φ along the orbit of x. Consequently, the level sets $K_\alpha^{\mathcal{E}}$ are determined by the level sets $K_\alpha^{\mathcal{B}}(\varphi)$, and Theorem 3.1.1 for the Birkhoff spectrum translates directly into Theorem 5.2.1 for the entropy spectrum. Writing $\varphi_1 = \varphi - P_X(\varphi)$, we find $\mathcal{E}(\alpha)$ as the Legendre transform of the function $T_{\mathcal{E}} \colon q \mapsto P_X(-q\varphi_1)$, ***provided*** $T_{\mathcal{E}}$ ***is continuously differentiable and equilibrium measures exist***.

So far, the three multifractal spectra considered—for Birkhoff averages, Lyapunov exponents, and local entropies—all boiled down to the same result concerning level sets of Birkhoff averages. The final spectrum we consider, the dimension spectrum, is a different sort of beast. Suppose that μ is a Gibbs measure for φ and that f is a conformal map for which we write $\psi(x) = \log f'(x)$. Then it can be shown (in the spirit of (1.1)) that

$$d_\mu(x) = \lim_{n\to\infty} \frac{-(\varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^n(x)))}{\psi(x) + \psi(f(x)) + \cdots + \psi(f^n(x))},$$

and so we are not studying the convergence of a single sequence of Birkhoff sums, but the convergence of the *ratio* of two such sequences.

In this case, the proper way to define a thermodynamic function $T(q)$ that is related to $\mathcal{D}(\alpha)$ by the Legendre transform is not to simply work with the one-dimensional subspace of $C(X)$ spanned by the single potential in question, but to work with the two-dimensional subspace spanned by φ and ψ. In particular, following Pesin and Weiss [PW97], one passes to φ_1 so that $P_X(\varphi_1) = 0$ and then defines $T_\mathcal{D}(q)$ for every $q \in \mathbb{R}$ as the smallest value of t such that

$$P_X(q\varphi_1 - t\psi) = 0.$$

Theorem 5.2.2 shows that if f is conformal without critical points or singularities and if X satisfies the tempered contraction condition mentioned before, then the implicitly defined function $T_\mathcal{D}(q)$ is the Legendre transform of the dimension spectrum $\mathcal{D}(\alpha)$, ***without any further conditions on*** f ***or*** φ. Furthermore, we show that $\mathcal{D}(\alpha)$ is the Legendre transform of $T_\mathcal{D}(q)$, completing the multifractal formalism, ***provided*** $T_\mathcal{D}$ ***is continuously differentiable and equilibrium measures exist***.

1.3.3 Relationship with other results

The key ingredients of the multifractal formalism were introduced in [HJK+86], and the use of the topological pressure as the crucial tool in the analysis goes back to [Ran89]. Since then, the multifractal spectra for many classes of systems, potentials, and measures have been studied. The most complete application of the theory has been to Gibbs measures for Hölder continuous potentials on uniformly hyperbolic systems, using thermodynamic results due to Bowen [Bow75] on existence and uniqueness of equilibrium states.

Other classes of systems have also been studied, typically on a case-by-case basis, following the general principle of obtaining multifractal results from knowledge of the thermodynamics of the system. To date, the general strategy informed by this philosophy has been as follows:

(1) Fix a specific class of systems—uniformly hyperbolic maps, conformal repellers, parabolic rational maps, Manneville–Pomeau maps, multimodal interval maps, etc.

(2) Using tools specific to that class of systems (Markov partitions, specification, inducing schemes), establish thermodynamic results—existence and uniqueness of equilibrium states, differentiability of the pressure function, etc.

(3) Using these thermodynamic results *together with the original toolkit*, study the multifractal spectra, and show that they can be given in terms of the Legendre transform of various pressure functions.

The principal novelty of the results presented in this work is to establish the multifractal formalism for a very general class of systems (including, but not limited to, most known examples) as a direct corollary of the thermodynamic formalism, rendering Step (3) above automatic, and eliminating the need for the use of a specific toolkit to study the multifractal formalism itself. Where these results duplicate known results, the proofs here are in some cases more direct than the original proofs.

We will mention previously known results for specific classes of systems in the later chapters, when we state the theorems that relate to them. For now, we observe that the only other results that address multifractal formalism for arbitrary topological dynamical systems are those recently announced by Feng and Huang in [FH10], which deal with asymptotically sub-additive sequences of potentials, and which imply Theorem 3.1.1 for continuous potentials φ as a special case. However, their results do not apply to the broader class of bounded measurable potentials for which we obtain partial results in Theorems 3.1.1 and 3.1.4, nor do they consider the dimension spectrum. To the best of the author's knowledge, the present results are the first rigorous multifractal results for general discontinuous potentials and for the dimension spectrum of weak Gibbs measures on arbitrary conformal systems.

Chapter 2

Dimension theory and thermodynamic formalism

2.1 Definitions

2.1.1 Dimensions of sets

In this chapter, we recall the definitions of Hausdorff and box dimension, topological entropy, and topological pressure in the general framework of Carathéodory dimension characteristics introduced by Pesin in [Pes98]. We begin with the Hausdorff and box dimensions, which are defined without reference to any underlying dynamics, before moving on to the definitions at the heart of the thermodynamic formalism.

Definition 2.1.1. Let X be a separable metric space. Given $Z \subset X$ and $\varepsilon > 0$, let $\mathcal{D}(Z, \varepsilon)$ denote the collection of countable open covers $\{U_i\}_{i=1}^{\infty}$ of Z for which $\operatorname{diam} U_i \leq \varepsilon$ for all i. For each $s \geq 0$, consider the set functions

$$m_H(Z, s, \varepsilon) = \inf_{\mathcal{D}(Z,\varepsilon)} \sum_{U_i} (\operatorname{diam} U_i)^s, \tag{2.1}$$

$$m_H(Z, s) = \lim_{\varepsilon \to 0} m_H(Z, s, \varepsilon). \tag{2.2}$$

The *Hausdorff dimension* of Z is

$$\dim_H Z = \inf\{s > 0 \mid m_H(Z,s) = 0\} = \sup\{s > 0 \mid m_H(Z,s) = \infty\}.$$

It is straightforward to show that $m_H(Z,s) = \infty$ for all $s < \dim_H Z$, and that $m_H(Z,s) = 0$ for all $s > \dim_H Z$.

One may equivalently define Hausdorff dimension using covers by open balls rather than arbitrary open sets; let $\mathcal{D}^b(Z,\varepsilon)$ denote the collection of countable sets $\{(x_i,r_i)\} \subset Z \times (0,\varepsilon]$ such that $Z \subset \bigcup_i B(x_i,r_i)$, and then define m_H^b by

$$m_H^b(Z,s,\varepsilon) = \inf_{\mathcal{D}^b(Z,\varepsilon)} \sum_i (\operatorname{diam} B(x_i,r_i))^s. \tag{2.3}$$

Finally, define $m_H^b(Z,s)$ and $\dim_H^b Z$ by the same procedure as above; then Proposition A.1.1 shows that $\dim_H^b Z = \dim_H Z$, so we are free to use either definition.

It is natural to replace (2.3) with

$$m_H^{b'}(Z,s,\varepsilon) = \inf_{\mathcal{D}^b(Z,\varepsilon)} \sum_i (2r_i)^s; \tag{2.4}$$

however, the two quantities are not necessarily equal, as there are cases in which $\operatorname{diam} B(x,r) < 2r$ (if x is an isolated point, for example, or if X is homeomorphic to a Cantor set). Nevertheless, Proposition A.1.1 shows that the resulting critical value $\dim_H^{b'} Z$ is equal to $\dim_H Z$.

The Hausdorff dimension is the most well-known example of a *Carathéodory* dimensional characteristic. If we restrict our attention to covers by balls of a fixed radius r, the sum in (2.4) takes a particularly simple form, and the critical point may be computed as a growth rate, giving us the corresponding example of a *capacity* dimensional characteristic.

Definition 2.1.2. Given $Z \subset X$ and $r > 0$, let

$$N(Z,r) = \min\left\{\#E_r \,\middle|\, E_r \subset Z, \bigcup_i B(x_i,r) \supset Z\right\}$$

be the minimal cardinality of an r-dense set in Z. The *lower and upper box di-*

mensions (or the *lower and upper capacity dimensions*) of Z are

$$\underline{\dim}_B Z = \varliminf_{r\to 0} \frac{\log N(Z,r)}{\log(1/r)}, \qquad \overline{\dim}_B Z = \varlimsup_{r\to 0} \frac{\log N(Z,r)}{\log(1/r)}.$$

In the case $Z = X$, replacing metric balls $B(x,r)$ by Bowen balls $B(x,n,\delta)$ in this definition yields one of the classical definitions of topological entropy. For arbitrary subsets $Z \subset X$, we will refer to this as the *capacity* entropy.

Definition 2.1.3. Let X be a compact metric space and fix a map $f\colon X \to X$. The *Bowen ball* of radius δ and order n is

$$B(x,n,\delta) = \{y \in X \mid d(f^k(y), f^k(x)) < \delta \text{ for all } 0 \le k \le n\}.$$

A set $E \subset Z$ is (n,δ)-*spanning* if $Z \subset \bigcup_{x\in E} B(x,n,\delta)$. Let Q_n^δ be the cardinality of a minimal (n,δ)-spanning set—one for which no proper subset is (n,δ)-spanning—and define the *lower and upper capacity topological entropies* by

$$\underline{Ch}_{\text{top}}(Z,\delta) = \varliminf_{n\to\infty} \frac{1}{n} \log Q_n^\delta, \qquad \overline{Ch}_{\text{top}}(Z,\delta) = \varlimsup_{n\to\infty} \frac{1}{n} \log Q_n^\delta, \tag{2.5}$$

$$\underline{Ch}_{\text{top}}(Z) = \lim_{\delta\to 0} \underline{Ch}_{\text{top}}(Z,\delta), \qquad \overline{Ch}_{\text{top}}(Z) = \lim_{\delta\to 0} \overline{Ch}_{\text{top}}(Z,\delta). \tag{2.6}$$

Elementary arguments given in [Wal75] show that we obtain the same values for the capacity entropies if we take Q_n^δ to be the cardinality of a maximal (n,δ)-separated set—that is, a set $F \subset Z$ such that $B(x,n,\delta/2) \cap B(y,n,\delta/2) = \emptyset$ for all $x \neq y \in F$. We will occasionally have reason to use this definition as well.

The capacity entropies are a direct analogue of the capacity dimensions, obtained by replacing $B(x,r)$ with $B(x,n,\delta)$. The same procedure may be carried out with Hausdorff dimension: this was first done by Bowen [Bow73], who defined what we will call the *Carathéodory topological entropy*.

Definition 2.1.4. Given $Z \subset X$, $\delta > 0$, and $N \in \mathbb{N}$, let $\mathcal{P}(Z,N,\delta)$ denote the collection of countable sets $\{(x_i,n_i)\}_{i=1}^\infty \subset Z \times \mathbb{N}$ such that $\{B(x_i,n_i,\delta)\}$ covers Z

and $n_i \geq N$ for all i. For each $s \in \mathbb{R}$, consider the set functions

$$m_h(Z, s, N, \delta) = \inf_{\mathcal{P}(Z,N,\delta)} \sum_{(x_i,n_i)} e^{-n_i s}, \tag{2.7}$$

$$m_h(Z, s, \delta) = \lim_{N\to\infty} m_h(Z, s, N, \delta), \tag{2.8}$$

and put

$$h_{\text{top}}(Z, \delta) = \inf\{s > 0 \mid m_h(Z, s, \delta) = 0\} = \sup\{s > 0 \mid m_h(Z, s, \delta) = \infty\}.$$

As with Hausdorff dimension, we get $m_h(Z) = \infty$ for $s < h_{\text{top}}(Z, \delta)$, and $m_h(Z) = 0$ for $s > h_{\text{top}}(Z, \delta)$. The *topological entropy* of f on Z is

$$h_{\text{top}}(Z) = \lim_{\delta\to 0} h_{\text{top}}(Z, \delta).$$

Given a potential function $\varphi\colon X \to \mathbb{R}$, the classical topological entropy generalises to the *topological pressure* by giving each element in a minimal (n, δ)-spanning set a weight that depends on the nth Birkhoff sum of the potential φ at x. Once again, carrying out this procedure for an arbitrary subset $Z \subset X$ yields the *capacity* topological pressure.

Definition 2.1.5. Given a potential φ, we denote the Birkhoff sums by

$$S_n\varphi(x) = \varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{n-1}(x)).$$

Fix a subset $Z \subset X$. For every $n \in \mathbb{N}$, $\delta > 0$, let E_n^δ be a minimal (n, δ)-spanning set and define a *partition function* $R_n^\delta(\varphi)$ by

$$R_n^\delta(\varphi) = \sum_{x\in E_n^\delta} e^{S_n\varphi(x)}.$$

Then the *lower and upper capacity topological pressures* of φ on Z are given by

$$\underline{CP}_Z(\varphi, \delta) = \varliminf_{n\to\infty} \frac{1}{n} \log R_n^\delta(\varphi), \qquad \overline{CP}_Z(\varphi, \delta) = \varlimsup_{n\to\infty} \frac{1}{n} \log R_n^\delta(\varphi), \tag{2.9}$$

$$\underline{CP}_Z(\varphi) = \lim_{\delta\to 0} \underline{CP}_Z(\varphi, \delta), \qquad \overline{CP}_Z(\varphi) = \lim_{\delta\to 0} \overline{CP}_Z(\varphi, \delta). \tag{2.10}$$

In the case $\varphi = 0$, these reduce to $\underline{Ch}_{\text{top}}(Z)$ and $\overline{Ch}_{\text{top}}(Z)$, respectively.

By including a factor of $e^{S_{n_i}(x_i)}$ for each element of a cover by Bowen balls, a similar modification may be made to the definition of Carathéodory entropy; this was introduced by Pesin and Pitskel' in [PP84].

Definition 2.1.6. Given a potential function φ and a subset $Z \subset X$, consider the following set functions for every $\delta > 0$ and $N \in \mathbb{N}$:

$$\begin{aligned} m_P(Z, s, \varphi, N, \delta) &= \inf_{\mathcal{P}(Z,N,\delta)} \sum_{(x_i,n_i)} \exp\left(-n_i s + S_{n_i}\varphi(x_i)\right), \\ m_P(Z, s, \varphi, \delta) &= \lim_{N\to\infty} m_P(Z, s, \varphi, N, \delta). \end{aligned} \tag{2.11}$$

The latter function is non-increasing in s, and takes values ∞ and 0 at all but at most one value of s. Denoting the critical value of s by

$$P_Z(\varphi, \delta) = \inf\{s \in \mathbb{R} \mid m_P(Z, s, \varphi, \delta) = 0\} = \sup\{s \in \mathbb{R} \mid m_P(Z, s, \varphi, \delta) = \infty\},$$

we get $m_P(Z, s, \varphi, \delta) = \infty$ when $s < P_Z(\varphi, \delta)$, and 0 when $s > P_Z(\varphi, \delta)$.

The *Carathéodory topological pressure* of φ on Z is $P_Z(\varphi) = \lim_{\delta\to 0} P_Z(\varphi, \delta)$. Once again, in the case $\varphi = 0$ this reduces to $h_{\text{top}}(Z)$.

Basic properties and results concerning all these definitions are given in the next section. For now, we remark that the definitions of Carathéodory entropy and pressure given here differ slightly from the definitions in [Pes98]. Nevertheless, Proposition A.2.1 shows that they yield the same quantity.

2.1.2 Basic properties

Because the quantities defined in the previous section all fit into the general framework of Carathéodory characteristics, they all satisfy a number of general properties given in [Pes98]. We will use two of these repeatedly, so they are worth giving special mention here.

In the first place, given any countable family of sets $Z_i \subset X$, we have

$$\begin{aligned} \dim_H\left(\bigcup_i Z_i\right) &= \sup_i \dim_H Z_i, \\ h_{\text{top}}\left(\bigcup_i Z_i\right) &= \sup_i h_{\text{top}} Z_i, \\ P_{\bigcup_i Z_i}(\varphi) &= \sup_i P_{Z_i}(\varphi). \end{aligned} \tag{2.12}$$

It is important to note that this property of *countable stability* (or σ*-stability*) only holds for the Carathéodory dimensions, and not for the capacities. Indeed, this is one of the chief advantages of the Carathéodory dimensions for our purposes.

The second important general property is that the Carathéodory dimensions are bounded above by the capacities:

$$\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z, \tag{2.13}$$

$$h_{\text{top}} Z \leq \underline{Ch}_{\text{top}} Z \leq \overline{Ch}_{\text{top}} Z, \tag{2.14}$$

$$P_Z(\varphi) \leq \underline{CP}_Z(\varphi) \leq \overline{CP}_Z(\varphi). \tag{2.15}$$

It is then a question of interest to know when the three quantities coincide. For Hausdorff dimension and box dimension, there is no completely satisfactory general theory; for entropy and pressure, on the other hand, it is shown in [Pes98] that if Z is compact and f-invariant (for example, if $Z = X$), then we have equality in (2.14) and (2.15).

2.1.3 Dimensions of measures

Given a separable metric space X, let $\mathcal{M}(X)$ denote the set of all Borel probability measures on X. The Carathéodory dimensional characteristics defined in the previous section can be made meaningful for measures $\mu \in \mathcal{M}(X)$ as well.

Definition 2.1.7. Given $\mu \in \mathcal{M}(X)$, the Hausdorff dimension of μ is

$$\dim_H(\mu) = \inf\{\dim_H Z \mid Z \subset X, \mu(Z) = 1\}.$$

Similarly, the entropy of μ is

$$h(\mu) = \inf\{h_{\text{top}} Z \mid Z \subset X, \mu(Z) = 1\},$$

and the pressure of μ with respect to a potential φ is

$$P_\mu(\varphi) = \inf\{P_Z(\varphi) \mid Z \subset X, \mu(Z) = 1\}.$$

One could also define the above quantities for the corresponding capacities, but we will not use this fact.

We point out that these definitions are given for arbitrary probability measures μ, which may not be invariant. We will see shortly that in the case where μ is invariant and ergodic, then $h(\mu)$ is equal to the usual measure-theoretic entropy, and $P_\mu(\varphi) = h(\mu) + \int \varphi \, d\mu$.

In many cases, we can obtain information about these global quantities (and the corresponding Carathéodory dimensions for sets) by studying a related set of local quantities.

Definition 2.1.8. Given $\mu \in \mathcal{M}(X)$ and $x \in X$, the *lower and upper pointwise dimensions* of μ at x are

$$\underline{d}_\mu(x) = \varliminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}, \qquad \overline{d}_\mu(x) = \varlimsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

If the two quantities agree, we denote the common value by $d_\mu(x)$, and call it the *pointwise dimension*. Similarly, the *lower and upper local entropies* are

$$\underline{h}_\mu(x) = \lim_{\delta\to 0} \varliminf_{n\to\infty} -\frac{1}{n} \log \mu(B(x,n,\delta)), \qquad \overline{h}_\mu(x) = \lim_{\delta\to 0} \varlimsup_{n\to\infty} -\frac{1}{n} \log \mu(B(x,n,\delta)),$$

with common value $h_\mu(x)$ if the limit exists, and the *lower and upper local pressures* are

$$\underline{P}_\mu(x) = \lim_{\delta\to 0} \varliminf_{n\to\infty} \frac{1}{n}(-\log(\mu(B(x,n,\delta))) + S_n\varphi(x)),$$
$$\overline{P}_\mu(x) = \lim_{\delta\to 0} \varlimsup_{n\to\infty} \frac{1}{n}(-\log(\mu(B(x,n,\delta))) + S_n\varphi(x)).$$

We remark that this last piece of terminology is not standard: the local quantity introduced in [Pes98] corresponding to pressure is somewhat different, but we will be able to use this quantity to derive the same sorts of results.

2.1.4 Global information from local quantities

The following result uses easy arguments from [Pes98]; a proof for the case of pressure is given in Theorem B.1.1 in Appendix B.

Proposition 2.1.1. *Let* dim *denote any of the three Carathéodory dimensions* $\dim_H(\cdot)$, $h_{\text{top}}(\cdot)$, *or* $P_\cdot(\varphi)$, *and let* $\mu \in \mathcal{M}(X)$ *and* $Z \subset X$ *be arbitrary. Then*

$$\dim Z \leq \sup_{x\in Z} \overline{\dim}_\mu(x). \tag{2.16}$$

Furthermore, if $\mu(Z) > 0$, *then*

$$\dim Z \geq \inf_{x\in Z} \underline{\dim}_\mu(x). \tag{2.17}$$

The following corollary allows us to apply ergodic results to relate $h(\mu)$ and $P_\mu(\varphi)$ to familiar quantities.

Corollary 2.1.2. *Let* dim *be as in Proposition 2.1.1 and suppose that for some measure* $\mu \in \mathcal{M}(X)$ *we have*

$$\underline{\dim}_\mu(x) = \overline{\dim}_\mu(x) = \alpha \tag{2.18}$$

on a set of full measure (with respect to μ*). Then* $\dim(\mu) = \alpha$.

Proof. We see from (2.17) that $\dim Z \geq \alpha$ for every $Z \subset X$ with $\mu(Z) = 1$. Furthermore, let Z be the set of points on which (2.18) holds: then $\mu(Z) = 1$ and Proposition 2.1.1 implies that $\dim Z = \alpha$. □

Using the Brin–Katok entropy formula and Birkhoff's ergodic theorem, we see that if μ is ergodic and invariant, then we have the following for μ-a.e. x:

$$\underline{h}_\mu(x) = \overline{h}_\mu(x) = h_\mu(f),$$
$$\underline{P}_\mu(x) = \overline{P}_\mu(x) = h_\mu(f) + \int \varphi\, d\mu.$$

Here $h_\mu(f)$ is the usual measure-theoretic (Kolmogorov–Sinai) entropy: Corollary 2.1.2 shows that $h_\mu(f) = h(\mu)$, where the latter quantity is the dimensional quantity defined in the previous section, whenever μ is ergodic and invariant. (If μ is not ergodic, the two quantities may differ.)

It follows immediately from these remarks and the definition of $\dim \mu$ that $h(\mu) \leq h_{\text{top}}(X)$ for every ergodic invariant measure μ, and more generally,

$$h(\mu) + \int \varphi \, d\mu \leq P_X(\varphi). \tag{2.19}$$

This is one half of the *variational principle*, which we will explore more in the next section.

Proposition 2.1.1 and Corollary 2.1.2 show that knowledge of local dimensional quantities of a measure at almost every point gives us knowledge of global dimensional quantities of that same measure, and hence also gives a lower bound for the dimensional quantities of a set on which the measure sits. Furthermore, the proposition shows that if we have knowledge of the local quantity at *every* point in the set, then we obtain an upper bound for the dimensional quantity of the set. In particular, if $h_\mu(x) = h(\mu)$ for *every* $x \in X$, then $h(\mu) = h_{\text{top}}(X)$, and so μ is a measure of maximal entropy. A similar statement holds for $P_\mu(x)$, and we will investigate this in due course.

2.2 Thermodynamic formalism

2.2.1 The variational principle

We now consider a compact metric space X and a continuous map $f\colon X \to X$. Once again, let $\mathcal{M}(X)$ denote the set of all Borel probability measures on X; furthermore, let $\mathcal{M}^f(X)$ be the set of f-invariant measures in $\mathcal{M}(X)$, and let $\mathcal{M}^f_E(X)$ be the set of all ergodic measures in $\mathcal{M}^f(X)$.

Definition 2.2.1. Let $\varphi\colon X \to \mathbb{R}$ be measurable and bounded above and below. Given $Z \subset X$, the *(variational) pressure* of φ on Z is

$$P^*_Z(\varphi) = \sup \left\{ h(\mu) + \int \varphi \, d\mu \,\middle|\, \mu \in \mathcal{M}^f(X), \mu(Z) = 1 \right\}. \tag{2.20}$$

A measure $\nu \in \mathcal{M}^f(X)$ is an *equilibrium state* for the potential φ if it achieves this supremum; that is, if

$$P^*(\varphi) = h(\nu) + \int \varphi \, d\nu.$$

Every equilibrium state is a convex combination of ergodic equilibrium states, and it follows using the ergodic decomposition that (2.20) is equivalent to

$$P_Z^*(\varphi) = \sup \left\{ h(\mu) + \int \varphi \, d\mu \,\middle|\, \mu \in \mathcal{M}_E^f(X), \mu(Z) = 1 \right\}.$$

The following well-known result lies at the very heart of the thermodynamic formalism. We do not give a complete proof here, but mention the key ideas of an approach due to Misiurewicz.

Theorem 2.2.1 (Variational Principle)**.** *Let X be a compact metric space, $f\colon X \to X$ a continuous map, and $\varphi\colon X \to \mathbb{R}$ a continuous potential. Then $P^*(\varphi) = P_X(\varphi)$.*

Proof. It follows from (2.19) that $P^*(\varphi) \leq P_X(\varphi)$, and so it remains only to construct a family of invariant measures μ for which $P_\mu(\varphi) = h(\mu) + \int \varphi \, d\mu$ is arbitrarily close to $P_X(\varphi)$. This is done as follows, using the fact that $P_X(\varphi) = \underline{CP}_X(\varphi) = \overline{CP}_X(\varphi)$ since X is compact and invariant.

Let $\delta > 0$ be arbitrary, and let $E_n \subset X$ be a maximal (n, δ)-separated set. Define an atomic measure σ_n on E_n by

$$\sigma_n = \frac{\sum_{y \in E_n} e^{S_n \varphi(y)} \delta_y}{\sum_{z \in E_n} e^{S_n \varphi(z)}}, \tag{2.21}$$

and define μ_n by

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \sigma_n \circ f^{-i}. \tag{2.22}$$

Let μ be any weak* limit of the sequence μ_n—then μ is invariant, and it is shown in the proof of [Wal75, Theorem 9.10] that

$$h(\mu) + \int \varphi \, d\mu \geq \overline{CP}_X(\varphi, \delta). \tag{2.23}$$

Letting δ go to 0 gives the result. $\square$

2.2.2 Pressure, entropy, and the Legendre transform

We now introduce a key mathematical tool that is commonly used in thermodynamics (although the version used here differs slightly).

Definition 2.2.2. Let V be a topological vector space, and let V^* be the dual space comprising all continuous linear functionals $v^*\colon V \to \mathbb{R}$. Given a function $T\colon V \to \mathbb{R}\cup\{+\infty\}$ (which we will usually take to be convex), the *Legendre transform* of T is the function $T^{L_1}\colon V^* \to \mathbb{R}\cup\{-\infty\}$ given by

$$T^{L_1}(v^*) = \inf_{v\in V}(T(v) - v^*(v)). \tag{2.24}$$

Conversely, given a function $S\colon V^* \to \mathbb{R}\cup\{-\infty\}$ (which we will usually take to be concave), the Legendre transform of S is the function $S^{L_2}\colon V \to \mathbb{R}\cup\{+\infty\}$ given by

$$S^{L_2}(v) = \sup_{v^*\in V^*}(S(v^*) + v^*(v)). \tag{2.25}$$

Figure 3.1 shows two functions that are related by a Legendre transform.

Recall that a function $S\colon V^* \to \mathbb{R}$ is concave if

$$S(tv^* + (1-t)w^*) \geq tS(v^*) + (1-t)S(w^*)$$

for every $v^*, w^* \in V^*$ and $t \in [0,1]$, and S is *upper semi-continuous* if

$$S(v^*) \geq \lim_{w^*\to v^*} S(w^*)$$

for every $v^* \in V^*$. It is easy to show that the infimum of a family of concave and upper semi-continuous functions is itself concave and upper semi-continuous. In particular, for every $v \in V$, the function $v^* \mapsto T(v) - v^*(v)$ is affine, and hence concave and upper semi-continuous; it follows that $T^{L_1}\colon V^* \to \mathbb{R}$ is concave and upper semi-continuous, no matter what T is. Similarly, $S^{L_2}\colon V \to \mathbb{R}$ is automatically convex and lower semi-continuous (that is, the above inequalities are reversed).

In fact, one may show that $(S^{L_2})^{L_1}$ is the concave and upper semi-continuous hull of S—that is, the infimum of all concave and upper semi-continuous functions greater than or equal to S—and that $(T^{L_1})^{L_2}$ is the convex and lower semi-continuous hull of T.

Remark. The standard definition of the Legendre–Fenchel transform in thermodynamics differs from (2.24) by a sign, being given by $T^L(v^*) = \sup_{v \in V}(v^*(v) - T(v)) = -T^{L_1}(v^*)$. This takes convex functions to convex functions, rather than to concave functions, as does our definition, and acts as its own inverse on the space of convex functions. The difference in the present definition is due to the fact that we wish to deal with some functions that are naturally convex (topological pressure) and some that are naturally concave (measure-theoretic entropy).

We can use the language of Legendre transforms to gain further insight into the relationship between topological pressure and measure-theoretic entropy. Indeed, setting $V = C(X)$ so that $V^* = C(X)^* \supset \mathcal{M}^f(X)$, we see that the variational pressure function $\varphi \mapsto P_X^*(\varphi)$ is nothing but the Legendre transform of the entropy function $\mu \mapsto h_\mu(f)$. Thus by the variational principle, the topological pressure function $\varphi \mapsto P_X(\varphi)$ is given by this same Legendre transform.

What we would like to do, though, is to understand the space of invariant measures in terms of the pressure function, not the other way round. By the above remarks, the Legendre transform of the pressure function is the upper semi-continuous hull of the entropy function (which is already concave, even affine). Thus we have the following result: *if the entropy function is upper semi-continuous, then it is completely determined by the pressure function.*

The entropy function is an infinite-dimensional beast, being defined on a Choquet simplex whose set of extreme points is the collection of all ergodic invariant measures. The principle guiding our multifractal results in Chapters 3 and 5 is that the various multifractal spectra are in some sense one-dimensional projections of this infinite-dimensional entropy function, and that the aforementioned result on the Legendre transform can be applied to them as well by finding the appropriate families of equilibrium states.

Chapter 3

Multifractal analysis of Birkhoff averages

3.1 Main results

3.1.1 Nearly continuous potentials, differentiable pressure

In this chapter, we state our main results for the Birkhoff spectrum defined in (1.5) and (1.6). Our goal is to obtain $\mathcal{B}(\alpha)$ as the Legendre transform of the function $T_{\mathcal{B}}(q) = P_X(q\varphi) = P^*(q\varphi)$; because we are dealing now with functions of a single variable, rather than functions defined on some larger topological vector space, the (convex to concave) Legendre transform (2.24) takes the form

$$T^{L_1}(\alpha) = \inf_{q\in\mathbb{R}}(T(q) - q\alpha), \tag{3.1}$$

where $T\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is any function, which in our case will be the convex function $T_{\mathcal{B}}$. Similarly, the (concave to convex) Legendre transform of $S\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined in (2.25) is given here by

$$S^{L_2}(q) = \sup_{\alpha\in\mathbb{R}}(S(\alpha) + q\alpha). \tag{3.2}$$

In what follows, we will consider situations in which the function T is given in terms of the pressure function, and is known to be convex and lower semi-continuous (usually even continuous), but the function S is one of the multifractal spectra, about which we have no *a priori* knowledge. Thus while $(T^{L_1})^{L_2}$ will be automatically equal to T, all we can say in general is that $(S^{L_2})^{L_1}$ is the concave and upper semi-continuous hull of S (note that concavity implies upper semi-continuity except possibly at points on the boundary of $S^{-1}(-\infty)$).

If $S(x) \geq 0$ for every $x \in \mathbb{R}$, then S^{L_2} is infinite everywhere. Thus for purposes of defining the various multifractal spectra, we adopt the (non-standard) convention that $h_{\text{top}}(\emptyset) = \dim_H(\emptyset) = -\infty$.

We recall that if T is known to be convex, then left and right derivatives exist at every point that has a neighbourhood on which T is finite; we will denote these by

$$D^- T(q) = \lim_{q' \to q^-} \frac{T(q) - T(q')}{q - q'}, \qquad D^+ T(q) = \lim_{q' \to q^+} \frac{T(q') - T(q)}{q' - q}.$$

Existence follows from monotonicity of the slopes of the secant lines. Given a convex function T, define a map from $\mathbb{R}$ to closed intervals in $\mathbb{R}$ by $A(q) = [D^- T(q), D^+ T(q)]$. Extend this in the natural way to a map from subsets of $\mathbb{R}$ to subsets of $\mathbb{R}$; we will again denote this map by A. This map has the following useful property: given any set $I_Q \subset \mathbb{R}$ and $\alpha \in A(I_Q)$, we have

$$T^{L_1}(\alpha) = \inf_{q \in I_Q} (T(\alpha) - q\alpha).$$

This will be important for us in settings where we only have partial information about the functions T and S. We will also make use of a map in the other direction: given a set $I_A \subset \mathbb{R}$ (in the domain of S), we denote the set of corresponding values of q by

$$Q(I_A) = \{q \in \mathbb{R} \mid A(q) \cap I_A \neq \emptyset\}.$$

In particular, if $\alpha = T'(q)$, then $\alpha = A(q)$, and if $q = -S'(\alpha)$, then $q = Q(\alpha)$. If (q_1, q_2) is an interval on which T is affine, then $A((q_1, q_2))$ is the slope of T on that interval; furthermore, T^{L_1} has a point of non-differentiability at $A((q_1, q_2))$.

In the results below, it will sometimes be important to know whether or not T is differentiable. A standard cardinality argument shows that $D^- T(q) = D^+ T(q)$ at all but countably many values of q; however, the values of q at which differentiability fails may *a priori* be dense in $\mathbb{R}$.

Before stating our most general result, we describe the class of functions to which it applies. Given a function $\varphi\colon X \to \mathbb{R}$, let $\mathcal{C}(\varphi) \subset X$ denote the set of points at which φ is discontinuous. Then we let $\mathcal{A}_f$ denote the class of Borel measurable functions $\varphi\colon X \to \mathbb{R}$ which satisfy the following conditions:

(A) φ is bounded (both above and below);

(B) $\mu(\overline{\mathcal{C}(\varphi)}) = 0$ for all $\mu \in \mathcal{M}^f(X)$.

In particular, $\mathcal{A}_f$ includes all continuous functions $\varphi \in C(X, \mathbb{R})$. It also includes all bounded measurable functions φ for which $C(\varphi)$ is finite and contains no periodic points, and more generally, all bounded measurable functions for which $\overline{\mathcal{C}(\varphi)}$ is disjoint from all its iterates.

We will see later (Proposition 3.3.1) that passing from $C(X, \mathbb{R})$ to $\mathcal{A}_f$ does not change the weak* topology at measures in $\mathcal{M}^f(X)$, which is the key to including these particular discontinuous functions in our results.

Theorem 3.1.1 (The entropy spectrum for Birkhoff averages)**.** *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi \in \mathcal{A}_f$. Then*

I. $T_{\mathcal{B}}$ is the Legendre transform of the Birkhoff spectrum:

$$T_{\mathcal{B}}(q) = \mathcal{B}^{L_2}(q) = \sup_{\alpha \in \mathbb{R}} (\mathcal{B}(\alpha) + q\alpha) \tag{3.3}$$

for every $q \in \mathbb{R}$.

II. The set $\{\alpha \in \mathbb{R} \mid \mathcal{B}(\alpha) > -\infty\}$ is bounded by the following:

$$\alpha_{\min} = \inf\{\alpha \in \mathbb{R} \mid T_{\mathcal{B}}(q) \geq q\alpha \text{ for all } q\}, \tag{3.4}$$

$$\alpha_{\max} = \sup\{\alpha \in \mathbb{R} \mid T_{\mathcal{B}}(q) \geq q\alpha \text{ for all } q\}, \tag{3.5}$$

That is, $K_\alpha^{\mathcal{B}} = \emptyset$ for every $\alpha < \alpha_{\min}$ and every $\alpha > \alpha_{\max}$.

III. Suppose that $T_{\mathcal{B}}$ is C^r on (q_1, q_2) for some $r \geq 1$, and that for each $q \in (q_1, q_2)$, there exists a (not necessarily unique) equilibrium state ν_q for the potential function $q\varphi$. Let $\alpha_1 = D^+T_{\mathcal{B}}(q_1)$ and $\alpha_2 = D^-T_{\mathcal{B}}(q_2)$; then

$$\mathcal{B}(\alpha) = T_{\mathcal{B}}^{L_1}(\alpha) = \inf_{q \in \mathbb{R}} (T_{\mathcal{B}}(q) - q\alpha) \tag{3.6}$$

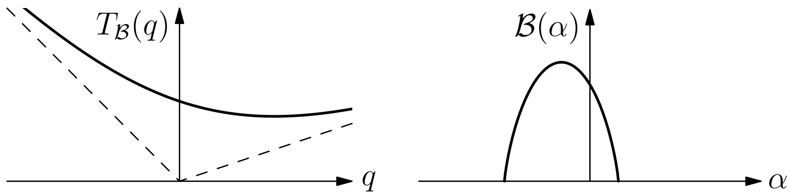

Figure 3.1. The Birkhoff spectrum for a map with no phase transitions.

for all $\alpha \in (\alpha_1, \alpha_2)$*. In particular,* $\mathcal{B}(\alpha)$ *is strictly concave on* (α_1, α_2)*, and* C^r *except at points corresponding to intervals on which* $T_{\mathcal{B}}$ *is affine.*

Observe that the first two statements hold for *every* continuous map f, without any assumptions on the system, thermodynamic or otherwise. For discontinuous potentials in $\mathcal{A}_f$, these are the first rigorous multifractal results of any sort known to the author.

Using the maps A and Q introduced above, Part III can be stated as follows: if $T_{\mathcal{B}}$ is C^r on an open interval I_Q and equilibrium states exist for all $q \in I_Q$, then (3.6) holds for all $\alpha \in A(I_Q)$. If in addition $T_{\mathcal{B}}$ is strictly convex on I_Q, then $\mathcal{B}(\alpha)$ is C^r on $A(I_Q)$.

We will show later that if the entropy map is upper semi-continuous, then the conclusion of Part III holds at α_1 and α_2 as well. We will also see (Proposition 3.2.1) that existence of a *unique* equilibrium state on an interval (q_1, q_2) is enough to guarantee differentiability, and hence to apply Theorem 3.1.1. As shown in Example 3.1.2 below, though, we may have differentiability without uniqueness.

3.1.2 Phase transitions, non-differentiable pressure

If $T_{\mathcal{B}}$ is continuously differentiable for all q, then Theorem 3.1.1 gives the complete Birkhoff spectrum, as shown in Figure 3.1. However, there are many physically interesting systems which display *phase transitions*—that is, values of q at which $T_{\mathcal{B}}$ is non-differentiable. For example, if $f\colon [0,1] \to [0,1]$ is the Manneville–Pomeau map and φ is the geometric potential $\log |f'|$, then $T_{\mathcal{B}}$ is as shown in Figure 3.2 [Nak00]; in particular, $T_{\mathcal{B}}$ is not differentiable at q_0. Thus Theorem 3.1.1 gives the Birkhoff spectrum on the interval $[\alpha_1, \alpha_2]$, where $\alpha_1 = \lim_{q \to q_0^+} T_{\mathcal{B}}'(q)$, but says nothing about the interval $[0, \alpha_1)$, on which $T_{\mathcal{B}}^{L_1}(\alpha) = -q_0 \alpha$ is linear.

Figure 3.2. A phase transition in the Manneville–Pomeau map.

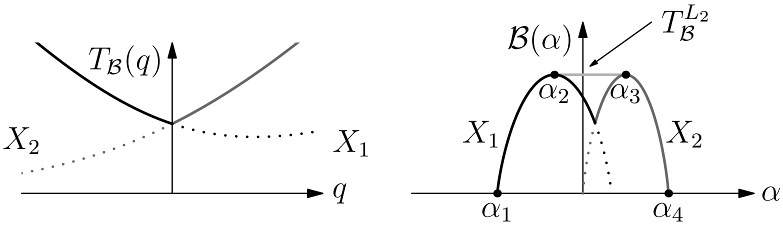

Figure 3.3. A different sort of phase transition.

In fact, it is known that for this particular example, we have $\mathcal{B}(\alpha) = T_\mathcal{B}^{L_1}$ even on the linear stretch corresponding to the point of non-differentiability of $T_\mathcal{B}$ [Nak00]. However, this is not universally the case, as may be seen by "gluing together" two unrelated maps. Consider two maps $f_1 \colon X_1 \to X_1$ and $f_2 \colon X_2 \to X_2$, where X_1 and X_2 are disjoint, and suppose that the thermodynamic functions are as shown in Figure 3.3. Let $X = X_1 \cup X_2$, and define a map $f \colon X \to X$ such that the restriction of f to X_i is f_i for $i = 1, 2$. Then $T_\mathcal{B}(q) = P^*(q\varphi) = \max\{P_1^*(q\varphi|_{X_1}), P_2^*(q\varphi|_{X_2})\}$, where P_i^* denotes the pressure of f_i, and furthermore $\mathcal{B}(\alpha)$ is the maximum of $h_{\text{top}}(K_\alpha^\mathcal{B} \cap X_1)$ and $h_{\text{top}}(K_\alpha^\mathcal{B} \cap X_2)$. Thus $T_\mathcal{B}$ is non-differentiable at $q = 0$, which corresponds to the interval $[\alpha_2, \alpha_3]$ on which $T_\mathcal{B}^{L_1}$ is constant. Applying Theorem 3.1.1 to each of the subsystems f_i, we see that $\mathcal{B}(\alpha) = T_\mathcal{B}^{L_1}(\alpha)$ on $[\alpha_1, \alpha_2]$ and $[\alpha_3, \alpha_4]$, but that the two are not equal on (α_2, α_3), and that $\mathcal{B}(\alpha)$ is not concave on this interval.

Example 3.1.2. Given $m, n \in \mathbb{N}$, let $(X_1, f_1) = (\Sigma_m^+, \sigma)$ and $(X_2, f_2) = (\Sigma_n^+, \sigma)$ be the full one-sided shifts on m and n symbols, respectively, and construct $f \colon X \to X$ as above, where $X = X_1 \cup X_2$. Choose two vectors $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$, and let

$\varphi\colon X \to \mathbb{R}$ be given by

$$\varphi(x) = \begin{cases} v_{x_1} & x = x_1 x_2 \cdots \in X_1 = \Sigma_m^+, \\ w_{x_1} & x = x_1 x_2 \cdots \in X_2 = \Sigma_n^+. \end{cases}$$

Then an easy computation using the classical definition of pressure and the variational principle shows that

$$\begin{aligned} T_{\mathcal{B}}(q) = P^*(q\varphi) &= \max(P_1^*(q\varphi), P_2^*(q\varphi)) \\ &= \max\left(\log\left(\sum_{i=1}^m e^{qv_i}\right), \log\left(\sum_{j=1}^n e^{qw_j}\right)\right). \end{aligned}$$

In particular, we see that $P_1^*(0) = \log m$ and $P_2^*(0) = \log n$, and also that

$$\begin{aligned} \frac{d^k}{dq^k} P_1^*(q\varphi)|_{q=0} &= \log\left(\sum_i v_i^k\right), \\ \frac{d^k}{dq^k} P_2^*(q\varphi)|_{q=0} &= \log\left(\sum_j w_j^k\right). \end{aligned} \tag{3.7}$$

By judicious choices of v and w, we can observe a variety of behaviours in the Birkhoff spectrum $\mathcal{B}(\alpha)$. If $m = n$ but $\sum_i v_i \neq \sum_j w_j$, we obtain the picture shown in Figure 3.3.

If $m = n$ and $\sum_i v_i = \sum_j w_j$, but $\sum_i v_i^2 > \sum_j w_j^2$, then the two pressure functions $P_1^*(q\varphi)$ and $P_2^*(q\varphi)$ are tangent at $q = 0$, corresponding to the existence of two ergodic measures of maximal entropy (one on X_1 and one on X_2), but for values of q near 0, there is a unique equilibrium state supported on X_1.

Finally, if $m = n$ and $\sum_i v_i^k = \sum_j w_j^k$ for $k = 1, 2$, but not for $k = 3$, then the two pressure functions are still tangent at $q = 0$, but now the equilibrium state passes from X_1 to X_2 as q passes through 0. Despite this transition and the non-uniqueness of the measure of maximal entropy, the pressure function $T_{\mathcal{B}}$ is still differentiable at 0.

Remark. The "gluing" in the previous example is introduced quite artificially. However, a similar phenomenon occurs naturally when one considers renormalisable maps; these display phase transitions in which the support of the equilibrium state jumps from one compact invariant set to another [Dob09].

Having seen two very different manifestations of phase transitions (Figures 3.2 and 3.3), we see that any generalisation of Theorem 3.1.1 that treats phase transitions must somehow distinguish between these two sorts of behaviour. The key difference is that in the first case, the system $f\colon X \to X$ can be approximated from within by a sequence of subsystems X_n on which there is no phase transition—that is, the following condition holds [Nak00, GR09]:

(A) There exists a sequence of compact f-invariant subsets $X_n \subset X$ such that the pressure function $q \mapsto P^*_{X_n}(q\varphi)$ is continuously differentiable for all $q \in \mathbb{R}$ (and equilibrium states exist), and furthermore,

$$\lim_{n\to\infty} P^*_{X_n}(q\varphi) = P^*(q\varphi). \tag{3.8}$$

This condition fails for the example in Figure 3.3, in which the phase transition represents a jump from one half of the system to the other half, which is disconnected from the first, rather than an escaping of measures to an adjacent fixed point. Using Condition **(A)**, we can state a general theorem which extends Theorem 3.1.1 to maps for which $T_{\mathcal{B}}$ has points of non-differentiability.

Theorem 3.1.3. *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi \in \mathcal{A}_f$. If Condition **(A)** holds, then we have* (3.6) *for all $\alpha \in (\alpha_{\min}, \alpha_{\max})$.*

3.1.3 General discontinuous potentials

As mentioned just before Theorem 3.1.1, the key property of potentials $\varphi \in \mathcal{A}_f$ is that weak* convergence to an invariant measure implies convergence of the integrals of φ; this is the only place in the proof where we use the requirement that φ lie in $\mathcal{A}_f$.

For potentials outside of $\mathcal{A}_f$, we can try to regain approximate convergence results at certain relevant measures by using the topological entropy of $\overline{\mathcal{C}(\varphi)}$ to give a bound on how much weight a neighbourhood of $\overline{\mathcal{C}(\varphi)}$ carries.

To this end, given $h \geq 0$, consider the set

$$I_A(h) = \{\alpha \in \mathbb{R} \mid T_{\mathcal{B}}^{L_1}(\alpha) > h\},$$

and also its counterpart

$$I_Q(h) = Q(I_A(h)).$$

Geometrically, $I_Q(h)$ may be described as the set of values $q \in \mathbb{R}$ such that there is a line through $(q, T_{\mathcal{B}}(q))$ that lies on or beneath the graph of $T_{\mathcal{B}}$ and intersects the y-axis somewhere above $(0, h)$.

Theorem 3.1.4. *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi\colon X \to \mathbb{R}$ be measurable and bounded (above and below). Let $\mathcal{C}(\varphi)$ be the set of discontinuities of φ, and let $h_0 = \underline{Ch}_{\text{top}}(\mathcal{C}(\varphi))$. Then*

I. For every $q \in I_Q(h_0)$, we have the following version of (3.3)*:*

$$T_{\mathcal{B}}(q) = \sup_{\alpha \in I_A(h_0)} (\mathcal{B}(\alpha) + q\alpha). \tag{3.9}$$

II. $\mathcal{B}(\alpha) \leq h_0$ for every $\alpha \notin I_A(h_0)$.

III. Suppose that $T_{\mathcal{B}}$ is C^r on $(q_1, q_2) \subset I_Q(h_0)$ for some $r \geq 1$, and that for each $q \in (q_1, q_2)$ there exists a (not necessarily unique) equilibrium state ν_q for the potential function $q\varphi$. Then (3.6) *holds for all $\alpha \in (\alpha_1, \alpha_2) = A((q_1, q_2))$.*

Finally, although we are not yet able to give a complete treatment of unbounded potential functions, we can show that everything works if our potential function is bounded below and we only consider $q \leq 0$.

Theorem 3.1.5. *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be continuous where finite (and hence bounded below). Let $\alpha_0 = D^- T_{\mathcal{B}}(0)$, and let $\alpha_{\min}$ be given by* (3.4)*, so $(\alpha_{\min}, \alpha_0) = A((-\infty, 0))$. Then*

I. For every $q \leq 0$, (3.3) *holds.*

II. For $\alpha < \alpha_{\min}$, we have $K_\alpha^{\mathcal{B}} = \emptyset$.

III. Suppose that $T_{\mathcal{B}}$ is C^r on (q_1, q_2) for some $r \geq 1$ and $q_1 < q_2 \leq 0$, and that for each $q \in (q_1, q_2)$ there exists a (not necessarily unique) equilibrium state ν_q for the potential $q\varphi$. Then (3.6) *holds for all $\alpha \in (\alpha_1, \alpha_2) = A((q_1, q_2))$.*

An analogous result holds for $q \geq 0$ if φ is bounded above but not below. Also, as with Theorem 3.1.1, Part III extends to the endpoints α_i if the entropy map is upper semi-continuous.

3.2 Applications and relation to other results

3.2.1 Verifying the hypotheses

3.2.1.1 Nearly continuous potentials

Parts I and II of Theorem 3.1.1 do not place any thermodynamic requirements on the function $T_{\mathcal{B}}$, and thus hold in full generality: for every continuous map f and every potential $\varphi \in \mathcal{A}_f$, the pressure function $T_{\mathcal{B}}$ is the Legendre transform of $\mathcal{B}(\alpha)$ (and hence $T_{\mathcal{B}}^{L_1}$ is the concave hull of $\mathcal{B}(\alpha)$), and the domain of the Birkhoff spectrum is the interval $[\alpha_{\min}, \alpha_{\max}]$.

There are two thermodynamic requirements in Part III—existence of an equilibrium state, and differentiability of $T_{\mathcal{B}}$. The latter is used in order to guarantee the existence of values $q \in \mathbb{R}$ for which $T'_{\mathcal{B}}(q)$ exists, and hence $A(q) = \{T'_{\mathcal{B}}(q)\}$ is a singleton. In fact, because T is continuous and convex, $A(q)$ is a singleton for all but at most countably many values of q, and consequently, once existence of equilibrium states is established, it follows that the Birkhoff spectrum is equal to the Legendre transform of the pressure function everywhere except possibly on some countable union of intervals, on each of which that Legendre transform is affine and gives an upper bound for $\mathcal{B}(\alpha)$.

Existence of equilibrium states is easy to verify in the following rather common setting.

Definition 3.2.1. The entropy map $\mu \mapsto h(\mu)$ is *upper semi-continuous* if for every sequence $\mu_n \in \mathcal{M}^f(X)$ that converges to μ in the weak* topology, we have

$$\overline{\lim_{n\to\infty}}\, h(\mu_n) \leq h(\mu).$$

If the entropy map is upper semi-continuous and φ is continuous, then the map

$$\mu \mapsto h(\mu) + \int q\varphi \, d\mu$$

is upper semi-continuous for every $q \in \mathbb{R}$, and thus attains its maximum, since the space of invariant measures is compact. In particular, there exists an equilibrium state for every $q\varphi$.

Definition 3.2.2. f is *expansive* if there exists $\varepsilon > 0$ such that for all $x \neq y$ there exists $n \in \mathbb{Z}$ (if f is invertible) or $n \in \mathbb{N}$ (if f is non-invertible) such that $d(f^n(x), f^n(y)) \geq \varepsilon$.

For expansive homeomorphisms, the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous [Wal75, Theorem 8.2], and so existence is guaranteed for continuous φ. Similarly, the entropy map is upper semi-continuous for C^∞ maps of compact smooth manifolds [New89], and we once again get existence for free.

Proposition 3.2.1. *Let X be a compact metric space, $f\colon X \to X$ a continuous map, and $\varphi \in \mathcal{A}_f$. Suppose that the entropy map is upper semi-continuous and that there exists an interval $(q_1, q_2) \subset \mathbb{R}$ such that for every $q \in (q_1, q_2)$, the potential $q\varphi$ has a unique equilibrium state. Then $T_\mathcal{B}$ is C^1 on (q_1, q_2).*

Proof. Suppose for a contradiction that the pressure function $q \mapsto P^*(q\varphi)$ is not differentiable at $q_0 \in (q_1, q_2)$. Let μ_n^- be the unique equilibrium state for $(q - \frac{1}{n})\varphi$, and let μ^- be a weak* limit of some subsequence $\mu_{n_j}^-$. By upper semi-continuity and Proposition 3.3.1 below, we have

$$\begin{aligned} h(\mu^-) + \int q\varphi \, d\mu^- \geq \overline{\lim_{n_j \to \infty}} h(\mu_{n_j}) + \int q\varphi \, d\mu_{n_j}^- \\ = \overline{\lim_{n_j \to \infty}} P^* \left(\left(q - \frac{1}{n_j} \right) \varphi \right) = P^*(q\varphi). \end{aligned}$$

Thus μ^- is an equilibrium state for $q\varphi$ with

$$\int q\varphi \, d\mu^- = D^- T_\mathcal{B}(q) = \lim_{q' \to q^-} T_\mathcal{B}'(q')$$

by Proposition 3.4.3 below. Similarly, one can construct an equilibrium state μ^+ such that $\int q\varphi \, d\mu^+$ is the right derivative of $T_\mathcal{B}$ at q. If the two derivatives do not agree, then we have two distinct equilibrium states for $q\varphi$, a contradiction. □

Using Proposition 3.2.1, one approach to verifying the hypotheses of Theorem 3.1.1 for a map with upper semi-continuous entropy is to show that the equilibrium state for each $q\varphi$ is unique.

We also observe that in the context of Part III of Theorem 3.1.1, the construction in the proof above gives equilibrium states for $q_1\varphi$ and $q_2\varphi$ that are supported on the sets $K^{\mathcal{B}}_{\alpha_1}$ and $K^{\mathcal{B}}_{\alpha_2}$, respectively, and which establish (3.6) for the endpoints α_1 and α_2, just as in the proof of Proposition 3.4.2 below.

3.2.1.2 General discontinuous potentials

If φ is discontinuous, the map from $\mathcal{M}(X)$ to $\mathbb{R}$ defined by

$$\mu \mapsto \int \varphi \, d\mu \tag{3.10}$$

is not continuous on all of $\mathcal{M}(X)$. For discontinuous potentials lying in $\mathcal{A}_f$, continuity still holds at measures in $\mathcal{M}^f(X)$ by Proposition 3.3.1 below, which suffices for all the proofs here.

However, if $\varphi \notin \mathcal{A}_f$, then there may be invariant measures at which the map is discontinuous. In particular, if $\mu(\mathcal{C}(\varphi)) > 0$, then the map in (3.10) is discontinuous at μ. If φ is unbounded, then it is relatively straightforward to show that the map is not continuous at *any* measure in $\mathcal{M}(X)$. In many cases, it is not even enough to restrict our attention to invariant measures [BK98, Proposition 2.8]. Thus for $\varphi \notin \mathcal{A}_f$, upper semi-continuity of the entropy is not enough to guarantee existence of equilibrium states without further information.

For potentials which are bounded above but not below, we observe in Proposition 3.3.2 that the map in (3.10) is upper semi-continuous, and thus the free energy function $\mu \mapsto h(\mu) + \int \varphi \, d\mu$ is upper semi-continuous as well. It follows that it attains its maximum, and we once again are guaranteed existence. This is also enough to prove Proposition 3.2.1 for these potentials, showing that existence and uniqueness imply differentiability of the pressure function (for the appropriate sign of q) if the entropy map is upper semi-continuous.

3.2.2 The Lyapunov spectrum

An important special case of the Birkhoff spectrum occurs when f is a conformal map (this is automatic if f is a C^1 interval map) and φ is the geometric potential $\varphi(x) = \log \|Df(x)\|$. Then the level sets $K^{\mathcal{L}}_{\alpha}$ for the Lyapunov exponents are as defined in (1.7), and coincide with the level sets $K^{\mathcal{B}}_{\alpha}$ for φ. The *entropy spectrum* *for Lyapunov exponents* is

$$\mathcal{L}_E(\alpha) = h_{\text{top}}\,(K_\alpha^{\mathcal{L}}), \tag{3.11}$$

and we see immediately that $\mathcal{L}_E(\alpha) = \mathcal{B}(\alpha)$ for the geometric potential, so all the results in this chapter apply to $\mathcal{L}_E(\alpha)$ as well.

The *dimension spectrum for Lyapunov exponents* is

$$\mathcal{L}_D(\alpha) = \dim_H(K_\alpha^{\mathcal{L}}). \tag{3.12}$$

Later on, we will see in Theorem 4.2.1 that $\mathcal{L}_E(\alpha) = \mathcal{L}_D(\alpha)/\alpha$ whenever f is conformal without critical points or singularities (we already mentioned this fact in (1.8)). In this setting, then, knowing one of the Lyapunov spectra suffices to tell us everything about both of them, with the possible exception of the point $\alpha = 0$. (Note that since $\mathcal{L}_D(\alpha)$ is not given by a Legendre transform, but is obtained by a rescaling, it may not be convex—see [IK09] for examples where this occurs.)

3.2.3 Uniform hyperbolicity

In [Bow75], Bowen showed that if M is a C^∞ Riemannian manifold and $f\colon M \to M$ is an Axiom A diffeomorphism, then any Hölder continuous potential function $\varphi\colon M \to \mathbb{R}$ has a unique equilibrium state. Since such maps are expansive on the hyperbolic set [KH95, Corollary 6.4.10], this suffices to check the hypotheses of Theorem 3.1.1, as shown in the previous section, and hence the Birkhoff spectrum is equal to the Legendre transform of the pressure function: in particular, it is concave and C^1 (see Figure 3.1). Versions of this result may be extracted from the results in [TV99, PW01], but Theorem 3.1.1 provides a more direct proof.

3.2.3.1 Non-Hölder potentials

Non-Hölder potentials were studied by Pesin and Zhang in [PZ06] (see also [Hu08]). They consider a uniformly piecewise expanding full-branched Markov map f of the unit interval, and use inducing schemes and tools from the theory of countable Markov shifts to study the existence and uniqueness of equilibrium states for a large class of potentials. In particular, they give the following example of a non-Hölder potential:

$$\varphi(x) = \begin{cases} -(1 - \log x)^{-\alpha} & x \in (0,1], \\ 0 & x = 0. \end{cases} \tag{3.13}$$

It is shown in [PZ06] that for any $\alpha > 1$ and $q \in \mathbb{R}$, the potential $q\varphi$ has a unique equilibrium state. Since f is expansive, by the comments in the previous section this suffices to check the hypotheses of Theorem 3.1.1, and we have the following result.

Proposition 3.2.2. *Let f be a uniformly expanding full-branched Markov map of the unit interval, and let φ be the potential function given in* (3.13)*, $\alpha > 1$. Then the Birkhoff spectrum $\mathcal{B}(\alpha)$ is smooth and concave, has domain $[\alpha_{\min}, \alpha_{\max}]$, and is the Legendre transform of $T_{\mathcal{B}}$.*

Indeed, Proposition 3.2.2 also holds for any potential φ such that all $q\varphi$ are in the class considered by Pesin and Zhang.

For $0 < \alpha \leq 1$, it is shown in [PZ06] that $T_{\mathcal{B}}$ has a phase transition at some value $q_0 > 0$. Applying Theorem 3.1.1, we obtain a result for the non-linear part of the Birkhoff spectrum (see Figure 3.2); to obtain a complete result, we would need to apply Theorem 3.1.3 by establishing Condition **(A)**. Although this remains open, one might attempt to do this by using the fact that for a potential with summable variations, the Gurevich pressure on a topologically mixing countable Markov shift X is the supremum of the classical topological pressure over topologically mixing finite Markov subshifts of X [Sar99]; these finite subshifts give natural candidates for the compact invariant sets X_n in Condition **(A)**.

Remark. In [PS07], Pfister and Sullivan prove a variational principle for the topological entropy of saturated sets, which include in particular the level sets $K_\alpha^{\mathcal{B}}$, under the assumption that the system in question satisfies two properties, which they call the *g-almost product property* and the *uniform separation property*. Expansive systems satisfy the latter, and uniformly hyperbolic systems satisfy the former. For such systems, they prove (among other things) the following multifractal result for any continuous φ [PS07, Proposition 7.1]:

$$\mathcal{B}(\alpha) = h_{\text{top}}(K_\alpha^{\mathcal{B}}) = \sup \left\{ h(\mu) \,\middle|\, \mu \in \mathcal{M}^f(X), \int \varphi \, d\mu = \alpha \right\}. \tag{3.14}$$

Given (3.14), it is not difficult to show that (3.6) holds, which establishes the multifractal formalism for systems with the g-almost product property and uniform separation, provided the potential is continuous. In particular, this includes the example given above, as well as some (but by no means all) of the examples mentioned below.

3.2.4 Non-uniform hyperbolicity

3.2.4.1 Parabolic maps

An important class of non-uniformly expanding maps is the Manneville–Pomeau maps; these are non-uniformly expanding interval maps with an indifferent fixed point. The primary potential of interest in this case is the geometric potential $\log|f'|$, which corresponds to studying a non-Hölder potential on a *uniformly* expanding interval map via an appropriate change of coordinates; thus this is closely related to the previous example.

The thermodynamic properties and Lyapunov spectra of these maps were studied in [PW99, Nak00, GR09]; once again, Theorem 3.1.1 provides a direct proof of the multifractal results using the thermodynamic results, although as above, one would need to establish Condition **(A)** to deal with the linear parts of the spectrum using Theorem 3.1.3. We also remark that a significant achievement of [GR09] is to deal with the endpoints of the spectrum ($\lambda = 0$ and $\lambda = \infty$), which cannot be dealt with using the present results.

Moving to two (real) dimensions, let $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a parabolic rational map of the Riemann sphere; that is, a rational map such that the Julia set $J(f)$ contains at least one indifferent fixed point (that is, a fixed point z_0 for which $|f'(z_0)| = 1$), but does not contain any critical points. Following Makarov and Smirnov [MS00], we say that f is *exceptional* if there is a finite, non-empty set $\Sigma \subset \overline{\mathbb{C}}$ such that $f^{-1}(\Sigma) \setminus \mathrm{Crit}\, f = \Sigma$, where $\mathrm{Crit}\, f$ is the set of critical points of f.

Let $\varphi(z) = \log|f'(z)|$ be the geometric potential. By combining the results in [MS00] with [Hu08, Corollary D.1 and Theorem G], we see that if f is non-exceptional, then the graph of the function $T_{\mathcal{B}}$ is as shown in Figure 3.2. In particular, $T_{\mathcal{B}}$ is analytic and strictly convex on (q_0, ∞), where $q_0 = -\dim_H J(f)$, and so writing

$$\alpha_1 = D^+ T_{\mathcal{B}}(q_0), \qquad \alpha_2 = \lim_{q\to\infty} T'_{\mathcal{B}}(q),$$

it follows from Theorem 3.1.1 that $\mathcal{B}(\alpha) = T_{\mathcal{B}}^{L_1}$ on (α_1, α_2). Since we are dealing with the geometric potential, this is also the entropy spectrum for Lyapunov exponents, and we may apply (1.8) to obtain the dimension spectrum for Lyapunov exponents, $\mathcal{L}_D(\alpha) = \frac{1}{\alpha} T_{\mathcal{B}}^{L_1}(\alpha)$.

This result is obtained by other methods in [GPR09], where it is also shown that the spectra are linear on $[0, \alpha_1]$ (the dotted line in Figure 3.2). As before, giving an alternate proof of this using Theorem 3.1.3 would require establishing Condition **(A)**.

Once again, Pfister and Sullivan's results establish the multifractal formalism for the Birkhoff spectrum here, but *not* for the dimension spectrum for Lyapunov exponents, as they only consider topological entropy.

3.2.4.2 Maps with contracting regions

The existence and uniqueness of equilibrium states for a broad class of non-uniformly expanding maps in a higher dimensional setting was studied by Oliveira and Viana [OV08] and by Varandas and Viana [VV08]. To the best of the author's knowledge, the multifractal properties of these systems have not been studied at all, and so they provide an ideal application of Theorem 3.1.1. It does not appear to be known whether or not these systems, which may have contracting regions, satisfy specification or any other property that would imply Pfister and Sullivan's g-almost product property, and so the results of [PS07] cannot be applied.

We describe the systems studied in [VV08] and use the results of that paper to apply Theorem 3.1.1. Let M be a compact manifold of dimension m with distance function d (more generally, Varandas and Viana consider metric spaces in which the Besicovitch covering lemma holds). Let $f\colon M \to M$ be a local homeomorphism, and let $L(x)$ be a bounded function such that for every $x \in M$ there exists a neighbourhood $U_x \ni x$ such that $f_x = f|_{U_x}\colon U_x \to f(U_x)$ is invertible, with

$$d(f(y), f(z)) \geq \frac{1}{L(x)} d(y, z)$$

for all $y, z \in U_x$. Thus if $L(x) < 1$, then f is expanding at x, while if $L(x) \geq 1$, then L controls how much contraction can happen near x.

Assuming that every point has finitely many preimages, we write $\deg_x(f) = \#f^{-1}(x)$. Assume also that level sets for the degree are closed and that M is connected; then is it shown in [VV08] that up to considering some iterate f^N of f, we can assume that $\deg_x(f) \geq e^{h_{\text{top}}(f)}$ for all x.

The final conditions on the map f are as follows: there exist constants $\sigma > 1$ and $L > 0$ and an open region $\mathcal{A} \subset M$ such that

(H1) $L(x) \leq L$ for every $x \in \mathcal{A}$ and $L(x) \leq \sigma^{-1}$ for all $x \in M \setminus \mathcal{A}$, and L is close to 1 (see [VV08] for precise conditions).

(H2) There exists $k_0 \geq 1$ and a covering $\mathcal{P} = \{P_1, \ldots, P_{k_0}\}$ of M by domains of injectivity for f such that $\mathcal{A}$ can be covered by $r < e^{h_{\text{top}}(f)}$ elements of $\mathcal{P}$.

That is, f is uniformly expanding outside of $\mathcal{A}$, and does not display too much contraction inside $\mathcal{A}$; furthermore, since there are at least $e^{h_{\text{top}}(f)}$ preimages of any given point x, and only r of these can lie in covering of $\mathcal{A}$ by elements of $\mathcal{P}$, every point has at least one preimage in the expanding region.

The requirement on the potential φ is that

(P) $\varphi\colon M \to \mathbb{R}$ is Hölder continuous and $\sup\varphi - \inf\varphi < h_{\text{top}}(f) - \log r$.

It is proved in [VV08] that for any map f and potential φ satisfying these conditions, there exists a unique equilibrium state for φ. In particular, if (P) holds for φ, then there exists $q_0 > 1$ such that (P) holds for $q\varphi$ as well, for all $q \in (-q_0, q_0)$. Thus Theorem 3.1.1 applies, and we have the following result on the Birkhoff spectrum.

Proposition 3.2.3. *Given a map $f\colon M \to M$ satisfying (H1) and (H2) and a Hölder continuous potential $\varphi\colon M \to \mathbb{R}$ satisfying (P), there exists $q_0 > 1$ such that $T_{\mathcal{B}}$ is C^1 on the interval $(-q_0, q_0)$, and writing*

$$\alpha_1 = \lim_{q \to -q_0^+} T'_{\mathcal{B}}(q), \qquad \alpha_2 = \lim_{q \to q_0^-} T'_{\mathcal{B}}(q),$$

we have $\mathcal{B}(\alpha) = T_{\mathcal{B}}^{L_1}(\alpha) = \inf_{q\in\mathbb{R}}(T_{\mathcal{B}}(q) - q\alpha)$ for every $\alpha \in [\alpha_1, \alpha_2]$.

See [VV08] for examples of specific systems to which their conditions, and hence Proposition 3.2.3, apply.

3.2.4.3 Maps with critical points

Ever since the family of logistic maps was introduced, unimodal and multimodal maps have received a great deal of attention. Existence and uniqueness of equilibrium states for a certain class of bounded potentials were established in [BT08]. In particular, let $\mathcal{H}$ denote the collection of topologically mixing C^∞ interval maps $f\colon [0,1] \to [0,1]$ with hyperbolically repelling periodic points and non-flat critical points; given $f \in \mathcal{H}$, let $\varphi\colon [0,1] \to \mathbb{R}$ be a Hölder continuous potential such that

$$\sup \varphi - \inf \varphi < h_{\mathrm{top}}(f). \tag{3.15}$$

It is shown in [BT08] that there exists a unique equilibrium state for φ, and so the analogue of Proposition 3.2.3 holds here.

In fact, it was shown by Blokh that any continuous topologically mixing interval map has the specification property (see, for example, [Buz97]), which implies the g-almost product property, and so Pfister and Sullivan's result applies here, showing that the multifractal formalism holds for *any* continuous potential φ on the entire spectrum. However, their result does not apply to unbounded potentials such as the geometric potential $\varphi(x) = -\log|f'(x)|$.

The potentials $q\varphi$, where φ is the geometric potential, were studied in [PS08, BT09, IT09b]. In the last of these papers, Iommi and Todd showed that for a related class of maps f, the potential $q\varphi$ has a unique equilibrium state for all $q \in (-\infty, 0]$. (In fact, they obtain results for $q > 0$ as well, but we do not yet have the tools to use these here.) Thus we may apply Theorem 3.1.5 and show that if $\alpha_0 = \lim_{q\to 0^-} T_{\mathcal{B}}'(q)$ and $\alpha_{\min} = \lim_{q\to -\infty} T_{\mathcal{B}}'(q)$, then for all $q \leq 0$, we have

$$T_{\mathcal{B}}(q) = \sup_{\alpha\in\mathbb{R}} (\mathcal{B}(\alpha) + q\alpha),$$

and for all $\alpha \in [\alpha_{\min}, \alpha_0]$, we have

$$\mathcal{B}(\alpha) = \inf_{q\in\mathbb{R}} (T_{\mathcal{B}}(q) - q\alpha).$$

In particular, $\mathcal{B}(\alpha)$ is strictly concave and C^1 on $[\alpha_{\min}, \alpha_0]$, and furthermore, $K_\alpha^{\mathcal{B}} = \emptyset$ for $\alpha < \alpha_{\min}$.

3.3 Preparatory results

3.3.1 Convergence results

Proposition 3.3.1. *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi \in \mathcal{A}_f$. Let $\mu \in \mathcal{M}^f(X)$ be an invariant measure, and consider a sequence of (not necessarily f-invariant) measures $\{\mu_n\} \subset \mathcal{M}(X)$ such that $\mu_n \to \mu$ in the weak* topology. Then*

$$\lim_{n\to\infty} \int \varphi \, d\mu_n = \int \varphi \, d\mu. \tag{3.16}$$

Proof. If φ is continuous, then this is immediate. If $\varphi \in \mathcal{A}_f$ is discontinuous, then let $M \in \mathbb{R}$ be such that $|\varphi(x)| \le M$ for all $x \in X$, and fix $\varepsilon > 0$. Condition (B) in the definition of $\mathcal{A}_f$ tells us that $\mu(\overline{\mathcal{C}(\varphi)}) = 0$, and thus there exists an open neighbourhood $B \supset \overline{\mathcal{C}(\varphi)}$ such that $\mu(\overline{B}) < \varepsilon$. Since $\overline{B}$ is closed, we have

$$\mu(\overline{B}) \ge \overline{\lim_{n\to\infty}}\, \mu_n(\overline{B}),$$

and so there exists N such that $\mu_n(B) \le \mu_n(\overline{B}) < 2\varepsilon$ for all $n \ge N$. Now we have

$$\left| \int_X \varphi \, d\mu - \int_X \varphi \, d\mu_n \right| \le \left| \int_{X\setminus B} \varphi \, d\mu - \int_{X\setminus B} \varphi \, d\mu_n \right| + \left| \int_B \varphi \, d\mu - \int_B \varphi \, d\mu_n \right|.$$

Since φ is continuous on the compact set $X \setminus B$, the first difference goes to 0 as $n \to \infty$. Furthermore, by the above estimates, the second difference is less than $3M\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this completes the proof of (3.16). □

Observe that if X is a compact metric space and $\psi\colon X \to \mathbb{R} \cup \{-\infty\}$ is continuous, then it must be bounded above (since it never takes the value $+\infty$). We use this fact in the next proposition.

Proposition 3.3.2. *Let X be a compact metric space and $\psi\colon X \to \mathbb{R} \cup \{-\infty\}$ be continuous (and hence bounded above). Consider a sequence of measures $\{\mu_n\}$*

converging to μ *in the weak* topology, and suppose that* $\int \psi \, d\mu > -\infty$. *Then*

$$\int \psi \, d\mu \geq \overline{\lim_{n\to\infty}} \int \psi \, d\mu_n. \tag{3.17}$$

Proof. Given $M < 0$, define a continuous function $\psi_M \colon X \to \mathbb{R}$ by

$$\psi_M(x) = \max(\psi(x), M).$$

Because ψ is integrable with respect to μ, we have for every $\varepsilon > 0$ some $M < 0$ such that

$$\int (\psi_M - \psi) \, d\mu < \varepsilon,$$

from which we deduce that

$$\int \psi \, d\mu \geq \int \psi_M \, d\mu - \varepsilon = \lim_{n\to\infty} \int \psi_M \, d\mu_n - \varepsilon \geq \overline{\lim_{n\to\infty}} \int \psi \, d\mu_n - \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, this establishes (3.17). □

Observe that there are no dynamics in Proposition 3.3.2, so there is no requirement that any of the measures μ_n or μ be invariant.

3.3.2 Measures associated with approximate level sets

Recall that the level sets $K_\alpha^{\mathcal{B}}$ are defined by

$$K_\alpha^{\mathcal{B}}(\varphi) = \left\{ x \in X \;\middle|\; \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha \right\},$$

where we write $K_\alpha^{\mathcal{B}}(\varphi)$ to emphasise the role of the function φ. This may be rewritten as

$$\begin{aligned} K_\alpha^{\mathcal{B}}(\varphi) &= \left\{ x \in X \;\middle|\; \forall \varepsilon > 0 \exists N \text{ such that } \left| \frac{1}{n} S_n\varphi(x) - \alpha \right| < \varepsilon \text{ for all } n \geq N \right\} \\ &= \bigcap_{\varepsilon>0} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \left\{ x \in X \;\middle|\; \left| \frac{1}{n} S_n\varphi(x) - \alpha \right| < \varepsilon \right\}. \end{aligned}$$

In the proofs of our main results, we will need to consider the following "approximate level sets":

$$\begin{aligned} F_\alpha^{\varepsilon,N}(\varphi) &= \bigcap_{n\geq N} \left\{ x\in X \;\middle|\; \left|\frac{1}{n}S_n\varphi(x)-\alpha\right| < \varepsilon\right\} \\ F_\alpha^{\varepsilon}(\varphi) &= \bigcup_{N\in\mathbb{N}} F_\alpha^{\varepsilon,N}(\varphi). \end{aligned} \tag{3.18}$$

For these we have

$$K_\alpha^{\mathcal{B}}(\varphi) = \bigcap_{\varepsilon>0} F_\alpha^{\varepsilon}(\varphi),$$

In particular, the following relations will be quite useful:

$$\begin{aligned} h_{\text{top}}\left(K_\alpha^{\mathcal{B}}(\varphi)\right) &\leq h_{\text{top}}\left(F_\alpha^{\varepsilon}(\varphi)\right) = \sup_N \left(h_{\text{top}}\, F_\alpha^{\varepsilon,N}(\varphi)\right), \\ \dim_H(K_\alpha^{\mathcal{B}}(\varphi)) &\leq \dim_H(F_\alpha^{\varepsilon}(\varphi)) = \sup_N \left(\dim_H F_\alpha^{\varepsilon,N}(\varphi)\right). \end{aligned}$$

Observe that for a continuous function φ, the sets $F_\alpha^{\varepsilon,N}(\varphi)$ are obtained via a sort of Cantor construction, being a countable intersection of sets $\{x \in X \mid |(1/n)S_n\varphi(x) - \alpha| < \varepsilon\}$. When Z is obtained via such a construction, it is reasonable to approximate $h_{\text{top}} Z$ with $\underline{Ch}_{\text{top}}Z$, which gives us an upper bound. A similar upper bound applies when we study the topological pressure.

The utility of the capacity quantities (entropy and pressure) for our purposes is in the following lemma, which shows that when we deal with sets like $F_\alpha^{\varepsilon,N}$ on which the Birkhoff averages converge *uniformly* to a given range of values, then we can build measures with large free energy and with the expected integrals.

Lemma 3.3.3. *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\psi, \zeta \in \mathcal{A}_f$. Fix $Z \subset X$ and let $\beta_1, \beta_2 \in \mathbb{R}$ be given by*

$$\beta_1 = \varliminf_{n\to\infty} \inf_{x\in Z} \frac{1}{n} S_n\psi(x), \qquad \beta_2 = \varlimsup_{n\to\infty} \sup_{x\in Z} \frac{1}{n} S_n\psi(x).$$

Then for every $\gamma > 0$ there exists $\mu \in \mathcal{M}^f(X)$ satisfying the following:

$$\int \psi \, d\mu \in [\beta_1, \beta_2], \tag{3.19}$$

$$h(\mu) + \int \zeta \, d\mu \geq \overline{CP}_Z(\zeta) - \gamma. \tag{3.20}$$

Proof. The construction of μ satisfying (3.20) is given in part 2 of the proof of [Wal75, Theorem 9.10], which is due to Misiurewicz, and goes as follows. Choose $\delta > 0$ such that

$$\varlimsup_{n\to\infty} \frac{1}{n} \sum_{x\in E_n} e^{S_n\zeta(x)} > \overline{CP}_Z(\zeta) - \gamma,$$

where E_n is a maximal (n, δ)-separated set, and define an atomic measure σ_n on E_n by

$$\sigma_n = \frac{\sum_{y\in E_n} e^{S_n\zeta(y)} \delta_y}{\sum_{z\in E_n} e^{S_n\zeta(z)}}. \tag{3.21}$$

Define μ_n by

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \sigma_n \circ f^{-i}, \tag{3.22}$$

and let μ be any weak* limit of the sequence μ_n—then μ is invariant, and the estimate (3.19) follows from Proposition 3.3.1 upon observing that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\int \psi \, d\mu_n \in [\beta_1 - \varepsilon, \beta_2 + \varepsilon]$ for all $n \geq N$. The estimate (3.20) is shown in the proof in [Wal75]; although the proof there assumes that ζ is continuous, this is only used to guarantee the convergence $\int \zeta \, d\mu_{n_j} \to \int \zeta \, d\mu$, which in our case is given by Proposition 3.3.1. □

The full strength of Lemma 3.3.3 is only needed in the proof of Theorem 5.2.2 (for the dimension spectrum). For the proof of Theorem 3.1.1 (for the Birkhoff spectrum), we only need the case $\zeta = 0$. In particular, in order to prove Theorems 3.1.4 and 3.1.5, we only need the following two versions of Lemma 3.3.3.

Lemma 3.3.4. *Let X be a compact metric space, $f : X \to X$ be continuous, and $\varphi : X \to \mathbb{R}$ be Borel measurable and bounded above and below. Suppose $Z \subset X$ is such that $\underline{Ch}_{\text{top}}(Z) > \overline{Ch}_{\text{top}}(\mathcal{C}(\varphi))$. Fix $Z \subset X$ and let $\beta_1, \beta_2 \in [-\infty, \infty]$ be given by*

$$\beta_1 = \varliminf_{n\to\infty} \inf_{x\in Z} \frac{1}{n} S_n\varphi(x), \qquad \beta_2 = \varlimsup_{n\to\infty} \sup_{x\in Z} \frac{1}{n} S_n\varphi(x).$$

Then for every $\gamma > 0$ there exists $\mu \in \mathcal{M}^f(X)$ satisfying the following:

$$\int \varphi \, d\mu \in (\beta_1 - \gamma, \beta_2 + \gamma), \tag{3.23}$$

$$h(\mu) \geq \underline{Ch}_{\text{top}}(Z) - \gamma. \tag{3.24}$$

Proof. For $n \in \mathbb{N}$ and $\delta > 0$, let P_n^δ be the maximal cardinality of an (n, δ)-separated subset of Z, and recall that

$$\underline{Ch}_{\text{top}}(Z) = \lim_{\delta \to 0} \varliminf_{n \to \infty} \frac{1}{n} \log P_n^\delta.$$

In particular, decreasing γ if necessary, we may choose $\delta > 0$ such that

$$\overline{Ch}_{\text{top}}(\mathcal{C}(\varphi), \delta) < \underline{Ch}_{\text{top}}(Z) - \gamma < \varliminf_{n \to \infty} \frac{1}{n} \log P_n^\delta. \tag{3.25}$$

Writing $h_0 = \varliminf_{n \to \infty} \frac{1}{n} \log P_n^\delta$, we choose $\eta > 0$ such that $h_0 - \eta > \overline{Ch}_{\text{top}}(\mathcal{C}(\varphi), \delta)$. Thus there exists $C > 0$ such that for every $m \in \mathbb{N}$ there exists a set $F_m \subset \mathcal{C}(\varphi)$ such that

$$\# F_m \leq C e^{m(h_0 - \eta)} \tag{3.26}$$

and $U_m = \bigcup_{x \in F_m} B(x, m, \delta) \supset \mathcal{C}(\varphi)$. Observe that U_m is open because f is continuous.

Given $n \in \mathbb{N}$, let E_n be an (n, δ)-separated subset of Z with maximum cardinality $\# E_n = P_n^\delta$. Following the previous proof, consider the measures σ_n given by (3.21) with $\zeta = 0$:

$$\sigma_n = \frac{\sum_{x \in E_n} \delta_x}{\# E_n}. \tag{3.27}$$

Now we vary the construction slightly; given $0 \leq m < n$, we go $n - m$ steps (not n) along each orbit:

$$\mu_n^m = \frac{1}{n - m} \sum_{k=0}^{n-m-1} \sigma_n \circ f^{-k}. \tag{3.28}$$

That is, μ_n^m is a convex combination of δ-measures evenly distributed across the first $n - m$ points in each orbit that begins in E_n.

For every $0 \leq k < n - m - 1$, consider the set

$$B_n^m(k) = \{x \in E_n \mid f^k \in U_m\} = \bigcup_{z \in F_m} f^{-k}(B(z, m, \delta)) \cap E_n.$$

Observe that for every $z \in F_m$ and every pair $x \neq y \in f^{-k}(B(z, m, \delta)) \cap E_n$, we have $d(f^i(x), f^i(y)) < \delta$ for all $n - m \leq i < n$, and since E_n is (n, δ)-separated, it follows that $d(f^i(x), f^i(y)) \geq \delta$ for some $0 \leq i < n - m$. In particular, $f^{-k}(B(z, m, \delta)) \cap E_n \subset Z$ is $(n-m, \delta)$-separated, and hence has cardinality at most P_{n-m}^δ. It follows that

$$\#B_n^m(k) \leq Ce^{m(h_0 - \eta)} P_{n-m}^\delta,$$

where C is as in (3.26), and hence

$$\sigma_n(f^{-k}(U_m)) = \frac{\#B_n^m(k)}{\#E_n} \leq Ce^{-\eta m + mh_0} \frac{P_{n-m}^\delta}{P_n^\delta}.$$

This holds for all $0 \leq k < n - m$, and hence

$$\mu_n^m(U_m) \leq Ce^{-\eta m} \frac{e^{mh_0} P_{n-m}^\delta}{P_n^\delta}. \tag{3.29}$$

Thus in order to bound $\mu_n^m(U_m)$, we need some control of the ratio $P_{n-m}^\delta / P_n^\delta$. Observe that if P_n^δ is actually equal to e^{nh_0} for all n, then (3.29) immediately yields the bound $\mu_n^m(U_m) \leq Ce^{-\eta m}$. However, P_n^δ may not grow as uniformly as we would like, so we must be more careful.

Given $m \in \mathbb{N}$, consider the quantity

$$L(m) = \overline{\lim_{n \to \infty}} (\log(P_n^\delta) - \log(P_{n-m}^\delta) - mh_0).$$

Suppose $L(m) < 0$ for some m. Then there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\log(P_n^\delta) - \log(P_{n-m}^\delta) - mh_0 < -\varepsilon.$$

In particular, this gives the following for every $k \in \mathbb{N}$:

$$\log(P_{N+km}^\delta) < \log(P_N^\delta) + kmh_0 - k\varepsilon.$$

Dividing by km and taking the limit as $k \to \infty$, we get

$$h_0 = \varliminf_{n\to\infty} \frac{1}{n} \log(P_n^\delta) \leq \varliminf_{k\to\infty} \frac{1}{N+km}(P_{N+km}^\delta) < h_0 - \frac{\varepsilon}{m},$$

a contradiction. This proves that $L(m) \geq 0$ for all m, from which we deduce that for every $m \in \mathbb{N}$, there exists a sequence $n_j = n_j(m) \to \infty$ such that

$$\varliminf_{j\to\infty} (\log(P_{n_j}^\delta) - \log(P_{n_j-m}^\delta) - mh_0) \geq 0,$$

or equivalently,

$$\varliminf_{j\to\infty} \frac{P_{n_j}^\delta}{e^{mh_0} P_{n_j-m}^\delta} \geq 1. \tag{3.30}$$

In combination with (3.29), this will soon give us the bound we need.

As in the proof of Lemma 3.3.3, let μ^m be a weak* limit point of the sequence $\mu_{n_j}^m$ (by passing to a subsequence if necessary, we assume that $\mu_{n_j}^m \to \mu^m$). Invariance of μ^m and the entropy estimate (3.24) hold just as before, so it only remains to show (3.23).

Let $M = \sup_{x\in X} |\varphi(x)|$, and choose m large enough so that $Ce^{-\eta m} < \gamma/2M$. Carry out the above construction for this value of m, and observe that because U_m is open, we have

$$\mu^m(U_m) \leq \varlimsup_{n_j\to\infty} \mu_{n_j}^m(U_m) \leq Ce^{-\eta m} < \frac{\gamma}{2M}, \tag{3.31}$$

where the middle inequality follows from (3.29) and (3.30). Consequently,

$$\left| \int_X \varphi \, d\mu^m - \int_X \varphi \, d\mu_{n_j}^m \right| \leq \\ \left| \int_{X\setminus U_m} \varphi \, d\mu^m - \int_{X\setminus U_m} \varphi \, d\mu_{n_j}^m \right| + \left| \int_{U_m} \varphi \, d\mu^m - \int_{U_m} \varphi \, d\mu_{n_j}^m \right|. \tag{3.32}$$

Since φ is continuous on the compact set $X \setminus U_m$, the first difference goes to 0 as $j \to \infty$, and by (3.31), the second term is less than γ; this proves (3.23) for μ^m. □

Lemma 3.3.5. *Let X be a compact metric space, let $f\colon X \to X$ be continuous,* *and let* $\psi\colon X \to \mathbb{R} \cup \{-\infty\}$ *be continuous (and hence bounded above). Fix* $Z \subset X$ *and let* $\beta \in \mathbb{R}$ *be given by*

$$\beta = \varliminf_{n\to\infty} \inf_{x\in Z} \frac{1}{n} S_n\psi(x).$$

Then for every $\gamma > 0$ *there exists* $\mu \in \mathcal{M}^f(X)$ *satisfying the following:*

$$\int \psi \, d\mu \geq \beta, \tag{3.33}$$

$$h(\mu) \geq \overline{Ch}_{\text{top}}(Z) - \gamma. \tag{3.34}$$

Proof. The proof is exactly as in Lemma 3.3.3 with the choice $\zeta = 0$, $\eta = \psi$, with Proposition 3.3.2 taking the place of Proposition 3.3.1. □

3.4 Proof of Theorem 3.1.1

The proof of Theorem 3.1.1 proceeds in three parts, corresponding to the three parts of the theorem. In the first part, we show that $T_{\mathcal{B}}$ is the Legendre transform of $\mathcal{B}$, thus establishing (3.3). From this, it immediately follows by standard properties of the Legendre transform that $T_{\mathcal{B}}^{L_1}$ is the concave hull of $\mathcal{B}$; that is, it is the smallest concave function greater than or equal to $\mathcal{B}$ at all α.

Part II of the theorem is an easy consequence of the following proposition.

Proposition 3.4.1. *Suppose that* $K_\alpha^{\mathcal{B}}$ *is non-empty; that is, there exists* $x \in X$ *such that* $\varphi^+(x) = \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha$. *Then* $P^*(q\varphi) \geq \alpha q$ *for all* $q \in \mathbb{R}$.

Once Part I is established, Part III of the theorem is proved via the following series of intermediate results.

Proposition 3.4.2. *Let* φ *be Borel measurable and suppose that* ν_q *is an ergodic equilibrium state for* $q\varphi$. *Let* $\alpha = \int \varphi \, d\nu_q$. *Then*

$$\mathcal{B}(\alpha) \geq T_{\mathcal{B}}^{L_1}(\alpha). \tag{3.35}$$

Note the requirement in Proposition 3.4.2 that the equilibrium state ν_q be ergodic. It will often be the case that general arguments will give the existence of *non*-ergodic equilibrium states with $\alpha(\nu_q) = \alpha$, but this is not sufficient for our purposes.

The following important result is well-known.

Proposition 3.4.3 (Ruelle's formula for the derivative of pressure)**.** *Let ψ and ϕ be Borel measurable functions. If the function*

$$q \mapsto P^*(\psi + q\phi)$$

is differentiable at q, and if in addition ν_q is an equilibrium state for $\psi + q\phi$, then

$$\frac{d}{dq} P^*(\psi + q\phi) = \int_X \phi \, d\nu_q. \tag{3.36}$$

Corollary 3.4.4. *Suppose $T_{\mathcal{B}}$ is continuously differentiable on (q_1, q_2) and $q\varphi$ has an equilibrium state ν_q for each $q \in (q_1, q_2)$. Let $\alpha_1 = D^+T_{\mathcal{B}}(q_1)$ and $\alpha_2 = D^-T_{\mathcal{B}}(q_2)$; then for every $\alpha \in (\alpha_1, \alpha_2)$ there exists $q \in \mathbb{R}$ such that $q\varphi$ has an ergodic equilibrium state ν_q with $\alpha = \int \varphi \, d\nu_q$.*

Once these results are established, (3.6) is a direct consequence of Proposition 3.4.2 and Corollary 3.4.4. It then follows from basic properties of the Legendre transform that $\mathcal{B} = T_{\mathcal{B}}^{L_1}$ has the same regularity as $T_{\mathcal{B}}$ (except for values of α corresponding to intervals on which $T_{\mathcal{B}}$ is affine).

Proof of part I. We prove (3.3) by establishing the following two inequalities:

$$T_{\mathcal{B}} \leq \mathcal{B}^{L_2}, \tag{3.37}$$

$$T_{\mathcal{B}} \geq \mathcal{B}^{L_2}. \tag{3.38}$$

First we prove (3.37). Recall that

$$T_{\mathcal{B}}(q) = P^*(q\varphi) = \sup_{\nu \in \mathcal{M}_E^f(X)} \left\{ h(\nu) + q \int_X \varphi \, d\nu \right\}.$$

By Birkhoff's ergodic theorem, every ergodic measure ν has $\nu(K_\alpha^{\mathcal{B}}) = 1$ for some α, and so for ν-almost every $x \in K_\alpha^{\mathcal{B}}$ (in particular, for *some* $x \in K_\alpha^{\mathcal{B}}$), we have

$\int_X \varphi\, d\nu = \varphi^+(x) = \alpha$. It follows that

$$T_{\mathcal{B}}(q) = \sup_{\alpha \in \mathbb{R}} \left(\sup_{\nu \in \mathcal{M}_E^f(K_\alpha^{\mathcal{B}})} \left\{ h(\nu) + q \int_X \varphi\, d\nu \right\} \right)$$
$$\leq \sup_{\alpha \in \mathbb{R}} \left(h_{\text{top}}\left(K_\alpha^{\mathcal{B}}\right) + q\alpha \right) = \mathcal{B}^{L_2}(q),$$

where the inequality $h(\nu) \leq h_{\text{top}}\left(K_\alpha^{\mathcal{B}}\right)$ follows from Theorem A2.1 in [Pes98].

Now we prove the reverse inequality (3.38), by showing that $T_{\mathcal{B}}(q) = P^*(q\varphi) \geq \mathcal{B}(\alpha) + q\alpha$ for all $q, \alpha \in \mathbb{R}$. To this end, we fix $\varepsilon > 0$ and consider the sets F_α^ε, $F_\alpha^{\varepsilon,N}$ defined in (3.18).

Applying Lemma 3.3.3 with $\zeta = 0$, $\psi = \varphi$, $Z = F_\alpha^{\varepsilon,N}$, and some $\gamma > 0$, we obtain a measure $\mu \in \mathcal{M}^f(X)$ with $h(\mu) \geq \underline{Ch}_{\text{top}}(F_\alpha^{\varepsilon,N}) - \gamma$ and $\int \varphi\, d\mu \geq \alpha - \varepsilon$. It follows that

$$P^*(q\varphi) \geq h(\mu) + q \int \varphi\, d\mu \geq \underline{Ch}_{\text{top}}(F_\alpha^{\varepsilon,N}) - \gamma + q\alpha - q\varepsilon,$$

and since Lemma 3.3.3 can be applied with arbitrarily small γ, we get

$$P^*(q\varphi) \geq h_{\text{top}}\left(F_\alpha^{\varepsilon,N}\right) + q\alpha - q\varepsilon.$$

Taking the supremum over all N yields

$$P^*(q\varphi) \geq h_{\text{top}}\left(F_\alpha^{\varepsilon}\right) + q\alpha - q\varepsilon \geq h_{\text{top}}\left(K_\alpha^{\mathcal{B}}\right) + q\alpha - q\varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, this implies

$$P^*(q\varphi) \geq h_{\text{top}}\left(K_\alpha^{\mathcal{B}}\right) + q\alpha.$$

This holds for all $q, \alpha \in \mathbb{R}$, which establishes (3.38). $\square$

We now proceed to the proof of Part II.

Proof of Proposition 3.4.1. Suppose $\alpha \in \mathbb{R}$ is such that there exists $x \in K_\alpha^{\mathcal{B}}$. Consider the empirical measures

$$\mu_{n,x} = \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

Choose any subsequence n_k such that $\mu_{n_k,x}$ converges in the weak* topology to some $\mu \in \mathcal{M}^f(X)$. Then by Proposition 3.3.1, we have $\int \varphi \, d\mu = \alpha$, and in particular,

$$P^*(q\varphi) \geq h(\mu) + \int q\varphi \, d\mu \geq q \int \varphi \, d\mu \geq q\alpha$$

for every $q \in \mathbb{R}$. □

Finally, we prove the string of propositions which implies Part III.

Proof of Proposition 3.4.2. Observe that since ν_q is ergodic, we have $\nu_q(K_\alpha^{\mathcal{B}}) = 1$, and hence $h(\nu_q) \leq h_{\text{top}}\,(K_\alpha^{\mathcal{B}})$. Thus

$$\begin{aligned} T_{\mathcal{B}}^{L_1}(\alpha) &= \inf_{q' \in \mathbb{R}} (T_{\mathcal{B}}(q') - q'\alpha') \\ &\leq T_{\mathcal{B}}(q) - q\alpha' = P^*(q\varphi) - q\alpha \\ &= h(\nu_q) + \int_X q\varphi \, d\nu_q - q\alpha \\ &\leq h_{\text{top}}\,(K_\alpha^{\mathcal{B}}) = \mathcal{B}(\alpha). \end{aligned}$$

□

Proof of Proposition 3.4.3. Write $g(q') = P^*(\psi + q'\phi)$. Then for all $q' \in \mathbb{R}$, we have

$$\begin{aligned} g(q') &= P^*(\psi + q'\phi) \\ &= \sup_\nu \left\{ h(\nu) + \int_X \psi \, d\nu + \int_X q'\phi \, d\nu \right\} \\ &\geq h(\nu_q) + \int_X \psi \, d\nu_q + q' \int_X \phi \, d\nu_q \\ &= P^*(\psi + q\phi) + (q' - q) \int_X \phi \, d\nu_q, \\ &= g(q) + (q' - q) \int_X \phi \, d\nu_q, \end{aligned}$$

whence

$$g(q') - g(q) \geq (q' - q) \int_X \phi \, d\nu_q.$$

In particular, for $q' > q$, we get

$$\frac{g(q') - g(q)}{q' - q} \geq \int_X \phi \, d\nu_q,$$

and hence $g'(q) \geq \int_X \phi \, d\nu_q$ (recall that differentiability of g was one of the hypotheses of the theorem), while for $q' < q$,

$$\frac{g(q') - g(q)}{q' - q} \leq \int_X \phi \, d\nu_q,$$

and hence $g'(q) \leq \int_X \phi \, d\nu_q$, which establishes equality. □

Proof of Corollary 3.4.4. Since $T'_{\mathcal{B}}$ is continuous, the Intermediate Value Theorem implies that for every such α there exists q such that $T'_{\mathcal{B}}(q) = \alpha$. Thus applying Proposition 3.4.3 with $\psi = 0$ and $\phi = \varphi$, we see that any equilibrium state ν for $q\varphi$ has $\nu(q\varphi) = \alpha$. Choose some such ν; if ν is not ergodic, then any element in its ergodic decomposition is also an equilibrium state, and we are done. □

3.5 Proof of Theorems 3.1.3, 3.1.4, and 3.1.5

Given the proof of Theorem 3.1.1 in the previous section, the proofs of Theorems 3.1.3, 3.1.4, and 3.1.5 are relatively straightforward.

Proof of Theorem 3.1.3. Recall that the first two parts of Theorem 3.1.1 hold without any assumptions on f, and thus we already have $T_{\mathcal{B}} = \mathcal{B}^{L_2}$. It remains only to show that $\mathcal{B}(\alpha) \geq T_{\mathcal{B}}^{L_1}(\alpha)$ for every $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, given Condition **(A)**.

Given such an α, if there exists $q \in \mathbb{R}$ such that $T'_{\mathcal{B}}(q) = \alpha$, then the proof of Theorem 3.1.1 shows that $\mathcal{B}(\alpha) = T_{\mathcal{B}}^{L_1}(\alpha)$. Thus we suppose that no such q exists; in this case, let $q_0 = Q(\alpha)$ be the (unique) value of q such that

$$T_{\mathcal{B}}(q) \geq T_{\mathcal{B}}(q_0) + (q - q_0)\alpha$$

for all $q \in \mathbb{R}$. (Equivalently, we have $q_0 = -(T_{\mathcal{B}}^{L_1})'(\alpha)$.)

Applying Theorem 3.1.1 to the subsystem X_n, we see that

$$h_{\text{top}}\,(K_\alpha^{\mathcal{B}} \cap X_n) = \inf_{q\in\mathbb{R}} (P^*_{X_n}(q\varphi) - q\alpha);$$

since $q \mapsto P^*_{X_n}(q\varphi)$ is assumed to be differentiable on $\mathbb{R}$, for every $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ there exists $q_n \in \mathbb{R}$ such that $A_n(q_n) = \frac{d}{dq} P^*_{X_n}(q\varphi)|_{q=q_n} = \alpha$. Let μ_n be an ergodic equilibrium state for $q_n\varphi$ on X_n; then $\int \varphi\, d\mu_n = \alpha$ by Proposition 3.4.3, and so $\mu_n(K_\alpha^{\mathcal{B}}) = 1$. Thus we have

$$h_{\text{top}}\, K_\alpha^{\mathcal{B}} \geq h(\mu_n) = P^*_{X_n}(q_n\varphi) - q_n\alpha. \tag{3.39}$$

It follows from convexity of the pressure function that $q_n \to q_0$ as n goes to ∞, and by continuity of the pressure function and Condition **(A)**, this implies that

$$\lim_{n\to\infty} P^*_{X_n}(q_n\varphi) = P^*(q_0\varphi),$$

which together with (3.39) shows that $\mathcal{B}(\alpha) \geq T_{\mathcal{B}}(q_0) - q_0\alpha \geq T_{\mathcal{B}}^{L_1}(\alpha)$. □

Proof of Theorem 3.1.4. The proof of Theorem 3.1.4 mirrors the proof of Theorem 3.1.1; the primary difference is that Lemma 3.3.4 replaces Lemma 3.3.3 in the proof of Part I, where we show (3.9).

The proof of Proposition 3.4.1 does not go through in this setting, and so Part II is weakened from the corresponding statement in Theorem 3.1.1.

The series of propositions in Part III goes through unchanged, as Proposition 3.4.2, Proposition 3.4.3, and Corollary 3.4.4 all hold without regard to continuity of the potential φ.

Observe that (3.37) holds here as well without modification, since its proof does not require any hypotheses on φ. Thus to prove (3.9), it suffices to establish the following inequality for every $q \in I_Q(h_0)$:

$$T_{\mathcal{B}}(q) \geq \sup_{\alpha\in I_A(h_0)} (\mathcal{B}(\alpha) + q\alpha). \tag{3.40}$$

That is, we show that $T_{\mathcal{B}}(q) = P^*(q\varphi) \geq \mathcal{B}(\alpha) + q\alpha$ for all $q \in I_Q(h_0)$ and $\alpha \in I_A(h_0)$. Observe that if $\mathcal{B}(\alpha) \leq h_0$, then since $\alpha \in I_A(h_0)$ we have $T_{\mathcal{B}}^{L_1}(\alpha) =$

$\inf_{q\in\mathbb{R}}(T_{\mathcal{B}}(q) - q\alpha) > h_0 \geq \mathcal{B}(\alpha)$, and so in particular $T_{\mathcal{B}}(q) \geq \mathcal{B}(\alpha) + q\alpha$ for $q \in I_Q(h_0)$. Thus it remains only to consider the case $\mathcal{B}(\alpha) > h_0$.

As in the proof of (3.38) in Theorem 3.1.1, we fix $\varepsilon > 0$ and consider the sets F_α^ε, $F_\alpha^{\varepsilon,N}$ defined in (3.18). Because $h_0 < \mathcal{B}(\alpha) = h_{\text{top}}\, K_\alpha^{\mathcal{B}} \leq h_{\text{top}}\, F_\alpha^\varepsilon = \sup_N h_{\text{top}}\, F_\alpha^{\varepsilon,N}$, we can find $N \in \mathbb{N}$ such that $h_{\text{top}}\, F_\alpha^{\varepsilon,N} > h_0$, and then apply Lemma 3.3.4 with $\psi = \varphi$, $Z = F_\alpha^{\varepsilon,N}$, and some $\gamma > 0$ to obtain a measure μ with $h(\mu) \geq \underline{Ch}_{\text{top}}(F_\alpha^{\varepsilon,N}) - \gamma$ and $\int \varphi\, d\mu \geq \alpha - \varepsilon - \gamma$. It follows that

$$P^*(q\varphi) \geq h(\mu) + q\int \varphi\, d\mu \geq \underline{Ch}_{\text{top}}(F_\alpha^{\varepsilon,N}) - \gamma + q\alpha - q(\varepsilon + \gamma),$$

and since Lemma 3.3.4 can be applied with arbitrarily small γ, we get

$$P^*(q\varphi) \geq h_{\text{top}}\,(F_\alpha^{\varepsilon,N}) + q\alpha - q\varepsilon.$$

Taking the supremum over all such N yields

$$P^*(q\varphi) \geq h_{\text{top}}\,(F_\alpha^\varepsilon) + q\alpha - q\varepsilon \geq h_{\text{top}}\,(K_\alpha^{\mathcal{B}}) + q\alpha - q\varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, this implies

$$P^*(q\varphi) \geq h_{\text{top}}\,(K_\alpha^{\mathcal{B}}) + q\alpha.$$

This holds for all $q \in I_Q(h_0)$ and $\alpha \in I_A(h_0)$, which establishes (3.40).

For Part II of Theorem 3.1.4, we observe that if $\mathcal{B}(\alpha) > h_0$, then we can apply Lemma 3.3.4 exactly as above to obtain $T_{\mathcal{B}}(q) \geq \mathcal{B}(\alpha) + q\alpha$ for all $q \in \mathbb{R}$, and hence $T_{\mathcal{B}}^{L_1}(\alpha) \geq \mathcal{B}(\alpha) > h_0$ as well, so $\alpha \in I_A(h_0)$.

As remarked above, the propositions in Part III go through unchanged, and we are done. □

Proof of Theorem 3.1.5. The proof of Parts I of Theorem 3.1.5 is nearly identical to the proof of Theorem 3.1.4, with Lemma 3.3.5 replacing Lemma 3.3.4 in the proof of (3.3), and with $(-\infty, 0)$ and $(-\infty, \alpha_0]$ replacing $I_Q(h_0)$ and $I_A(h_0)$.

Part II of Theorem 3.1.5 follows from the observation that Proposition 3.4.1 *does* apply in this setting as follows: if $K_\alpha^{\mathcal{B}}$ is non-empty for some $\alpha \in \mathbb{R}$, then $P^*(q\varphi) \geq \alpha q$ for all $q \leq 0$. The proof only requires replacing Proposition 3.3.1

with Proposition 3.3.2.

Again, (3.37) holds here without modification. Furthermore, for any $q \leq 0$ and $\alpha \in \mathbb{R}$, we may fix $\varepsilon > 0$ and apply Lemma 3.3.5 with $\psi = q\varphi$, $Z = F_\alpha^{\varepsilon,N}$, and some $\gamma > 0$ to obtain a measure $\mu \in \mathcal{M}^f(X)$ with $h(\mu) \geq \underline{Ch}_{\text{top}}(F_\alpha^{\varepsilon,N}) - \gamma$ and $\int q\varphi \, d\mu \geq q\alpha - q\varepsilon$. It follows that

$$P^*(q\varphi) \geq h(\mu) + \int q\varphi \, d\mu \geq \underline{Ch}_{\text{top}}(F_\alpha^{\varepsilon,N}) - \gamma + q\alpha - q\varepsilon,$$

and just as in the proof of Theorem 3.1.1, we obtain

$$P^*(q\varphi) \geq h_{\text{top}}\,(K_\alpha^{\mathcal{B}}) + q\alpha.$$

This holds for all $q \leq 0$ and $\alpha \in \mathbb{R}$, which establishes (3.38).

Part III is once again just as before. □

Chapter 4

Conformal maps and Bowen's equation

4.1 Pressure and dimension: known results

In order to establish multifractal results for spectra defined using Hausdorff dimension, we need to obtain a relationship between topological pressure and Hausdorff dimension. The first such connection was given by Bowen [Bow79], who showed that for certain compact sets (quasi-circles) $J \subset \mathbb{C}$ which arise as invariant sets of fractional linear transformations f of the Riemann sphere, the Hausdorff dimension $t = \dim_H J$ is the unique root of the equation

$$P_J(-t\varphi) = 0, \tag{4.1}$$

where P_J is the topological pressure of the map $f\colon J \to J$, and φ is the geometric potential $\varphi(z) = \log |f'(z)|$. Later, Ruelle showed that Bowen's equation (4.1) gives the Hausdorff dimension of J whenever f is a $C^{1+\varepsilon}$ conformal map on a Riemannian manifold and J is a repeller. More precisely, he proved the following [Rue82, Proposition 4]:

Theorem 4.1.1. *Let M be a Riemannian manifold and $V \subset M$ be open, and let $f\colon V \to M$ be $C^{1+\varepsilon}$ and conformal (that is, $Df(x)$ is a scalar multiple of an isometry for every $x \in V$). Suppose $J \subset V$ is a repeller—that is, it has the following properties:*

(1) J is compact.

(2) J is maximal: $J = \{x \in V \mid f^n(x) \in V$ *for all* $n > 0\}$.

(3) f is topologically mixing on J: *For every open set* $U \subset V$ *such that* $U \cap J \neq \emptyset$, *there exists* n *such that* $f^n(U) \supset J$.

(4) f is uniformly expanding on J: *There exist* $C > 0$ *and* $r > 1$ *such that* $\|Df^n v\| \geq Cr^n \|v\|$ *for every tangent vector* $v \in T_x M$ *and every* $n \geq 1$.

Let $\varphi(x) = \log \|Df(x)\|$. *Then Bowen's equation* (4.1) *has a unique root, and this root is equal to the Hausdorff dimension of* J.

This result was eventually extended to the case where f is C^1 by Gatzouras and Peres [GP97]. One can also give a definition of conformal map in the case where X is a metric space (not necessarily a manifold), and the analogue of Theorem 4.1.1 in this setting was proved by Rugh [Rug08].

In all of these settings, one of the essential tools is the availability of geometric bounds that relate statically defined metric balls $B(x, r)$ (used in the definition of dimension) to dynamically defined Bowen balls $B(x, n, \delta)$ (used in the definition of pressure). However, the above proofs differ in how these bounds are used. The proofs given by Bowen, Ruelle, and Gatzouras and Peres all rely on the construction of a measure of full dimension (in particular, a measure that is equivalent to Hausdorff measure), which in turn relies on the aforementioned geometric bounds (among other things). Rugh's proof, on the other hand, does not use measures and instead applies these bounds directly to the definitions of dimension and pressure.

These two methods of proof represent different approaches to the problem of using Bowen's equation to find the Hausdorff dimension of dynamically significant sets. In this chapter, we will follow the second approach (Rugh's) and avoid the use of measures; this will allow us to establish the analogue of Theorem 4.1.1 for a broad class of subsets of a repeller on which we may not have uniform expansion, and which need not carry any invariant measures. First, however, we will mention some of the other settings in which the approach using measures of full dimension has been successful.

Working with maps in one real dimension, Urbański [Urb96] proved that the smallest root of (4.1) gives the Hausdorff dimension of a repeller J that is expanding except on some set of indifferent fixed points, by finding a conformal measure that is the measure of full dimension. Similar results for Julia sets of maps in one complex dimension were proved in [DU91, Urb91]. In fact, Bowen's equation is also known to give the Hausdorff dimension of the Julia set for a broad class of rational maps (those satisfying the *topological Collet-Eckmann* condition) whose Julia sets even contain critical points [PRLS03, PRLS04]. There are also situations where conformal measures can be built when J is a non-compact set; for instance, when J is the radial Julia set of a meromorphic function satisfying certain conditions [UZ04, MU08, MU10].

Given a map f, all of the above results give the Hausdorff dimension of one very particular dynamically significant set J via Bowen's equation. It is natural to ask if one can find the Hausdorff dimension of subsets $Z \subset J$ via a similar approach.

For certain subsets, results in this direction are given by the multifractal analysis. In the uniformly expanding case, the multifractal results in [BPS97, PW97, Wei99] all boil down to the following result. If J is a conformal repeller and $\varphi \colon J \to \mathbb{R}$ is any Hölder continuous function, then for the one-parameter family of sets $K_\alpha \subset J$ given by

$$K_\alpha = \left\{ x \in J \,\middle|\, \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n \log \|Df(x)\|} = \alpha \right\},$$

we may define a convex analytic function $T \colon \mathbb{R} \to \mathbb{R}$ implicitly by

$$P_J(q\varphi - T(q)\log\|Df\|) = 0, \tag{4.2}$$

and obtain $\dim_H K_\alpha$ as the Legendre transform of T:

$$\dim_H K_\alpha = \inf_{q\in\mathbb{R}} (T(q) - q\alpha). \tag{4.3}$$

In the case $\varphi \equiv 0$, $\alpha = 0$, this reduces to Bowen's equation; for other values of φ and α, this may be seen as a sort of (indirect) generalisation of Theorem 4.1.1. Analogous results for certain almost-expanding conformal maps with neutral fixed points are at the heart of the multifractal analyses in [PW99, Nak00, GR09, GPR09, MU10].

Once again, these results all rely on the construction of measures of full dimension on the sets K_α as Gibbs states ν_q for the family of potentials $q\varphi - T(q)\log\|Df\|$, and so they do not generalise to more arbitrary subsets $Z \subset J$ (which may not support any invariant measures). A more natural generalisation of Theorem 4.1.1 would be to obtain $\dim_H Z$ as the root of $P_Z(-t\log\|Df\|) = 0$ using Pesin and Pitskel's notion of topological pressure *on the set* Z as a Carathéodory dimension characteristic. Such a generalisation is the goal of this chapter; this will allow us to prove the above multifractal results in a more general setting in Chapter 5, where the measures ν_q are only required to be equilibrium states, and not necessarily Gibbs.

Using the general theory of Carathéodory dimension characteristics introduced in [Pes98], Barreira and Schmeling [BS00] introduced the notion of the *u-dimension* $\dim_u Z$ for positive functions u, showing that $\dim_u Z$ is the unique number t such that $P_Z(-tu) = 0$. They also showed that for a subset Z of a conformal repeller J, where we may take $u = \log\|Df\| > 0$, we have $\dim_u Z = \dim_H Z$, and hence upon replacing P_J with P_Z, the Hausdorff dimension of any subset $Z \subset J$ is given by Bowen's equation, whether or not Z is compact or invariant.

Thus it has already been shown that in the uniformly expanding case, Theorem 4.1.1 holds not just for J itself, but for any subset $Z \subset J$. Furthermore, the aforementioned works of Urbański *et al* show that when we consider J itself, there are many cases in which the requirement that f be uniformly expanding can be replaced with rather weaker expansion properties. However, there do not appear to be any results at present that combine these two directions, and give a Bowen's equation result for arbitrary sets Z under properties weaker than uniform expansion (the closest results to this appear to be the multifractal results mentioned above). We give such a result in this chapter, showing that the applicability of Bowen's equation to arbitrary Z extends beyond the uniformly expanding case.

Indeed, given a conformal map f without critical points or singularities, the only requirement we place on the expansion properties of f is that every point x of Z has positive lower Lyapunov exponent, and that there not be too much contraction along the orbit of x (see (4.5) below—this is automatically satisfied if the Lyapunov exponent of x exists or if f is nowhere contracting). We do not require any uniformity in these hypotheses; Z may contain points with arbitrarily small or large Lyapunov exponents. Furthermore, these hypotheses are only required to hold at points in Z, and not for other points in phase space.

This result has an immediate application to the multifractal formalism, as already mentioned in Section 3.2.2 and the sections following it (see also (1.8)): for any conformal map without critical points or singularities (no expansion properties are required), it allows us to compute the dimension spectrum for Lyapunov exponents directly from the entropy spectrum for Lyapunov exponents, which can in turn be obtained from the pressure function, provided the latter has nice properties, as described in Chapter 3. Furthermore, we will use this result in Chapter 5 to compute the dimension spectrum for pointwise dimensions given certain thermodynamic information.

4.2 Definitions and statement of result

We consider a continuous map f acting on a compact metric space X.

Definition 4.2.1. We say that $f\colon X \to X$ is *conformal* with factor $a(x)$ if for every $x \in X$ we have

$$a(x) = \lim_{y\to x} \frac{d(f(x), f(y))}{d(x,y)}, \tag{4.4}$$

where $a\colon X \to [0,\infty)$ is continuous. We denote the Birkhoff sums of $\log a$ by

$$\lambda_n(x) = \frac{1}{n} S_n(\log a)(x) = \frac{1}{n}\sum_{k=0}^{n-1} \log a(f^k(x));$$

the lower and upper limits of this sequence are the *lower Lyapunov exponent* and *upper Lyapunov exponent*, respectively:

$$\underline{\lambda}(x) = \varliminf_{n\to\infty} \lambda_n(x), \qquad \overline{\lambda}(x) = \varlimsup_{n\to\infty} \lambda_n(x).$$

If the two agree (that is, if the limit exists), then their common value is the *Lyapunov exponent*:

$$\lambda(x) = \lim_{n\to\infty} \lambda_n(x).$$

Because a is assumed to be continuous on a compact space X, it is bounded above (and hence $\overline{\lambda}(x)$ is as well); we do not allow maps with singularities.

If $a(x) = 0$, then we say that x is a *critical point* for the map f. We will exclude such points from the maps we consider.

Note that in the case where X is a smooth Riemannian manifold, the definition of conformality may be restated as the requirement that $Df(x)$ is $a(x)$ times some isometry, and the definition of Lyapunov exponent becomes the usual one from smooth ergodic theory. In particular, if X is one-dimensional, then any differentiable map is conformal.

We relate the Hausdorff dimension of Z to the topological pressure of $\log a$ on Z, provided every point in Z has positive lower Lyapunov exponent and satisfies the following *tempered contraction* condition:

$$\inf_{\substack{n\in\mathbb{N}\\ 0\leq k\leq n}} \{S_{n-k}\log a(f^k(x)) + n\varepsilon\} > -\infty \text{ for every } \varepsilon > 0. \tag{4.5}$$

Denote by $\mathbf{B}$ the set of all points in X which satisfy (4.5). The following three criteria are useful for checking that $x \in \mathbf{B}$.

1. Proposition 4.4.3 shows that if the Lyapunov exponent of x exists and is positive—that is, if $\underline{\lambda}(x) = \overline{\lambda}(x) > 0$—then x satisfies (4.5).

2. If $a(x) \geq 1$ for all $x \in X$, then (4.5) is automatically satisfied for all $x \in X$, and so in this case $\mathbf{B} = X$.

3. We say that x has *bounded contraction* if $\inf\{S_{n-k}\log a(f^k(x)) \mid n \in \mathbb{N}, 0 \leq k \leq n\} > -\infty$. Any such point x satisfies (4.5).

Given $E \subset \mathbb{R}$, we denote by $\mathcal{A}(E)$ the set of points along whose orbits all the asymptotic exponential expansion rates of the map f lie in E:

$$\mathcal{A}(E) = \{x \in X \mid [\underline{\lambda}(x), \overline{\lambda}(x)] \subset E\}.$$

In particular, $\mathcal{A}((0,\infty))$ is the set of all points for which $\underline{\lambda}(x) > 0$. Our main result deals with subsets $Z \subset X$ that lie in both $\mathcal{A}((0,\infty))$ and $\mathbf{B}$. (Observe that by Proposition 4.4.3, $\mathcal{A}(\alpha) = \mathcal{A}(\{\alpha\}) \subset \mathbf{B}$ for every $\alpha > 0$.)

Theorem 4.2.1. *Let X be a compact metric space and $f\colon X \to X$ be continuous and conformal with factor $a(x)$. Suppose that f has no critical points and no* *singularities—that is, that* $0 < a(x) < \infty$ *for all* $x \in X$*. Consider* $Z \subset \mathcal{A}((0,\infty)) \cap \mathbf{B}$*. Then the Hausdorff dimension of* Z *is given by*

$$\begin{aligned} \dim_H Z = t^* &= \sup\{t \geq 0 \mid P_Z(-t \log a) > 0\} \\ &= \inf\{t \geq 0 \mid P_Z(-t \log a) \leq 0\}. \end{aligned} \tag{4.6}$$

Furthermore, if $Z \subset \mathcal{A}((\alpha,\infty)) \cap \mathbf{B}$ *for some* $\alpha > 0$ *(that is, the lower Lyapunov exponents of points in* Z *are* uniformly *positive), then* t^* *is the unique root of Bowen's equation*

$$P_Z(-t \log a) = 0. \tag{4.7}$$

Finally, if $Z \subset \mathcal{A}(\alpha)$ *for some* $\alpha > 0$*, then* $P_Z(-t \log a) = h_{\text{top}} Z - t\alpha$*, and hence*

$$\dim_H Z = \frac{1}{\alpha} h_{\text{top}} Z. \tag{4.8}$$

Before proceeding to specific examples and to the proofs, we make a few remarks on Theorem 4.2.1 in some standard settings.

1. For expanding conformal maps ($a(x) > 1$ for all x), we have $\mathbf{B} = X$, and Theorem 4.2.1 reduces to Barreira and Schmeling's generalisation of Theorem 4.1.1, although we work in the slightly more general setting where X need not be a manifold.

2. For almost expanding conformal maps (maps which are expanding away from a collection of indifferent periodic points), we have $a(x) \geq 1$ for all x, and so $\mathbf{B} = X$; thus the theorem applies, showing that Bowen's formula gives the Hausdorff dimension of any set which does not contain any points with zero lower Lyapunov exponent. This complements the results in [DU91, Urb91, Urb96], which give the Hausdorff dimension of the *entire* Julia set for a large family of almost expanding conformal maps, but have nothing to say about arbitrary subsets of the Julia set. (Observe that because the Julia set contains points with zero Lyapunov exponent, Theorem 4.2.1 does not give the Hausdorff dimension of the entire Julia set.)

3. For maps with some contracting regions ($a(x) < 1$) but no critical points ($a(x) = 0$), we cannot rule out the possibility that $\mathbf{B} \neq X$. However, the

result still holds for $Z \subset X$ as long as every point $x \in Z$ satisfies (4.5) and has positive lower Lyapunov exponent. In particular, if the Lyapunov exponent is constant and positive on Z, then (4.8) relates the Hausdorff dimension of Z to the topological entropy of Z; this proves the equality claimed earlier in (1.8).

4.3 Applications

4.3.1 Lyapunov spectra

The dimension spectrum for Lyapunov exponents of a conformal repeller was studied by Weiss [Wei99], who proved that it is real analytic on an interval (α_1, α_2), and may be obtained in terms of the Legendre transform of the pressure function.[1]

The proof in [Wei99] is roundabout, and analyses $\mathcal{L}_D(\alpha)$ in terms of the dimension spectrum for pointwise dimensions of a measure of maximal entropy, by showing that for such a measure the level sets of the pointwise dimension coincide with the level sets of the Lyapunov exponent (this may also be shown using the fact that the local entropy of such a measure is constant everywhere and applying Lemma 4.4.4 below), and then applying results from [PW97].

By using Theorems 3.1.1 and 4.2.1, we obtain a more direct proof of this result. When f is a $C^{1+\varepsilon}$ expanding conformal map on a repeller J, it is well-known that the pressure function $T\colon t \mapsto P_J(-t \log a)$ is real analytic and strictly convex (provided $\log a$ is not cohomologous to a constant, or equivalently, that the measure of maximal dimension and the measure of maximal entropy do not coincide). Then Theorem 3.1.1 applies to the entropy spectrum for Lyapunov exponents, and Theorem 4.2.1 applies to the level sets $K_\alpha^{\mathcal{L}}$: together, they give a more direct proof of the following well-known result.

Proposition 4.3.1. *Let $f\colon V \to M$ be as in Theorem 4.1.1, and let J be a uniformly expanding repeller. Then the Lyapunov spectra of f are given in terms of*

[1] Weiss also claims that the spectrum is concave, but Iommi and Kiwi have shown that there are examples in which this is not the case [IK09].

the Legendre transform of the pressure function as follows:

$$\begin{aligned} \mathcal{L}_E(\alpha) &= \inf_{t\in\mathbb{R}}(P_J(-t\log a) - \alpha t), \\ \mathcal{L}_D(\alpha) &= \frac{1}{\alpha}\inf_{t\in\mathbb{R}}(P_J(-t\log a) - \alpha t). \end{aligned} \tag{4.9}$$

In particular, if $\log a$ is not cohomologous to a constant, then the spectrum $\mathcal{L}_E$ is strictly concave, and both spectra are real analytic (this follows from analyticity of the pressure function and standard properties of the Legendre transform).

For non-uniformly expanding conformal repellers, such as Manneville–Pomeau maps and parabolic rational maps, this approach gives the partial results described in Section 3.2.4.1. (To obtain the complete results given in [GPR09], we would need to apply Theorem 3.1.3.)

4.3.2 Symbolic dynamics

We now describe a class of systems to which these results may be applied, for which the phase space is not a manifold. Fix an integer $k \geq 2$, and let $X = \Sigma_k^+$ be the full one-sided shift on k symbols. Given $x, y \in X$, let $x \wedge y$ denote the common prefix of x and y—that is, if n is the unique integer such that $x_i = y_i$ for all $1 \leq i \leq n$, but $x_{n+1} \neq y_{n+1}$, then

$$x \wedge y = x_1 \dots x_n = y_1 \dots y_n.$$

Let $\psi\colon \bigcup_{n\geq 0}\{1, \dots, k\}^n \to \mathbb{R}^+$ be a function defined on the space of all finite words on the alphabet $\{1, \dots, k\}$, and suppose that ψ is such that for every $x \in X$, the sequence $\{\psi(x_1 \dots x_n)\}$ is non-increasing and approaches 0 as $n \to \infty$. Then $d(x, y) = \psi(x \wedge y)$ defines a metric on $X = \Sigma_k^+$; to prove this, one needs only verify the triangle inequality, or equivalently, show that

$$\psi(x \wedge z) \leq \psi(x \wedge y) + \psi(y \wedge z)$$

for every $x, y, z \in X$. This follows from the observation that if n is the length of the common prefix of x and z, then either $x_{n+1} \neq y_{n+1}$ or $y_{n+1} \neq z_{n+1}$: without loss of generality, suppose the first holds, and then we have

$$\psi(x \wedge y) = \psi(x_1 \dots x_m) \geq \psi(x_1 \dots x_n) = \psi(x \wedge z)$$

for some $0 \leq m \leq n$. Thus d is a metric, and the requirement that $\psi(x_1 \dots x_n) \to 0$ guarantees that d induces the product topology on $X = \{1, \dots, k\}^{\mathbb{N}}$.

In order for the shift σ to be conformal, we require the following limit to exist for every $x \in X$:

$$a(x) = \lim_{n\to\infty} \frac{\psi(x_2 \dots x_n)}{\psi(x_1 \dots x_n)}. \tag{4.10}$$

Furthermore, we demand that $a(x)$ depend continuously on x. If these conditions are satisfied, then the shift σ is conformal with factor $a(x)$ given by (4.10): indeed, given any $x, y \in X$ such that the length of the common prefix is n, we have

$$\frac{d(\sigma(x), \sigma(y))}{d(x,y)} = \frac{\psi(\sigma(x) \wedge \sigma(y))}{\psi(x \wedge y)} = \frac{\psi(x_2 \dots x_n)}{\psi(x_1 \dots x_n)},$$

and since $y \to x$ if and only if $n \to \infty$, this gives

$$\lim_{y\to x} \frac{d(\sigma(x), \sigma(y))}{d(x,y)} = \lim_{n\to\infty} \frac{\psi(x_2 \dots x_n)}{\psi(x_1 \dots x_n)} = a(x).$$

Given the above conditions, we may apply Theorem 4.2.1 to subsets $Z \subset X$ on which the lower Lyapunov exponents are positive and we have tempered contraction. We now describe some simple candidates for the function ψ, for which the results take on a straightforward form (and for which tempered contraction is automatic).

Example 4.3.2. Fix $\theta > 1$ and let $\psi(x_1 \dots x_n) = \theta^{-n}$, so that $d(x,y) = \theta^{-n}$, where $n+1$ is the first entry in which x and y differ. Then $a(x) = \theta$ for every $x \in X$, and it follows from (4.8) that for every $Z \subset X$, we have

$$\dim_H Z = \frac{h_{\text{top}} Z}{\log \theta}.$$

Example 4.3.3. Fix $\theta_1, \theta_2, \dots, \theta_k \geq 1$ and define ψ by

$$\psi(x_1 \dots x_n) = \frac{1}{n}(\theta_{x_1}\theta_{x_2} \cdots \theta_{x_n})^{-1}.$$

(The factor of $\frac{1}{n}$ is necessary to ensure that $\psi(x_1 \dots x_n) \to 0$ even if all but finitely many of the θ_{x_i} are equal to 1; it can be omitted if $\theta_j > 1$ for all j.) Then $a(x) = \theta_{x_1}$ is continuous, and $a(x) \geq 1$ for all x, so $\mathbf{B} = X$.

It follows that (4.6) gives the Hausdorff dimension of any set $Z \subset X$ on which the lower Lyapunov exponents are positive (in this case, the lower Lyapunov exponent of x is determined solely by the asymptotic frequency of the various symbols $1, \dots, k$ in the expansion of x).

If we consider the level sets of Lyapunov exponents, then we see that the entropy and dimension spectra for Lyapunov exponents are once again related by (1.8).

4.4 Proof of Theorem 4.2.1

4.4.1 Preparatory results

We proceed now to the proofs, beginning with some preparatory results that give information about the pressure on a set when we know something about the Birkhoff averages on that set.

Proposition 4.4.1. *Given $f\colon X \to X$, $\varphi\colon X \to \mathbb{R}$, and $Z \subset X$, suppose there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha \leq \varliminf_{n\to\infty} \frac{1}{n} S_n\varphi(x) \leq \varlimsup_{n\to\infty} \frac{1}{n} S_n\varphi(x) \leq \beta$$

for every $x \in Z$, and write $\gamma(t) = P_Z(t\varphi)$. Then the graph of γ lies between the lines of slope α and β through any point $(t, \gamma(t)) \subset \mathbb{R}^2$; that is,

$$\gamma(t) + \alpha h \leq \gamma(t+h) \leq \gamma(t) + \beta h \tag{4.11}$$

for all $t \in \mathbb{R}$, $h > 0$.

Proof. Let $\varepsilon > 0$ be arbitrary. Given $m \geq 1$, let

$$Z_m = \left\{ x \in Z \,\middle|\, \frac{1}{n} S_n\varphi(x) \in (\alpha - \varepsilon, \beta + \varepsilon) \text{ for all } n \geq m \right\},$$

and observe that $Z = \bigcup_{m=1}^{\infty} Z_m$. Now fix $t \in \mathbb{R}$, $h > 0$, and $N \geq m$. It follows from the definition of Z_m that for any $\delta > 0$ and $s \in \mathbb{R}$ we have

$$\begin{aligned} m_P(Z_m, s, &(t+h)\varphi, N, \delta) \\ &= \inf_{\mathcal{P}(Z_m,N,\delta)} \sum_{(x_i,n_i)} \exp(-n_i s + (t+h) S_{n_i}\varphi(x_i)) \\ &\geq \inf_{\mathcal{P}(Z_m,N,\delta)} \sum_{(x_i,n_i)} \exp(-n_i s + t S_{n_i}\varphi(x_i) + n_i h(\alpha - \varepsilon)) \\ &= m_P(Z_m, s - h(\alpha-\varepsilon), t\varphi, N, \delta). \end{aligned}$$

Letting $N \to \infty$, this gives

$$m_P(Z_m, s, (t+h)\varphi, \delta) \geq m_P(Z_m, s - h(\alpha - \varepsilon), t\varphi, \delta);$$

in particular, if the second quantity is equal to ∞, then the first is as well. Letting $\delta \to 0$, it follows that

$$P_{Z_m}((t+h)\varphi) \geq P_{Z_m}(t\varphi) + h(\alpha - \varepsilon).$$

Taking the supremum over all $m \geq 1$ and using the fact that topological pressure is countably stable—that is, that $P_Z(\varphi) = \sup_m P_{Z_m}(\varphi)$ (see [Pes98, Theorem 11.2(3)])—we obtain

$$\gamma(t+h) \geq \gamma(t) + h(\alpha - \varepsilon);$$

since $\varepsilon > 0$ was arbitrary, this establishes the first half of (4.12). The second half is proved similarly; an analogous computation shows that

$$m_P(Z_m, s, (t+h)\varphi, N, \delta) \leq m_P(Z_m, s - h(\beta + \varepsilon), t\varphi, N, \delta),$$

whence upon passing to the limits and taking the supremum, we have

$$\gamma(t+h) \leq \gamma(t) + h\beta. \qquad \square$$

Corollary 4.4.2. *Let $f\colon X \to X$ be as in Theorem 4.2.1. Fix $0 < \alpha \leq \beta < \infty$ and $Z \subset \mathcal{A}([\alpha, \beta])$, and write $\gamma(t) = P_Z(-t \log a)$. Then the following hold:*

1. *γ is Lipschitz continuous with Lipschitz constant β and strictly decreasing*

with rate at least α; that is, for every $t \in \mathbb{R}$ and $h > 0$ we have

$$\gamma(t) - \beta h \leq \gamma(t+h) \leq \gamma(t) - \alpha h. \tag{4.12}$$

2. *The equation* (4.7) *has a unique root t^*; furthermore,*

$$\frac{h_{\text{top}}(Z)}{\beta} \leq t^* \leq \frac{h_{\text{top}}(Z)}{\alpha}.$$

3. *If $\alpha = \beta$, so that $Z \subset \mathcal{A}(\alpha)$, then the unique root of* (4.7) *is $t^* = h_{\text{top}}(Z)/\alpha$.*

Proof. (1) follows from Proposition 4.4.1 with $\varphi = -\log a$. (2) follows from the Intermediate Value Theorem by observing that the map $\tau \mapsto P_Z(-\tau \log a)$ is continuous and strictly decreasing, and that by (4.12) applied with $t = 0$ and $h = \tau$, we have in the first place,

$$P_Z(-\tau \log a) \geq P_Z(0) - \tau\beta = h_{\text{top}}(Z) - \tau\beta,$$

so that $P_Z(-(h_{\text{top}}(Z)/\beta)\log a) \geq 0$, and in the second place,

$$P_Z(-\tau \log a) \leq P_Z(0) - \tau\alpha = h_{\text{top}}(Z) - \tau\alpha,$$

so that $P_Z(-(h_{\text{top}}(Z)/\alpha)\log a) \leq 0$. Then (3) follows immediately. □

Proposition 4.4.3. *Let $f \colon X \to X$ be as in Theorem 4.2.1, and suppose that $\lambda(x)$ exists and is positive. Then $x \in \mathbf{B}$.*

Proof. Fix $\varepsilon > 0$ such that $\lambda(x) > \varepsilon$, and choose $m \in \mathbb{N}$ such that $|\lambda_n(x) - \lambda| < \varepsilon$ for all $n \geq m$. Let $\eta > 0$ be such that

$$\log \eta = \min_{0 \leq j \leq m} \{S_j(\log a)(x)\} - \max_{0 \leq k \leq m} \{S_k(\log a)(x)\};$$

thus for every $n \leq m$ and $0 \leq k \leq n$, we have

$$S_{n-k} \log a(f^k(x)) \geq \log \eta.$$

Furthermore, for all $n \geq m$ and $0 \leq k \leq n$, we have

$$S_n(\log a)(x) \geq n(\lambda(x) - \varepsilon),$$

and either $0 \leq k \leq m$, in which case $S_k(\log a) \leq -\log \eta$, or $m \leq k \leq n$, in which case

$$S_k(\log a)(x) \leq k(\lambda(x) + \varepsilon).$$

Both these upper bounds are non-negative, and so together they imply

$$S_k(\log a)(x) \leq -\log \eta + k(\lambda(x) + \varepsilon),$$

which yields

$$\begin{aligned} S_{n-k}(\log a)(f^k(x)) &= S_n \log a(x) - S_k \log a(x) \\ &\geq n(\lambda(x) - \varepsilon) + \log \eta - k(\lambda(x) + \varepsilon) \\ &\geq \log \eta - 2n\varepsilon. \end{aligned}$$

It follows that $S_{n-k}(\log a)(f^k(x)) + 2n\varepsilon \geq \log \eta$ for every $0 \leq k \leq n$, and since $\varepsilon > 0$ was arbitrary, we have that x satisfies (4.5). □

4.4.2 A geometric lemma

In order to draw a connection between the Hausdorff dimension of Z and the topological pressure of $-t \log a$ on Z, we need to establish a relationship between the two collections of covers $\mathcal{D}(Z, \varepsilon)$ and $\mathcal{P}(Z, N, \delta)$. Thus we prove the following lemma, which relates regular balls $B(x, r)$ to Bowen balls $B(x, n, \delta)$.

Lemma 4.4.4. *Let $f\colon X \to X$ be as in Theorem 4.2.1. Then given any $x \in \mathbf{B}$ and $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ and $\eta = \eta(x, \varepsilon) > 0$ such that for every $n \in \mathbb{N}$ and $0 < \delta < \delta_0$,*

$$B\left(x, \eta\delta e^{-n(\lambda_n(x)+\varepsilon)}\right) \subset B(x, n, \delta) \subset B\left(x, \delta e^{-n(\lambda_n(x)-\varepsilon)}\right). \tag{4.13}$$

Proof. Since f is conformal with factor $a(x) > 0$, we have

$$\lim_{y \to x} \frac{d(f(x), f(y))}{d(x, y)} = a(x).$$

Since $a(x) > 0$ everywhere, we may take logarithms and obtain

$$\lim_{y \to x} (\log d(f(x), f(y)) - \log d(x, y)) = \log a(x).$$

The pre-limit expression is a function on the direct product $X \times X$ with the diagonal $D = \{(x, x) \in X \times X\}$ removed; because f is conformal, this function extends continuously to all of $X \times X$. That is, there exists a continuous function $\zeta \colon X \times X \to \mathbb{R}$ such that

$$\zeta(x, y) = \begin{cases} \log d(f(x), f(y)) - \log d(x, y) & x \neq y, \\ \log a(x) & x = y. \end{cases}$$

Because $X \times X$ is compact, ζ is uniformly continuous, hence given $\varepsilon > 0$ there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that for every $0 < \delta < \delta_0$ and $(x, y), (x', y') \in X \times X$ with

$$(d \times d)((x, y), (x', y')) = d(x, x') + d(y, y') < \delta,$$

we have $|\zeta(x, y) - \zeta(x', y')| < \varepsilon$. In particular, for $x, y \in X$ with $d(x, y) < \delta$, we have $(d \times d)((x, y), (x, x)) < \delta$, and hence

$$|\log d(f(x), f(y)) - \log d(x, y) - \log a(x)| = |\zeta(x, y) - \zeta(x, x)| < \varepsilon.$$

We may rewrite this inequality as

$$\log d(f(x), f(y)) - \log a(x) - \varepsilon < \log d(x, y) < \log d(f(x), f(y)) - \log a(x) + \varepsilon,$$

and taking exponentials, we obtain

$$d(f(x), f(y))e^{-(\log a(x) + \varepsilon)} < d(x, y) < d(f(x), f(y))e^{-(\log a(x) - \varepsilon)} \tag{4.14}$$

whenever the middle quantity is less than δ.

We now show the second half of (4.13), and then go back and prove the first half. Suppose $y \in B(x, n, \delta)$; that is, $d(f^k(y), f^k(x)) < \delta$ for all $0 \le k \le n$. Then repeated application of the second inequality in (4.14) yields

$$\begin{aligned} d(x,y) &< d(f(x), f(y))e^{-(\log a(x) - \varepsilon)} \\ &< d(f^2(x), f^2(y))e^{-(\log a(f(x)) - \varepsilon)}e^{-(\log a(x) - \varepsilon)} \\ &= d(f^2(x), f^2(y))e^{-S_2(\log a)(x) - 2\varepsilon} \\ &< \cdots \\ &< d(f^n(x), f^n(y))e^{-S_n(\log a)(x) - n\varepsilon} \\ &< \delta e^{-n(\lambda_n(x) - \varepsilon)}. \end{aligned}$$

The second inclusion in (4.13) follows.

To prove the first inclusion in (4.13), we observe that if $d(x, y) < \delta$, then the first inequality in (4.14) yields

$$d(f(x), f(y)) < d(x, y)e^{\log a(x) + \varepsilon}.$$

Then if $d(x, y) < \delta e^{-(\log a(x) + \varepsilon)}$, we have $d(f(x), f(y)) < \delta$, and so

$$\begin{aligned} d(f^2(x), f(y)) &< d(f(x), f(y))e^{\log a(f(x)) + \varepsilon} \\ &< d(x, y)e^{2(\lambda_2(x) + \varepsilon)}. \end{aligned}$$

Continuing in this manner, we see that if

$$d(x, y) < \delta e^{-k(\lambda_k(x) + \varepsilon)}$$

for every $0 \le k \le n$, we have $d(f^k(x), f^k(y)) < \delta$ for every $0 \le k \le n$, and hence $y \in B(x, n, \delta)$. Thus we have proved that

$$B\left(x, \delta \min_{0 \le k \le n} e^{-k(\lambda_k(x) + \varepsilon)}\right) \subset B(x, n, \delta), \tag{4.15}$$

which is almost what we wanted. If the minimum was always achieved at $k = n$, we would be done; however, this may not be the case. Indeed, if $\log a(f^n(x)) < -\varepsilon$

for some $n \in \mathbb{N}$, then the minimum will be achieved for some smaller value of k.

We now show that the tempered contraction assumption (4.5) allows us to replace $e^{-k(\lambda_k(x)+\varepsilon)}$ with the corresponding expression for $k = n$, at the cost of multiplying by some constant $\eta > 0$ and replacing ε with 2ε. To see what η should be, we observe that

$$\frac{e^{-n(\lambda_n(x)+2\varepsilon)}}{e^{-k(\lambda_k(x)+\varepsilon)}} = \frac{e^{-S_n \log a(x)-2n\varepsilon}}{e^{-S_k \log a(x)-k\varepsilon}} = e^{-(S_{n-k}\log a(f^k(x))+2n\varepsilon-k\varepsilon)} \leq e^{-(S_{n-k}\log a(f^k(x))+n\varepsilon)}.$$

Since x has tempered contraction, there exists $\eta = \eta(x, \varepsilon) > 0$ such that

$$\log \eta < S_{n-k}(\log a)(f^k(x)) + n\varepsilon \tag{4.16}$$

for all $n \in \mathbb{N}$, $0 \leq k \leq n$, and hence

$$e^{-(S_{n-k}(\log a)(f^k(x))+n\varepsilon)} < \frac{1}{\eta}.$$

Thus for every such n, k, we have

$$\eta e^{-n(\lambda_n(x)+2\varepsilon)} \leq e^{-k(\lambda_k(x)+\varepsilon)},$$

which along with (4.15) shows that

$$B(x, \delta\eta e^{-n(\lambda_n(x)+2\varepsilon)}) \subset B(x, n, \delta).$$

Taking $\delta_0 = \delta_0(\varepsilon/2)$ gives the stated version of the result. We remark that if x has *bounded* contraction, then $\eta = \eta(x)$ may be chosen independently of ε. Furthermore, if $a(x) \geq 1$ for all $x \in X$, then $\eta = 1$ suffices. □

4.4.3 Completion of the proof

Using Lemma 4.4.4, we can prove the theorem for sets $Z \subset \mathcal{A}((\alpha, \infty))$, where $0 < \alpha < \infty$; the general result will then follow from countable stability of topological pressure. (Note that writing $\beta = \sup_{x\in X} \log a(x)$, we have $\mathcal{A}((\alpha, \infty)) \subset \mathcal{A}((\alpha, \beta])$; we do not allow maps with singularities, so all Lyapunov exponents are finite.)

Lemma 4.4.5. *Let f satisfy the conditions of Theorem 4.2.1, and fix a set $Z \subset \mathcal{A}((\alpha,\infty)) \cap \mathbf{B}$, where $0 < \alpha < \infty$. Let t^* be the unique real number such that $P_Z(-t^* \log a) = 0$, whose existence and uniqueness is guaranteed by Corollary 4.4.2. Then $\dim_H Z = t^*$.*

Proof. First we show that $\dim_H Z \leq t^*$. Given $m \geq 1$, consider the set

$$Z_m = \{x \in Z \mid \lambda_n(x) > \alpha \text{ for all } n \geq m\},$$

and observe that $Z = \bigcup_{m=1}^{\infty} Z_m$. Fix $t > t^*$; since $P_Z(-t \log a) < 0$, there exists $\varepsilon \in (0, \alpha)$ such that $-t\varepsilon > P_Z(-t \log a)$. By Lemma 4.4.4, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that for every $x \in Z_m$, $0 < \delta \leq \delta_0$, and $n \geq m$, we have

$$\operatorname{diam} B(x,n,\delta) \leq 2\delta e^{-n(\lambda_n(x)-\varepsilon)} \leq 2\delta e^{-n(\alpha-\varepsilon)} \tag{4.17}$$

Thus given $N > m$ and $0 < \delta \leq \delta_0$, we have

$$\mathcal{P}(Z_m, N, \delta) \subset \mathcal{D}\left(Z_m, 2\delta e^{-N(\alpha-\varepsilon)}\right).$$

For any such N and δ, this allows us to relate the set functions which appear in the definitions of Hausdorff dimension and topological pressure as follows:

$$\begin{aligned}
&m_P(Z_m, -t\varepsilon, -t\log a, N, \delta) \\
&\qquad = \inf_{\mathcal{P}(Z_m,N,\delta)} \sum_{(x_i,n_i)} \exp(-n_i(-t\varepsilon) - tS_{n_i}(\log a)(x_i)) \\
&\qquad = \inf_{\mathcal{P}(Z_m,N,\delta)} \sum_{(x_i,n_i)} \exp(-n_i t(\lambda_{n_i}(x_i) - \varepsilon)) \\
&\qquad \geq \inf_{\mathcal{P}(Z_m,N,\delta)} \sum_{(x_i,n_i)} \left(\frac{1}{2\delta} \operatorname{diam} B(x_i,n_i,\delta)\right)^t \\
&\qquad \geq \inf_{\mathcal{D}(Z_m,2\delta e^{-N(\alpha-\varepsilon)})} \sum_{U_i} (2\delta)^{-t} (\operatorname{diam} U_i)^t \\
&\qquad = (2\delta)^{-t} m_H\left(Z_m, t, 2\delta e^{-N(\alpha-\varepsilon)}\right).
\end{aligned}$$

Taking the limit as $N \to \infty$ gives

$$m_P(Z_m, -t\varepsilon, -t\log a, \delta) \geq (2\delta)^{-t} m_H(Z_m, t), \tag{4.18}$$

for all $0 < \delta < \delta_0$. By our choice of ε, we have

$$-t\varepsilon > P_Z(-t\log a) \geq P_{Z_m}(-t\log a) = \lim_{\delta\to 0} P_{Z_m}(-t\log a, \delta),$$

and so for sufficiently small $\delta > 0$, we have $-t\varepsilon > P_{Z_m}(-t\log a, \delta)$, and hence $m_H(Z_m, t) = 0$ by (4.18), which implies $\dim_H(Z_m) \leq t$.

Since this holds for all $t > t^*$, we have $\dim_H(Z_m) \leq t^*$, and taking the union over all m gives $\dim_H(Z) \leq t^*$.

For the other inequality, $\dim_H Z \geq t^*$, we fix $t < t^*$ and show that $\dim_H Z \geq t$. We may assume that $t > 0$, or there is nothing to prove. By Corollary 4.4.2, t^* is the unique real number such that $P_Z(-t^*\log a) = 0$, and since the pressure function is decreasing, we have $P_Z(-t\log a) > 0$. Thus we can choose $\varepsilon > 0$ such that

$$0 < t\varepsilon < P_Z(-t\log a).$$

Let $\delta_0 = \delta_0(\varepsilon)$ be as in Lemma 4.4.4. Given $m \geq 1$, consider the set

$$Z_m = \{x \in Z \mid (4.13) \text{ holds with } \eta = e^{-m} \text{ for all } n \in \mathbb{N} \text{ and } 0 < \delta < \delta_0\}.$$

Observe that $Z = \bigcup_{m=1}^{\infty} Z_m$, and so $P_Z(-t\log a) = \sup_m P_{Z_m}(-t\log a)$, where we once again use countable stability [Pes98, Theorem 11.2(3)]. Thus there exists $m \in \mathbb{N}$ such that $t\varepsilon < P_{Z_m}(-t\log a)$, and we fix $0 < \delta < \delta_0$ such that

$$t\varepsilon < P_{Z_m}(-t\log a, \delta). \tag{4.19}$$

Let $\beta = \sup_{x\in X} \log a(x) < \infty$. Write $s_n(x) = e^{-m}\delta e^{-n(\lambda_n(x)+\varepsilon)}$, and note that

$$\frac{s_n(x)}{s_{n+1}(x)} = \frac{e^{-S_n \log a(x) - n\varepsilon}}{e^{-S_{n+1}\log a(x) - (n+1)\varepsilon}} = a(f^n(x))e^{\varepsilon} \leq e^{\beta+\varepsilon} \tag{4.20}$$

for every n and x. Furthermore, given $x \in Z_m$ and $r > 0$ small, there exists

$n = n(x, r)$ such that

$$s_n(x)e^{-(\beta+\varepsilon)} \leq s_{n+1}(x) \leq r \leq s_n(x) = e^{-m}\delta e^{-n(\lambda_n(x)+\varepsilon)}. \tag{4.21}$$

For this value of n, Lemma 4.4.4 implies that

$$B(x, r) \subset B(x, n, \delta);$$

consequently, given any $\{(x_i, r_i)\}$ such that $Z_m \subset \bigcup_i B(x_i, r_i)$, we also have $Z_m \subset \bigcup_i B(x_i, n_i, \delta)$, where $n_i = n(x_i, r_i)$ satisfies (4.21).

Furthermore, we have $\lambda_n(x) \leq \beta$ for all $n \in \mathbb{N}$ and $x \in X$, and so $s_n(x) \geq \delta e^{-(m+n(\beta+\varepsilon))}$. It follows from (4.21) that for $n = n(x, r)$, we have

$$\delta e^{-(m+(n+1)(\beta+\varepsilon))} \leq r,$$

and hence

$$n \geq \frac{-\log r + \log \delta - m}{\beta + \varepsilon} - 1.$$

Denote the quantity on the right by $N(r, \delta)$, and observe that for each fixed $\delta > 0$, we have $\lim_{r\to 0} N(r, \delta) = \infty$. We see that the map $\{(x_i, r_i)\} \mapsto \{(x_i, n_i)\}$ defined above is a map from $\mathcal{D}^b(Z_m, r)$ to $\mathcal{P}(Z_m, N(r, \delta), \delta)$; thus (4.21) allows us to make the following computation for all $r > 0$ and $0 < \delta < \delta_0$:

$$\begin{aligned}
m_H^{b'}(Z_m, t, r) &= \inf_{\mathcal{D}^b(Z_m, r)} \sum_{(x_i, r_i)} (2r_i)^t \\
&\geq \inf_{\mathcal{P}(Z_m, N(r,\delta),\delta)} \sum_{(x_i, n_i)} (2e^{-(\beta+\varepsilon)} s_{n_i}(x))^t \\
&= (2\delta)^t e^{-t(m+\beta+\varepsilon)} \inf_{\mathcal{P}(Z_m, N(r,\delta),\delta)} \sum_{(x_i, n_i)} e^{-n_i t(\lambda_{n_i}(x)+\varepsilon)} \\
&= (2\delta)^t e^{-t(m+\beta+\varepsilon)} m_P(Z_m, t\varepsilon, -t\log a, N, \delta).
\end{aligned}$$

Taking the limit as $r \to 0$ (and hence $N(r, \delta) \to \infty$), it follows from (4.19) that the quantity on the right goes to ∞, and so we have $m_H^{b'}(Z_m, t) = \infty$. Using

Proposition A.1.1, this yields

$$\dim_H Z \geq \dim_H Z_m \geq t,$$

and since $t < t^*$ was arbitrary, this establishes the lemma. □

Proof of Theorem 4.2.1. Fix a decreasing sequence of positive numbers α_k converging to 0, and let $Z_k = Z \cap \mathcal{A}((\alpha_k, \infty))$, so that Lemma 4.4.5 applies to Z_k, and we have $Z = \bigcup_{k=1}^{\infty} Z_k$. For each k, let t_k be the unique real number such that

$$P_{Z_k}(-t_k \log a) = 0;$$

existence and uniqueness of t_k are given by Corollary 4.4.2. Then Lemma 4.4.5 shows that

$$\dim_H Z_k = t_k.$$

Writing $t^* = \sup_k t_k$, it follows that $\dim_H Z = t^*$, and it remains to show that

$$t^* = \sup\{t \geq 0 \mid P_Z(-t \log a) > 0\}. \tag{4.22}$$

But given $t \geq 0$, we have

$$P_Z(-t \log a) = \sup_k P_{Z_k}(-t \log a),$$

and this is positive if and only if there exists k such that $P_{Z_k}(-t \log a) > 0$; that is, if and only if $t < t_k$. This establishes (4.22).

Finally, it follows from (4.22) and continuity of the function $t \mapsto P_Z(-t \log a)$ that $P_Z(-t^* \log a) = 0$. If $Z \subset \mathcal{A}((\alpha, \infty))$ for some $\alpha > 0$, then Corollary 4.4.2 guarantees that t^* is in fact the unique root of Bowen's equation. □

Chapter 5

Multifractal analysis of Gibbs measures

5.1 Objects of study

5.1.1 Entropy and dimension spectra

So far we have studied the Birkhoff and Lyapunov spectra; the former characterises a system together with an observable function φ, while the latter characterises the intrinsic properties of the system itself. The remaining two spectra that we will study—the entropy and dimension spectra—characterise a measure μ. The entropy spectrum will also depend on some underlying dynamics, but the dimension spectrum is defined without reference to any dynamical system.

We will work in the setting where there is a dynamical system in the background, because this allows us to use a certain (weak) Gibbs property for μ, described below, to gain information about the pointwise dimensions and local entropies of μ. In many cases μ may be taken to be an invariant probability measure for these dynamics, but the definitions and the results are valid for arbitrary Borel measures as long as the Gibbs property holds.

Given a compact metric space X, a continuous map $f\colon X \to X$, and a Borel measure μ, the local entropy $h_\mu(x)$ and pointwise dimension $d_\mu(x)$ are as given in Definition 2.1.8 (note that $d_\mu(x)$ does not depend on the dynamics of f). The associated level sets are

$$K_\alpha^{\mathcal{D}} = \{x \in X \mid d_\mu(x) = \alpha\},$$
$$K_\alpha^{\mathcal{E}} = \{x \in X \mid h_\mu(x) = \alpha\},$$

which lets us define the *entropy spectrum for local entropies*

$$\mathcal{E}(\alpha) = h_{\text{top}}\,(K_\alpha^{\mathcal{E}})$$

and the *dimension spectrum for pointwise dimensions*

$$\mathcal{D}(\alpha) = \dim_H K_\alpha^{\mathcal{D}}.$$

Following the general outline, each of the spectra we have met so far could also be defined using the alternate global dimensional quantity. That is, we could define the *entropy spectrum for pointwise dimensions* by

$$\mathcal{D}_E(\alpha) = h_{\text{top}}\,(K_\alpha^{\mathcal{D}}),$$

and similarly for the *dimension spectrum for local entropies* and the *dimension spectrum for Birkhoff averages*. It turns out that these *mixed multifractal spectra* are harder to deal with than the ones we have defined so far; see [BS01] for further details. We will restrict our attention to the spectra for which the local and global quantities are naturally related, and will simply refer to the *entropy spectrum* and the *dimension spectrum*.

5.1.2 Weak Gibbs measures

In order to study the spectra $\mathcal{E}(\alpha)$ and $\mathcal{D}(\alpha)$, we will consider measures μ for which the local scaling quantities of the measure are related to the Birkhoff averages of a potential function φ. This amounts to a weaker version of the classical Gibbs property; we observe that there are several cases in which weak Gibbs measures (of one definition or another) are known to exist [Yur00, Kes01, FO03, VV08, JR09], and Theorem B.2.1 in Appendix B gives a proof that for a large class of systems, every continuous φ admits a weak Gibbs measure (in our sense), although this measure is not necessarily invariant or fully supported.

Definition 5.1.1. Given a compact metric space X, a continuous map $f\colon X \to X$, and a potential $\varphi\colon X \to \mathbb{R}$ (not necessarily continuous), we say that a Borel probability measure μ is a *weak Gibbs measure* for φ with constant $P \in \mathbb{R}$ if for every $x \in X$ and $\delta > 0$ there exists a sequence $M_n = M_n(x, \delta) > 0$ such that

$$\frac{1}{M_n} \leq \frac{\mu(B(x, n, \delta))}{\exp(-nP + S_n\varphi(x))} \leq M_n \tag{5.1}$$

for every $n \in \mathbb{N}$, where we require the following growth condition on M_n to hold for every $x \in X$:

$$\lim_{\delta \to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log M_n(x, \delta) = 0. \tag{5.2}$$

There are various definitions in the literature of Gibbs measures of one sort or another; most of these definitions agree in spirit, but differ in some slight details. We note the differences between the above definition and other definitions in use.

1. The classical definition (see [Bow75]) requires M_n to be bounded, not just to have slow growth, as we require here. In that case the sequence M_n can be (and is) replaced by a single constant M. The notion of a weak Gibbs measure, for which the constant can vary slowly in n, is used in [Yur00, Kes01, FO03, JR09], among others.

2. The above definitions all require the constant M to be independent of x, whereas we require no such uniformity. Furthermore, they are given in terms of cylinder sets rather than Bowen balls; we follow [VV08] in using the latter, as this is what we need for the multifractal analysis.

3. Certain authors only require (5.1) to hold for μ-a.e. $x \in X$ [Yur00, VV08]. In order to do the multifractal analysis, we need conditions which hold everywhere, not just almost everywhere, and so we require (5.1) for *every* point $x \in X$.

4. Following Kesseböhmer [Kes01], we do not *a priori* require that a weak Gibbs measure be f-invariant. As we show in Theorem B.2.1, non-invariant weak Gibbs measures exist for *any* continuous function φ on a compact metric

space provided f is nowhere contracting (this is proved for one-sided shift spaces in [Kes01]), but it is not the case that such measures can always be taken to be invariant or fully supported.

We have given the definition in the above form because (5.1) is reminiscent of the classical definition of Gibbs measure. For our purposes, an alternate form of (5.1) will be more useful:

$$\left| -\frac{1}{n} \log \mu(B(x,n,\delta)) + \frac{1}{n} S_n\varphi(x) - P \right| \leq \frac{1}{n} \log M_n(x,\delta) \to 0, \tag{5.3}$$

where the limit is taken as $n \to \infty$ and then as $\delta \to 0$; in particular, $P_\mu(x)$ exists and is constant at every point $x \in X$.

Given an invariant weak Gibbs measure, it follows from (5.3) that $h_\mu(x)$ exists if and only if $\lim_{n\to\infty} \frac{1}{n} S_n\varphi(x)$ exists, and that in this case

$$h_\mu(x) + \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = P. \tag{5.4}$$

If φ is continuous and μ is a weak Gibbs measure for φ, then Theorem B.1.1 shows that P is equal to the topological pressure $P_X(\varphi)$, and thus the variational principle shows that it is equal to $P^*(\varphi)$. If μ is invariant, then integrating (5.4) with respect to μ gives $P^*(\varphi) = h(\mu) + \int \varphi \, d\mu$, and so μ is an equilibrium state. Thus an invariant weak Gibbs measure is an equilibrium state. (If a not fully supported invariant measure satisfies the weak Gibbs property at every point in its support, then it is an equilibrium state for the restriction of the original map to its support.)

5.2 Results for entropy and dimension spectra

Writing $\varphi_1(x) = \varphi(x) - P^*(\varphi)$, we observe that

$$K_\alpha^{\mathcal{B}}(\varphi_1) = K_{-\alpha}^{\mathcal{E}}, \tag{5.5}$$

and thus we expect to obtain $\mathcal{E}(\alpha)$ as a Legendre transform of the following function:

$$T_{\mathcal{E}}(q) = P^*(q\varphi_1).$$

As with $T_{\mathcal{B}}$, convexity of $T_{\mathcal{E}}$ is immediate from the definition of P^*.

The following theorem is a direct consequence of Theorem 3.1.1 and (5.4); because of the change of sign in (5.5), we must use the following versions of the Legendre transform:

$$\begin{aligned} T^{L_3}(\alpha) &= \inf_{q\in\mathbb{R}}(T(q) + q\alpha), \\ S^{L_4}(q) &= \sup_{\alpha\in\mathbb{R}}(S(\alpha) - q\alpha). \end{aligned} \tag{5.6}$$

Note that there is a corresponding change of sign in the definitions of the maps A and Q.

Theorem 5.2.1 (The entropy spectrum for local entropies)**.** *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi \in \mathcal{A}_f$. Then if μ is a weak Gibbs measure for φ, we have the following:*

I. $T_{\mathcal{E}}$ is the Legendre transform of the entropy spectrum:

$$T_{\mathcal{E}}(q) = \mathcal{B}^{L_4}(q) = \sup_{\alpha\in\mathbb{R}}(\mathcal{E}(\alpha) - q\alpha) \tag{5.7}$$

for every $q \in \mathbb{R}$.

II. The set $\{\alpha \in \mathbb{R} \mid \mathcal{E}(\alpha) > -\infty\}$ is bounded by the following:

$$\begin{aligned} \alpha_{\min} &= \inf\{\alpha \in \mathbb{R} \mid T_{\mathcal{E}}(q) \geq -q\alpha \text{ for all } q\}, \\ \alpha_{\max} &= \sup\{\alpha \in \mathbb{R} \mid T_{\mathcal{E}}(q) \geq -q\alpha \text{ for all } q\}, \end{aligned}$$

That is, $K^{\mathcal{E}}_{\alpha} = \emptyset$ for every $\alpha < \alpha_{\min}$ and every $\alpha > \alpha_{\max}$.

III. Suppose that $T_{\mathcal{E}}$ is C^r on (q_1, q_2) for some $r \geq 1$, and that for each $q \in (q_1, q_2)$, there exists a (not necessarily unique) equilibrium state ν_q for the potential function $q\varphi_1$. Let $\alpha_1 = -D^+T_{\mathcal{E}}(q_1)$ and $\alpha_2 = -D^-T_{\mathcal{E}}(q_2)$. Then

$$\mathcal{E}(\alpha) = T_{\mathcal{E}}^{L_3}(\alpha) = \inf_{q\in\mathbb{R}}(T_{\mathcal{E}}(q) + q\alpha) \tag{5.8}$$

for all $\alpha \in (\alpha_2, \alpha_1)$; in particular, $\mathcal{E}(\alpha)$ is strictly concave on (α_2, α_1), and C^r except at points corresponding to intervals on which $T_{\mathcal{E}}$ is affine.

In the case where f is conformal, we prove the analogous result for the dimension spectrum of μ. First recall that $\mathbf{B}$ is the set of points with tempered contraction; we will assume that $\mathbf{B} = X$. We will also need to eliminate points at which the Birkhoff averages of $\log a$ cluster around zero along a sequence of times at which the local entropy of μ is also negligible—that is, the following set:

$$\mathbf{Z}(\mu) = \left\{ x \in X \;\middle|\; \lim_{\delta \to 0} \varliminf_{n \to \infty} \left| \frac{1}{n} \log \mu(B(x, n, \delta)) \right| + \left| \frac{1}{n} S_n \log a(x) \right| = 0 \right\}. \tag{5.9}$$

When μ is a weak Gibbs measure for φ, we have

$$\mathbf{Z}(\mu) = \left\{ x \in X \;\middle|\; \varliminf_{n \to \infty} \left| \frac{1}{n} S_n \varphi_1(x) \right| + \left| \frac{1}{n} S_n \log a(x) \right| = 0 \right\}. \tag{5.10}$$

In the context of Theorem 5.2.2 below, we will suppress the dependence on μ and simply write $\mathbf{Z} = \mathbf{Z}(\mu)$. We will see that the set $\mathbf{Z}$ contains all points x for which $\underline{\lambda}(x) = 0$ but $\overline{d}_\mu(x) < \infty$; these are the only points our methods cannot deal with. In many cases we do not lose much by neglecting them; for example, if $\sup \varphi - \inf \varphi < h(\mu)$, then

$$\varlimsup_{n \to \infty} \frac{1}{n} S_n \varphi_1(x) < 0$$

for every $x \in X$, and so $\mathbf{Z} = \emptyset$. Even in cases when $\mathbf{Z}$ is non-empty, it often has zero Hausdorff dimension [JR09].

The remaining set of "good" points will be denoted by

$$X' = X \setminus \mathbf{Z}. \tag{5.11}$$

In the definition of $\mathcal{D}(\alpha)$, we adopt the convention that $\mathcal{D}(\alpha) = -\infty$ if $K_\alpha^{\mathcal{D}} \subset \mathbf{Z}$. Since there may be points at which μ has infinite pointwise dimension, we also include the value $\alpha = +\infty$ in (5.6), and follow the convention that if $K_\infty^{\mathcal{D}} \cap X' \neq \emptyset$, then $\mathcal{D}^{L_4}(q) = +\infty$ for all $q < 0$.

Now consider the centred potential $\varphi_1(x) = \varphi(x) - P^*(\varphi)$. Define a family of potentials by

$$\varphi_{q,t}(x) = q\varphi_1(x) - t \log a(x). \tag{5.12}$$

We will be particularly interested in the potentials with zero pressure; we would like to define a function $T_{\mathcal{D}}(q)$ by the equation

$$P^* \left(\varphi_{q,T_{\mathcal{D}}(q)}\right) = 0. \tag{5.13}$$

Formally, we write

$$T_{\mathcal{D}}(q) = \inf\{t \in \mathbb{R} \mid P^*(\varphi_{q,t}) \leq 0\} = \sup\{t \in \mathbb{R} \mid P^*(\varphi_{q,t}) > 0\}; \tag{5.14}$$

by continuity of P^*, $T_{\mathcal{D}}(q)$ solves (5.13) if it is finite, but is not necessarily the unique solution of (5.13). (Indeed, there may be values of q for which $P^*(\varphi_{q,t}) = 0$ for all $t > T_{\mathcal{D}}(q)$.)

For $T_{\mathcal{D}}(q) < \infty$ we write $\varphi_q = \varphi_{q,T_{\mathcal{D}}(q)}$, and observe that (5.13) may be written as $P^*(\varphi_q) = 0$.

Given $\eta > 0$ and $I_Q \subset \mathbb{R}$, we will need to consider the following region lying just under the graph of $T_{\mathcal{D}}(q)$:

$$R_\eta(I_Q) = \{(q,t) \in \mathbb{R}^2 \mid q \in I_Q, T_{\mathcal{D}}(q) - \eta < t < T_{\mathcal{D}}(q)\}.$$

We can now state a general result regarding the dimension spectrum.

Theorem 5.2.2 (The dimension spectrum for pointwise dimensions). *Let X be a compact metric space with $\dim_H X < \infty$, and let $f\colon X \to X$ be continuous and conformal with continuous non-vanishing factor $a(x)$. Suppose that $\mathbf{B} = X$ and that $\lambda(\nu) \geq 0$ for every $\nu \in \mathcal{M}^f(X)$. Let $\mu \in \mathcal{M}^f(X)$ be a weak Gibbs measure for a continuous potential φ. Finally, suppose that $\dim_H \mathbf{Z} = 0$. Then we have the following.*

I. $T_{\mathcal{D}}$ is the Legendre transform of the dimension spectrum:

$$T_{\mathcal{D}}(q) = \mathcal{D}^{L_4}(q) = \sup_{\alpha \in \mathbb{R}} (\mathcal{D}(\alpha) - q\alpha) \tag{5.15}$$

for every $q \in \mathbb{R}$.

II. Neglecting points in $\mathbf{Z}$, the set $\{\alpha \in \mathbb{R} \mid \mathcal{D}(\alpha) > -\infty\}$ is bounded by the *following:*

$$\alpha_{\min} = \inf\{\alpha \in \mathbb{R} \mid T_{\mathcal{D}}(q) \geq -q\alpha \text{ for all } q\},$$
$$\alpha_{\max} = \sup\{\alpha \in \mathbb{R} \mid T_{\mathcal{D}}(q) \geq -q\alpha \text{ for all } q\},$$

That is, $K_\alpha^{\mathcal{D}} \cap X' = \emptyset$ *for every* $\alpha < \alpha_{\min}$ *and every* $\alpha > \alpha_{\max}$.

III. *Suppose* $I_Q = (q_1, q_2)$ *and* $\eta > 0$ *are such that for every* $(q,t) \in R_\eta(I_Q)$, *the potential* $\varphi_{q,t}$ *has a (not necessarily unique) equilibrium state, and that the map* $(q,t) \mapsto P^*(\varphi_{q,t})$ *is* C^r *on* $R_\eta(I_Q)$ *for some* $r \geq 1$. *Then we have*

$$\mathcal{D}(\alpha) = T_{\mathcal{D}}^{L_3}(\alpha) = \inf_{q\in\mathbb{R}}(T_{\mathcal{D}}(q) + q\alpha) \tag{5.16}$$

for all $\alpha \in (\alpha_2, \alpha_1) = A(I_Q)$; *in particular,* $\mathcal{D}$ *is strictly concave on* (α_2, α_1), *and* C^r *except at points corresponding to intervals on which* $T_{\mathcal{D}}$ *is affine.*

We will see in the proof that the requirement on existence of equilibrium states for $\varphi_{q,t}$ with $(q,t) \in R_\eta(I_Q)$ can be replaced by the condition that there exist equilibrium states ν_q for $\varphi_q = \varphi_{q,T_{\mathcal{D}}(q)}$ such that $\lambda(\nu_q) > 0$. However, such measures do not necessarily exist, while upper semi-continuity of the entropy is enough to guarantee the existence of the measures required in the theorem.

If we do have equilibrium states ν_q with $\lambda(\nu_q) > 0$, then in Part III of the theorem, the requirement that $(q,t) \mapsto P^*(\varphi_{q,t})$ be C^r on $R_\eta(I_Q)$ can be replaced by the condition that $T_{\mathcal{D}}$ be C^r on I_Q.

5.3 Applications and relation to other results

5.3.1 Verifying the hypotheses

There are many cases in which equilibrium states are known to have the weak Gibbs property (5.1) or one which implies it. For example, equilibrium states for Hölder continuous potentials on uniformly hyperbolic systems are known to be Gibbs, as are equilibrium states for potentials satisfying a certain regularity property on expansive maps with specification [TV99]. Finally, Kesseböhmer proves the existence of weak Gibbs measures for continuous potentials on symbolic space [Kes01] (these measures are studied by Jordan and Rams [JR09] on parabolic interval maps).

Given a weak Gibbs measure, all the remarks in Section 3.2 regarding the Birkhoff spectrum apply to the entropy spectrum: upper semi-continuity, sufficiency of uniqueness, etc.

Because of the geometric implications of any result regarding the dimension spectrum, we must deal with a more restricted class of systems. In particular, the present approach is completely dependent upon conformality of the map f; without conformality, we have no analogue of Lemma 4.4.4 or Proposition 5.4.4. If analogues of these can be found in the non-conformal case, then it may be possible to establish a non-conformal version of the present result; however, this appears to require the use of a non-additive version of the thermodynamic formalism [Bar96, FH10].

We also presently lack the tools to deal with maps with critical points or singularities. To establish an analogue of Lemma 4.4.4 for such maps would require an estimate on the rate of recurrence of fairly arbitrary orbits to the critical set in order to control the distortion. This approach, however, has yet to bear fruit.

Nevertheless, the hypotheses of Theorem 5.2.2 are satisfied for quite general classes of maps. We discuss briefly each of the other hypotheses.

1. $\mathbf{B} = X$. If $a(x) \geq 1$ for all x, then this is automatically satisfied; we do not need $a(x) > 1$, nor any uniformity, and so the class of systems with this property includes Manneville–Pomeau maps and parabolic rational maps.

2. $\dim_H \mathbf{Z} = 0$. Points at which $\lambda(x) = 0$ and $d_\mu(x) < \infty$ are problematic for various reasons, and so we want to avoid having to deal with them. Since all such points lie in the set $\mathbf{Z}$, we can do this by neglecting $\mathbf{Z}$ in all our computations, and it turns out that this is not a very heavy price to pay. Of course if f is uniformly expanding, this set is empty, but even in the non-uniformly expanding case, it is shown in [JR09] that $\mathbf{Z}$ has zero Hausdorff dimension for a class of parabolic interval maps.

3. If the entropy map is upper semi-continuous, then existence of equilibrium states for $\varphi_{q,t}$ is guaranteed for all $q, t \in \mathbb{R}$, and we see once again that unique-

ness is enough to establish differentiability of the map $(q,t) \mapsto P^*(\varphi_{q,t})$, and hence to apply Theorem 5.2.2.

5.3.2 Uniform hyperbolicity

The uniformly hyperbolic situation was the first to be understood completely. The basic elements of the multifractal formalism were first proposed by Halsey *et al* in [HJK$^+$86], where they studied the dimension spectrum $\mathcal{D}(\alpha)$ (which they also referred to as the $f(\alpha)$*-spectrum for dimensions*), and argued that $\mathcal{D}(\alpha)$ is analytic and concave on its domain of definition and is related to the Rényi and Hentschel–Procaccia spectra for dimensions by a Legendre transform.

The dimension spectrum for Gibbs measures on hyperbolic cookie-cutters (dynamically defined Cantor sets) were studied by Rand [Ran89], who introduced the use of the topological pressure to define the function $T_{\mathcal{D}}(q)$. The more general class of uniformly hyperbolic conformal maps was studied by Pesin and Weiss [PW97]: modern expositions of the whole theory for uniformly hyperbolic systems can be found in [Pes98, BPS97, TV00].

The entropy spectrum was studied by Takens and Verbitskiy [TV99] in the more general case of expansive maps satisfying a specification property. For such systems, it can be shown that equilibrium states for Hölder continuous potentials are Gibbs measures, and so Theorem 5.2.1 applies to the entropy spectrum $\mathcal{E}(\alpha)$, giving an alternate proof of the results in [TV99].

Returning to the dimension spectrum, it is well-known that if f is a $C^{1+\varepsilon}$ conformal map with a uniform repeller X (the setting of Theorem 4.1.1), then the entropy map is upper semi-continuous and each of the potentials $q\varphi_1 - t\log\|Df\|$ has a unique equilibrium state when φ is Hölder continuous, so Theorem 5.2.2 gives an alternate proof of the main multifractal results in [PW97].

5.3.3 Non-uniform hyperbolicity

5.3.3.1 Parabolic maps

Because conformality is automatic for one-dimensional differentiable maps and for rational maps of the Riemann sphere, these provide an ideal setting to apply

Theorem 5.2.2. In this context, various non-uniformly hyperbolic systems have been studied in [Nak00, Tod08, JR09, IT09a].

Kesseböhmer proves the existence of (non-invariant) weak Gibbs measures for continuous potentials on shift spaces [Kes01]; in [JR09], Jordan and Rams examine these weak Gibbs measures as measures on interval maps with parabolic fixed points. Theorem 5.2.1 then gives results regarding the entropy spectra of these measures.

In one dimension, the dimension spectrum for Manneville–Pomeau maps has been studied in [Nak00, JR09]; once again, the present approach provides an alternate proof of some results.

5.3.3.2 Maps with critical points

Given a multimodal map $f \in \mathcal{H}$, the multifractal analysis of the dimension spectrum for Gibbs measures associated to the potentials described above is carried out in [Tod08, IT09b]. At present, these results *cannot* be obtained using the results here, due to the presence of the critical point, which the tools used here cannot yet handle.

5.4 Proof of Theorem 5.2.2

As in the proof of Theorem 3.1.1, we carry out the proof of Theorem 5.2.2 in three parts. First, we show that $T_\mathcal{D}$ is the Legendre transform of $\mathcal{D}$, establishing (5.15). From this, it immediately follows by standard properties of the Legendre transform that $T_\mathcal{D}^{L_3}$ is the concave hull of $\mathcal{D}$.

Part II of the theorem is an easy consequence of the following proposition.

Proposition 5.4.1. *Given $\alpha \in \mathbb{R}$, suppose that $K_\alpha^\mathcal{D} \cap X'$ is non-empty; that is, there exists $x \in X'$ such that $d_\mu(x) = \alpha$. Then $T_\mathcal{D}(q) \geq -\alpha q$ for all $q \in \mathbb{R}$. Furthermore, if there exists $x \in X'$ such that $d_\mu(x) = +\infty$, then $T_\mathcal{D}(q) = +\infty$ for all $q < 0$.*

Part III of the theorem is once again proved via intermediate results similar in spirit to those in the proof of Theorem 3.1.1.

Proposition 5.4.2. *Given $q \in \mathbb{R}$, let $q_n \to q$ and $t_n \to T_{\mathcal{D}}(q)$ be such that $t_n \leq T_{\mathcal{D}}(q_n)$ for all n. Fix $\alpha \in \mathbb{R}$, and suppose that for all $n \in \mathbb{N}$, there exists an ergodic equilibrium state ν_n for φ_{q_n,t_n} such that $\lambda(\nu_n) > 0$ and*

$$\alpha = \frac{-\int \varphi_1 \, d\nu_n}{\lambda(\nu_n)}. \tag{5.17}$$

Then $\mathcal{D}(\alpha) \geq T_{\mathcal{D}}^{L_3}(\alpha)$.

Proposition 5.4.3. *Given $\eta > 0$ and $I_Q = (q_1, q_2)$, suppose that the map $(q,t) \mapsto P^*(\varphi_{q,t})$ is continuously differentiable on $R_\eta(I_Q)$, and that $\varphi_{q,t}$ has an equilibrium state $\nu_{q,t}$ for every $(q,t) \in R_\eta(I_Q)$. Let $\alpha_1 = -D^+T_{\mathcal{D}}(q_1)$ and $\alpha_2 = -D^-T_{\mathcal{D}}(q_2)$. Then for every $\alpha \in (\alpha_2, \alpha_1)$ there exists a sequence $(q_n, t_n) \to (q, T_{\mathcal{D}}(q))$ such that each φ_{q_n,t_n} has an ergodic equilibrium state ν_n satisfying* (5.17).

As mentioned after the statement of Theorem 5.2.2, we can do away with the talk of sequences of potentials and measures in Propositions 5.4.2 and 5.4.3 if each φ_q has an equilibrium state ν_q with $\lambda(\nu_q) > 0$ and if $T_{\mathcal{D}}$ is C^r on (q_1, q_2). The proof in this case goes just like the proof we carry out below.

Before proceeding to the proof itself, we pause to collect pertinent results on the relationship between pointwise dimension, local entropy, and the Lyapunov exponent. Given an ergodic measure $\nu \in \mathcal{M}_E^f(X)$, the Lyapunov exponent $\lambda(x) = (\log a)^+(x)$ exists and is constant ν-a.e. as a consequence of Birkhoff's ergodic theorem. The analogous result for the local entropy $h_\nu(x)$ was proved by Brin and Katok [BK83]. The following proposition shows (among other things) that together, these imply exactness of the measure ν when the map f is conformal.

Proposition 5.4.4. *Let $f\colon X \to X$ be continuous and conformal with continuous non-vanishing factor $a(x)$, and fix $\nu \in \mathcal{M}^f(X)$. Suppose that the local entropy $h_\nu(x)$ and Lyapunov exponent $\lambda(x)$ both exist at some $x \in X$. If $\lambda(x) > 0$, then the pointwise dimension $d_\nu(x)$ also exists, and*

$$d_\nu(x) = \lim_{n\to\infty} \frac{-\log \nu(B(x,n,\delta))}{S_n \log a(x)} = \frac{h_\nu(x)}{\lambda(x)}. \tag{5.18}$$

If $\lambda(x) = 0$ and $h_\nu(x) > 0$, then $d_\nu(x)$ exists and is equal to $+\infty$.

Proof. Fix $\varepsilon > 0$; if $\lambda(x) > 0$, choose $\varepsilon < \lambda(x)$. Since $\lambda(x)$ exists we may apply Lemma 4.4.4 and obtain $\delta = \delta(\varepsilon) > 0$ and $\eta = \eta(x) > 0$ such that (4.13) holds for all $n \in \mathbb{N}$, and hence writing

$$r_n = \eta\delta e^{-n(\lambda_n(x)+\varepsilon)}, \qquad s_n = \delta e^{-n(\lambda_n(x)-\varepsilon)}, \tag{5.19}$$

we have

$$\nu(B(x, r_n)) \leq \nu(B(x, n, \delta)) \leq \nu(B(x, s_n)). \tag{5.20}$$

Observe that

$$\log r_n = \log(\eta\delta) - S_n \log a(x) - n\varepsilon, \tag{5.21}$$

and that furthermore,

$$\begin{aligned} \frac{\log r_{n+1}}{\log r_n} &= \frac{\log(\eta\delta) - S_{n+1} \log a(x) - (n+1)\varepsilon}{\log(\eta\delta) - S_n \log a(x) - n\varepsilon} \\ &= 1 - \frac{\varepsilon + \log a(f^n(x))}{\log(\eta\delta) - S_n \log a(x) - n\varepsilon}. \end{aligned} \tag{5.22}$$

Observe that the numerator is uniformly bounded, and that if $\lambda(x) > 0$, the denominator goes to $-\infty$ by the assumption that $\varepsilon < \lambda(x)$, while if $\lambda(x) = 0$, the denominator goes to $-\infty$ because $\left|\frac{1}{n} S_n \log a(x)\right| < \frac{\varepsilon}{2}$ for all sufficiently large n. It follows that the ratio in (5.22) converges to 1, and a similar result holds for s_n. The same argument shows that $r_n \to 0$ for all values of $\lambda(x)$, while $s_n \to 0$ provided $\lambda(x) > 0$.

For future reference, we point out that everything up to this point also holds if $x \in \mathbf{B}$ and $\underline{\lambda}(x) > 0$.

Now suppose that $\lambda(x) > 0$. It follows that

$$\lim_{n\to\infty} \frac{-\log r_n}{S_n \log a(x)} = \lim_{n\to\infty} \left(1 + \frac{n\varepsilon - \log(\eta\delta)}{S_n \log a(x)}\right) = 1 + \frac{\varepsilon}{\lambda(x)}. \tag{5.23}$$

and we see from the first inequality in (5.20) that

$$\frac{\log \nu(B(x, r_n))}{\log r_n} \left(\frac{-\log r_n}{S_n \log a(x)}\right) \geq \frac{-\log \nu(B(x, n, \delta))}{S_n \log a(x)},$$

where we observe that the quantity on the right is exactly the quantity that appears in (5.18). Letting n tend to infinity, this yields

$$\varliminf_{n\to\infty} \frac{\log \nu(B(x,r_n))}{\log r_n}\left(1+\frac{\varepsilon}{\lambda(x)}\right) \geq \frac{h_\nu(x)}{\lambda(x)}. \tag{5.24}$$

Now given an arbitrary $r > 0$, let n be such that $r_n \leq r \leq r_{n-1}$; it follows that

$$\frac{\log \nu(B(x,r))}{\log r} \geq \frac{\log \nu(B(x,r_n))}{\log r_{n-1}} = \frac{\log \nu(B(x,r_n))}{\log r_n}\frac{\log r_n}{\log r_{n-1}},$$

and since $\log r_n / \log r_{n-1} \to 1$, we may let r tend to 0 to obtain

$$\underline{d}_\nu(x)\left(1+\frac{\varepsilon}{\lambda(x)}\right) \geq \frac{h_\nu(x)}{\lambda(x)}.$$

Since $\varepsilon > 0$ was arbitrary, this gives

$$\underline{d}_\nu(x) \geq \frac{h_\nu(x)}{\lambda(x)}.$$

Using similar estimates on s_n, we obtain the upper bound

$$\overline{d}_\nu(x) \leq \frac{h_\nu(x)}{\lambda(x)},$$

which implies (5.18).

It only remains to consider the case $\lambda(x) = 0$. We first observe that in this case we can choose N sufficiently large that $|S_n \log a(x) - \log(\eta\delta)| < n\varepsilon$ for all $n \geq N$, and hence $0 > \log r_n > -2n\varepsilon$. Then the first inequality in (5.20) gives

$$\frac{\log \nu(B(x,r_n))}{\log r_n} > -\frac{1}{2n\varepsilon}\log \nu(B(x,n,\delta)),$$

and taking the limit as $n \to \infty$ gives

$$\underline{d}_\nu(x) > \frac{h_\nu(x)}{2\varepsilon},$$

just as above. Since $\varepsilon > 0$ was arbitrary, we have $d_\nu(x) = +\infty$. □

The following corollaries of Proposition 5.4.4 are easily proved by considering generic points for the measure ν.

Corollary 5.4.5. *Let $f: X \to X$ be continuous and conformal with continuous non-vanishing factor $a(x)$, and fix $\nu \in \mathcal{M}^f(X)$ with $\lambda(\nu) > 0$. Then $\dim_H \nu = h(\nu)/\lambda(\nu)$.*

Corollary 5.4.6. *Let $f: X \to X$ be continuous and conformal with continuous non-vanishing factor $a(x)$, and fix $\mu, \nu \in \mathcal{M}^f(X)$. Suppose that $\lambda(\nu) > 0$, and let $\alpha \in \mathbb{R}$ be given by*

$$\alpha = \frac{\int h_\mu(x)\, d\nu(x)}{\lambda(\nu)}.$$

Then $\nu(K_\alpha^{\mathcal{D}}(\mu)) = 1$, where $K_\alpha^{\mathcal{D}}(\mu)$ is the set of points $x \in X$ for which $d_\mu(x) = \alpha$.

Given a little more information about X, we can also say something about measures with zero Lyapunov exponent.

Corollary 5.4.7. *Let $f: X \to X$ be continuous and conformal with continuous non-vanishing factor $a(x)$, and suppose that $\dim_H X < \infty$. Then any $\nu \in \mathcal{M}^f(X)$ with $\lambda(\nu) = 0$ must have $h(\nu) = 0$ as well.*

Proof. First suppose that ν is ergodic and that $h(\nu) > 0$. Then by Birkhoff's ergodic theorem and the Brin–Katok entropy formula, there exists a set $Y \subset X$ such that $\nu(Y) = 1$ and for every $x \in Y$, we have $\lambda(x) = 0$ and $h_\nu(x) = h(\nu) > 0$. It follows from Proposition 5.4.4 that $d_\nu(x) = +\infty$, and hence

$$\dim_H X \geq \dim_H \nu = +\infty,$$

which contradicts the assumption in Theorem 5.2.2 that $\dim_H X < \infty$. □

A converse of sorts to Proposition 5.4.4 is given by the following, which addresses the case where $d_\mu(x)$ exists even though $h_\mu(x)$ and $\lambda(x)$ may not. We exclude points lying in $\mathbf{Z} = \mathbf{Z}(\mu)$.

Proposition 5.4.8. *Let $f: X \to X$ be continuous and conformal with continuous non-vanishing factor $a(x)$, and fix $\mu \in \mathcal{M}^f(X)$. Suppose that the pointwise dimension $d_\mu(x)$ exists at some point $x \in X' \cap \mathbf{B}$ and is equal to α. Then although the* *local entropy and Lyapunov exponent may not exist at x, the ratio of the pre-limit quantities still converges; in particular, we have*

$$\lim_{n\to\infty} \frac{-\log \mu(B(x,n,\delta))}{S_n \log a(x)} = \alpha = d_\mu(x) \tag{5.25}$$

whenever $\underline{\lambda}(x) > 0$, and $\alpha = \infty$ if $\underline{\lambda}(x) = 0$.

Proof. We deal first with the case $\underline{\lambda}(x) = 0$. In this case, there exists an increasing sequence n_k such that

$$\frac{1}{n_k} S_{n_k} \log a(x) \to 0,$$

and since $x \notin \mathbf{Z}$, there exists $\delta_0 > 0$ such that

$$\gamma(\delta) := \varliminf_{k\to\infty} -\frac{1}{n_k} \log \mu(B(x, n_k, \delta)) > \gamma(\delta_0) > 0$$

for any $0 < \delta < \delta_0$.

Fix $\varepsilon > 0$. Because $x \in \mathbf{B}$, we may apply Lemma 4.4.4 to get r_n as in (5.19) for which (5.20) holds for μ, and we have $r_{n_k} \to 0$ just as in the proof of Proposition 5.4.4. In particular, for all sufficiently large k, (5.20) gives

$$\frac{\log \mu(B(x, r_{n_k}))}{\log r_{n_k}} > -\frac{1}{2n_k\varepsilon} \log \mu(B(x, n_k, \delta)),$$

and it follows that

$$\alpha = \lim_{k\to\infty} \frac{\log \mu(B(x, r_{n_k}))}{\log r_{n_k}} \geq \frac{\gamma(\delta_0)}{2\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary, we see that $\alpha = \infty$. (Observe that since the hypothesis of the proposition tells us that $d_\mu(x)$ exists, it suffices to obtain $\underline{d}_\mu(x) = \infty$, as we do here.)

We turn now to the case $\underline{\lambda}(x) > 0$. As remarked in the proof of Proposition 5.4.4, the computations at the beginning of that proof are valid here as well; everything up to but not including (5.23) works in the present setting. (5.23) is replaced by the following inequality:

$$\varlimsup_{n\to\infty} \frac{-\log r_n}{S_n \log a(x)} \leq 1 + \frac{\varepsilon}{\underline{\lambda}(x)}.$$

Thus we have the following in place of (5.24):

$$\begin{aligned} d_\mu(x)\left(1+\frac{\varepsilon}{\underline{\lambda}(x)}\right) &= \lim_{n\to\infty}\frac{\log\mu(B(x,r_n))}{\log r_n}\left(1+\frac{\varepsilon}{\underline{\lambda}(x)}\right) \\ &\geq \overline{\lim_{n\to\infty}}\frac{-\log\mu(B(x,n,\delta))}{S_n\log a(x)}. \end{aligned}$$

Similar computations with s_n give

$$d_\mu(x)\left(1-\frac{\varepsilon}{\underline{\lambda}(x)}\right) \leq \underline{\lim_{n\to\infty}}\frac{-\log\mu(B(x,n,\delta))}{S_n\log a(x)},$$

and since $\varepsilon>0$ was arbitrary, this suffices to prove (5.25). □

Proof of Theorem 5.2.2. We prove part I of the theorem by establishing the following two inequalities:

$$T_{\mathcal{D}} \leq \mathcal{D}^{L_4}, \tag{5.26}$$

$$T_{\mathcal{D}} \geq \mathcal{D}^{L_4}. \tag{5.27}$$

We begin by proving (5.26). First, observe that we may have $T_{\mathcal{D}}(q)=+\infty$ for some values of q. Suppose that this is the case for some $q\in\mathbb{R}$; then for any sequence $t_n\to+\infty$, we have $P^*(\varphi_{q,t_n})>0$ for all n, and hence there exists a sequence of ergodic f-invariant measures ν_n such that

$$h(\nu_n)+q\int\varphi_1\,d\nu_n - t_n\lambda(\nu_n)>0. \tag{5.28}$$

Now there are two possibilities.

Case 1. $\lambda(\nu_n)>0$ for all n. In this case we obtain

$$\frac{h(\nu_n)}{\lambda(\nu_n)}+q\frac{\int\varphi_1\,d\nu_n}{\lambda(\nu_n)}>t_n.$$

Applying Corollary 5.4.5, we see that the first term is equal to $\dim_H\nu$; furthermore, Corollary 5.4.6 together with the weak Gibbs property of μ gives $\nu_n(K^{\mathcal{D}}_{\alpha_n})=1$, where $\alpha_n=\int\varphi_1\,d\nu_n/\lambda(\nu_n)$. Consequently, we have

$$\mathcal{D}(\alpha_n)+q\alpha_n\geq\dim_H\nu_n+q\alpha_n>t_n,$$

and it follows that $\mathcal{D}^{L_4}(q) = \sup_{\alpha\in\mathbb{R}}(\mathcal{D}(\alpha) + q\alpha) = +\infty$.

Case 2. There exists n such that $\lambda(\nu_n) = 0$. Then Corollary 5.4.7 implies that $h(\nu_n) = 0$ as well, and (5.28) gives us that $q\int \varphi_1\, d\nu_n > 0$. If $q \geq 0$, this is impossible, since $\int \varphi_1\, d\nu \leq 0$ for all $\nu \in \mathcal{M}^f(X)$. If $q < 0$, this implies that $\int \varphi_1\, d\nu_n < 0$, and hence $\nu_n(\mathbf{Z}) = 0$. Now for ν_n-a.e. $x \in X$, we may apply Proposition 5.4.8 to obtain $d_\mu(x) = +\infty$. It follows that $\nu_n(K_\infty^{\mathcal{D}}) = 1$ and $K_\infty^{\mathcal{D}} \cap X' \neq \emptyset$, and we once again have $\mathcal{D}^{L_4}(q) = +\infty$.

Having dealt with the case where $T_{\mathcal{D}}(q) = +\infty$, we now turn our attention to the case where $T_{\mathcal{D}}(q)$ is finite. Given $t < T_{\mathcal{D}}(q)$, we observe that any measure ν with $h(\nu) + \int \varphi_{q,t}\, d\nu > 0$ must also satisfy $\lambda(\nu) > 0$, otherwise we would have $T_{\mathcal{D}}(q) = +\infty$. It follows that

$$P^*(\varphi_{q,t}) = \sup\left\{h(\nu) + \int \varphi_{q,t}\, d\nu \,\middle|\, \nu \in \mathcal{M}_E^f(X), \lambda(\nu) > 0\right\}.$$

Given $\alpha, \lambda \geq 0$, consider the following set:

$$Z_{\alpha,\lambda} = \{x \in X \mid \varphi_1^+(x) = -\alpha\lambda, \lambda(x) = \lambda\}.$$

Every ergodic measure ν is supported on some $Z_{\alpha,\lambda}$, and so we have

$$0 < P^*(\varphi_{q,t}) = \sup_{\alpha\geq 0}\sup_{\lambda>0}\sup\left\{h(\nu) + \int \varphi_{q,t}\, d\nu \,\middle|\, \nu \in \mathcal{M}_E^f(X), \nu(Z_{\alpha,\lambda}) = 1\right\}.$$

It follows that there exists some α, λ, and ν for which $\nu(Z_{\alpha,\lambda}) = 1$ and

$$h(\nu) + q\int \varphi_1\, d\nu - t\lambda(\nu) > 0.$$

Applying Corollaries 5.4.5 and 5.4.6 as before, we see that $\nu(K_\alpha^{\mathcal{D}}) = 1$ and

$$(\dim_H \nu - q\alpha - t)\lambda > 0,$$

which immediately yields

$$t < \mathcal{D}(\alpha) - q\alpha.$$

Since $t < T_{\mathcal{D}}(q)$ was arbitrary, this proves (5.26).

In order to show (5.27), we show that

$$T_{\mathcal{D}}(q) \geq \mathcal{D}(\alpha) - q\alpha \tag{5.29}$$

for every $q \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. (Observe that Proposition 5.4.1 deals with the case $\alpha = \infty$.)

Recall from (5.14) that

$$T_{\mathcal{D}}(q) = \inf\{t \in \mathbb{R} \mid P^*(q\varphi_1 - t\log a) \leq 0\} = \sup\{t \in \mathbb{R} \mid P^*(q\varphi_1 - t\log a) > 0\},$$

and so to establish (5.29) (and hence (5.27)), it suffices to show that $P^*(q\varphi_1 - t\log a) > 0$ for every $t < \mathcal{D}(\alpha) - q\alpha$.

To this end, fix $q, t \in \mathbb{R}$ such that $t + q\alpha < \mathcal{D}(\alpha) = \dim_H K_\alpha^{\mathcal{D}}$. We will build a measure ν such that

$$h(\nu) + \int q\varphi_1 \, d\nu - t\lambda(\nu) > 0, \tag{5.30}$$

which will suffice to complete the proof of (5.27), by the above remarks. Observe that since $\dim_H \mathbf{Z} = 0$, we have

$$\dim_H(K_\alpha^{\mathcal{D}} \setminus \mathbf{Z}) = \dim_H K_\alpha^{\mathcal{D}} > t + q\alpha;$$

furthermore, it follows from Proposition 5.4.8 that $\underline{\lambda}(x) > 0$ for every $x \in K_\alpha^{\mathcal{D}} \setminus \mathbf{Z}$, and so we may apply Theorem 4.2.1 and obtain

$$P_{K_\alpha^{\mathcal{D}} \setminus \mathbf{Z}}(-(t + q\alpha)\log a) > 0,$$

where P_Z is the (Carathéodory dimension) topological pressure on Z. Fix $\gamma > 0$ small enough that we have

$$P_{K_\alpha^{\mathcal{D}} \setminus \mathbf{Z}}(-(t + q\alpha)\log a) - \gamma > \gamma > 0. \tag{5.31}$$

Now define a family of sets as in (3.18): for every $\varepsilon > 0$ and $n \in \mathbb{N}$, consider the set

$$G_\alpha^{\varepsilon,N} = \left\{ x \in X \;\middle|\; \left| \frac{-S_n\varphi_1(x)}{S_n \log a(x)} - \alpha \right| \le \varepsilon \text{ and } S_n \log a(x) > 0 \text{ for all } n \ge N \right\}. \tag{5.32}$$

We will also make use of the following sets:

$$G_\alpha^\varepsilon = \bigcup_{N\in\mathbb{N}} G_\alpha^{\varepsilon,N}. \tag{5.33}$$

Applying Proposition 5.4.8 and using the fact that μ is a weak Gibbs measure for φ, we see that for every $x \in K_\alpha^{\mathcal{D}} \setminus \mathbf{Z}$,

$$\alpha = d_\mu(x) = \lim_{n\to\infty} \frac{-S_n\varphi_1(x)}{S_n \log a(x)}.$$

Since $\underline{\lambda}(x) > 0$ for every $x \in K_\alpha^{\mathcal{D}} \setminus \mathbf{Z}$, this implies $K_\alpha^{\mathcal{D}} \setminus \mathbf{Z} \subset G_\alpha^\varepsilon$ for every $\varepsilon > 0$. In particular, this implies that

$$P_{K_\alpha^{\mathcal{D}}\setminus\mathbf{Z}}(-(t+q\alpha)\log a) \le P_{G_\alpha^\varepsilon}(-(t+q\alpha)\log a) = \sup_{N\in\mathbb{N}} P_{G_\alpha^{\varepsilon,N}}(-(t+q\alpha)\log a),$$

and so there exists $N \in \mathbb{N}$ such that

$$P_{G_\alpha^{\varepsilon,N}}(-(t+q\alpha)\log a) - \gamma > \gamma > 0. \tag{5.34}$$

Now we can apply the general inequality [Pes98, (11.9)] to obtain

$$\underline{CP}_{G_\alpha^{\varepsilon,N}}(-(t+q\alpha)\log a) - \gamma > \gamma > 0. \tag{5.35}$$

Let $\psi(x) = \varphi_1(x) + (\alpha+\varepsilon)\log a(x)$, and observe that for every $x \in G_\alpha^{\varepsilon,N}$ and $n \ge N$, we have

$$|-S_n\varphi_1(x) - \alpha S_n \log a(x)| \le \varepsilon S_n \log a(x),$$

which gives

$$-S_n\varphi_1(x) \le (\alpha+\varepsilon) S_n \log a(x),$$

and in particular, $S_n\psi(x) \ge 0$. We may now apply Lemma 3.3.3 with $\psi = \varphi_1 +$

$(\alpha+\varepsilon)\log a$, $\zeta = -(t+q\alpha)\log a$, $Z = G_\alpha^{\varepsilon,N}$, and γ as before, to obtain a measure $\nu \in \mathcal{M}^f(X)$ with the following properties:

$$\int \varphi_1 \, d\nu + (\alpha+\varepsilon)\lambda(\nu) \geq 0, \tag{5.36}$$

$$h(\nu) - (t+q\alpha)\lambda(\nu) \geq \underline{CP}_{G_\alpha^{\varepsilon,N}}(-(t+q\alpha)\log a) - \gamma > \gamma > 0. \tag{5.37}$$

If $q \geq 0$, then multiplying (5.36) by q yields

$$\int q\varphi_1 \, d\nu + (q\alpha + q\varepsilon)\lambda(\nu) \geq 0,$$

and adding this to (5.37) yields

$$h(\nu) + \int q\varphi_1 \, d\nu - t\lambda(\nu) \geq \gamma - q\varepsilon\lambda(\nu).$$

We can choose $\varepsilon > 0$ small enough such that $\gamma > q\varepsilon\lambda(\nu)$ for any invariant measure ν, and this establishes (5.30).

For $q \leq 0$, we do a similar computation with $\psi = \varphi_1 + (\alpha - \varepsilon)\log a$. □

We now proceed to the proof of Part II.

Proof of Proposition 5.4.1. Suppose there exists $x \in K_\alpha^{\mathcal{D}} \setminus \mathbf{Z}$, and let n_k be a subsequence such that the empirical measures μ_{x,n_k} converge to an invariant measure ν. Then $\lambda(\nu) > 0$ (otherwise $\alpha = \infty$ or $x \in \mathbf{Z}$) and $-\int \varphi_1 \, d\nu = \alpha \int \log a \, d\nu$ (by Proposition 5.4.8 and weak* convergence). It follows that

$$\begin{aligned} P^*(q\varphi_1 - t\log a) &\geq h(\nu) + \int q\varphi_1 \, d\nu - \int t\log a \, d\nu \\ &\geq -\lambda(\nu)(q\alpha + t) \end{aligned}$$

for every $q, t \in \mathbb{R}$. In particular, if $P^*(\varphi_{q,t}) \leq 0$, then $q\alpha + t \geq 0$, hence $t \geq -q\alpha$. This holds for all $t \geq T_{\mathcal{D}}(q)$, and consequently $T_{\mathcal{D}}(q) \geq -q\alpha$ as well.

As for the case $\alpha = \infty$, we use the above construction and Corollary 5.4.7 to obtain $\nu \in \mathcal{M}^f(X)$ with $\lambda(\nu) = h(\nu) = 0$. Furthermore, since $x \in X'$, we have $\int \varphi_1 \, d\nu < 0$, and it follows immediately that $P^*(\varphi_{q,t}) > 0$ for all $q < 0$ and $t \in \mathbb{R}$, hence $T_{\mathcal{D}}(q) = +\infty$ for all $q < 0$. □

It only remains to prove the propositions implying Part III.

Proof of Proposition 5.4.2. It follows from Corollary 5.4.6 and the weak Gibbs property of μ that $\nu_n(K_\alpha^{\mathcal{D}}) = 1$ for all n. Furthermore, from the assumption that $t_n \le T_{\mathcal{D}}(q_n)$, we have

$$0 \le P^*(\varphi_{q_n,t_n}) = h(\nu_n) + q_n \int \varphi_1 \, d\nu_n - t_n\lambda(\nu_n) = h(\nu_n) - q_n\alpha\lambda(\nu_n) - t_n\lambda(\nu_n),$$

and applying Corollary 5.4.5 (using the assumption that $\lambda(\nu_n) > 0$) gives

$$\dim_H \nu_n \ge q_n\alpha + t_n.$$

Since $\nu_n(K_\alpha^{\mathcal{D}}) = 1$, this in turn implies

$$\mathcal{D}(\alpha) \ge q_n\alpha + t_n,$$

and taking the limit as $n \to \infty$ yields

$$\mathcal{D}(\alpha) \ge q\alpha + T_{\mathcal{D}}(q) \ge T_{\mathcal{D}}^{L_3}(\alpha). \qquad \square$$

Proof of Proposition 5.4.3. As before, it follows from the finiteness of $T_{\mathcal{D}}(q)$ that $\frac{\partial}{\partial t} P^*(\varphi_{q,t}) = -\lambda(\nu_{q,t}) < 0$ for all $(q,t) \in R_\eta(I_Q)$, and consequently (assuming n is large enough) we may apply the Implicit Function Theorem to obtain a continuously differentiable function $T_n \colon (q_1, q_2) \to \mathbb{R}$ such that $(q, T_n(q)) \in R_\eta(I_Q)$ for all q, and such that

$$P^*(\varphi_{q,T_n(q)}) = \frac{1}{n}.$$

Furthermore, we have

$$\lim_{n\to\infty} D^+T_n(q_1) = D^+T_{\mathcal{D}}(q_1), \qquad \lim_{n\to\infty} D^-T_n(q_2) = D^-T_{\mathcal{D}}(q_2),$$

so for every α as in the statement of the proposition, and for all sufficiently large n, we have

$$-D^-T_n(q_2) < \alpha < -D^+T_n(q_1).$$

In particular, by the Intermediate Value Theorem, there exists q_n such that $-\alpha =$

$T_n'(q_n)$. Let $t_n = T_n(q_n)$; then by passing to a subsequence if necessary, we may assume that $(q_n, t_n) \to (q, T_{\mathcal{D}}(q))$ for some $q \in I_Q$. Let ν_n be an ergodic equilibrium state for φ_{q_n,t_n}; because $P^*(\varphi_{q_n,t_n}) > 0$ and $T_{\mathcal{D}}(q_n) < \infty$, we have $\lambda(\nu_n) > 0$.

Finally, we observe that since $P^*(\varphi_{q,t})$ is constant along the curve $(q, T_n(q))$, we have

$$\begin{aligned} 0 = \frac{d}{dq} P^*(\varphi_{q,T_n(q)})|_{q_n} &= \frac{\partial}{\partial q} P^*(\varphi_{q,t})|_{(q_n,t_n)} + T_n'(q_n) \frac{\partial}{\partial t} P^*(\varphi_{q,t})|_{(q_n,t_n)} \\ &= \int \varphi_1 \, d\nu_n + \alpha\lambda(\nu_n), \end{aligned}$$

and hence ν_n satisfies (5.17). □

Appendix A

Coincidence of various definitions

A.1 Definitions of Hausdorff dimension

We now prove a proposition that shows that our various definitions of Hausdorff dimension all agree with each other.

Proposition A.1.1. *Let X be a separable metric space, fix $Z \subset X$, and let $\dim_H Z$, $\dim_H^b Z$, and $\dim_H^{b'} Z$ be as in Definition 2.1.1. Then all three quantities are equal.*

Proof. To see that $\dim_H Z = \dim_H^b Z$, it suffices to show that

$$2^{-s} m_H^b(Z, s, \varepsilon) \leq m_H(Z, s, \varepsilon) \leq m_H^b(Z, s, \varepsilon/2).$$

The first inequality follows by associating to every ε-cover $\{U_i\} \in \mathcal{D}(Z, \varepsilon)$ the set $\{(x_i, r_i)\} \in \mathcal{D}^b(Z, \varepsilon)$, where $x_i \in U_i$ is arbitrary and $r_i = \operatorname{diam} U_i$. The second inequality follows by associating to every $\{(x_i, r_i)\} \in \mathcal{D}^b(Z, \varepsilon/2)$ the ε-cover $\{B(x_i, r_i)\}$. (This half of the proposition may be found in any standard reference on Hausdorff dimension.)

To show that $\dim_H^b Z = \dim_H^{b'} Z$, we show that

$$m_H^b(Z, s, \varepsilon) \leq m_H^{b'}(Z, s, \varepsilon) \leq 2^s m_H^b(Z, s, \varepsilon). \tag{A.1}$$

The first inequality is immediate since $\operatorname{diam} B(x, r) \leq 2r$ for every $x \in X$ and $r > 0$. For the second inequality, we observe that by separability, there are at most countably many isolated points in X, and that removing a countable number of isolated points does not affect the value of $m_H^{b'}(Z,s,\varepsilon)$ or $m_H^b(Z,s,\varepsilon)$; thus we may assume without loss of generality that X has no isolated points. Given (x_i,r_i), let t_i be given by

$$t_i = \sup\{t \in [0,r_i] \mid d(x_i,y) = t \text{ for some } y \in B(x_i,r_i)\};$$

because x_i is not isolated, we have $0 < t_i \leq r_i$. Furthermore, we have

$$\begin{aligned} \operatorname{diam} B(x_i,t_i) &\geq d(x_i,y) = t_i, \\ \sum_i (\operatorname{diam} B(x_i,t_i))^s &\geq \sum_i t_i^s = 2^{-s} \sum_i (2t_i)^s. \end{aligned}$$

Taking the infimum over all $\{(x_i,r_i)\} \in \mathcal{D}^b(Z,\varepsilon)$ gives the second inequality in (A.1), and we are done. □

A.2 Definitions of topological pressure

Similarly, we prove that our definition of topological pressure agrees with Pesin's.

Proposition A.2.1. *For continuous f and φ, the definition of pressure given in Definition 2.1.6 is equivalent to the definition given in [Pes98].*

Proof. In [Pes98], Pesin defines topological pressure as follows. Given a compact metric space X, a continuous map $f\colon X \to X$, and a continuous function $\varphi\colon X \to \mathbb{R}$, we fix a finite open cover $\mathcal{U}$ of X, and let $\mathcal{S}_m(\mathcal{U})$ denote the set of all strings $\mathbf{U} = \{U_{w_1} \dots U_{w_m} \mid U_{w_j} \in \mathcal{U}\}$ of length $m = m(\mathbf{U})$. We write $\mathcal{S} = \mathcal{S}(\mathcal{U}) = \bigcup_{m \geq 0} \mathcal{S}_m(\mathcal{U})$.

Now to each string $\mathbf{U} \in \mathcal{S}(\mathcal{U})$ we associate the set

$$X(\mathbf{U}) = \{x \in X \mid f^{j-1}(x) \in U_{w_j} \text{ for all } j = 1,\dots,m(\mathbf{U})\};$$

given $Z \subset X$ and $N \in \mathbb{N}$, we let $\mathcal{S}(Z,\mathcal{U},N)$ denote the set of all finite or countable collections $\mathcal{G}$ of strings of length at least N which cover Z; that is, $\mathcal{G} \subset \mathcal{S}(\mathcal{U})$ is in $\mathcal{S}(Z,\mathcal{U},N)$ if and only if

1. $m(\mathbf{U}) \geq N$ for all $\mathbf{U} \in \mathcal{G}$, and also

2. $\bigcup_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U}) \supset Z$.

Then we define a set function by

$$m'_P(Z,\varphi,\mathcal{U},s,N) = \inf_{\mathcal{S}(Z,\mathcal{U},N)} \left\{ \sum_{\mathbf{U}\in\mathcal{G}} \exp\left(-sm(\mathbf{U}) + \sup_{x\in X(\mathbf{U})} S_{m(\mathbf{U})}\varphi(x) \right) \right\} \tag{A.2}$$

and the critical value of $m'_P(Z,\varphi,\mathcal{U},s) = \lim_{N\to\infty} m'_P(Z,\varphi,\mathcal{U},s,N)$ by

$$P'_Z(\varphi,\mathcal{U}) = \inf\{s \mid m'_P(Z,\varphi,\mathcal{U},s) = 0\} = \sup\{s \mid m'_P(Z,\varphi,\mathcal{U},s) = \infty\}.$$

(We write m'_P and P' to distinguish these from our definitions given earlier.) The topological pressure is $P'_Z(\varphi) = \lim_{|\mathcal{U}|\to 0} P'_Z(\varphi,\mathcal{U})$, where $|\mathcal{U}| = \max\{\operatorname{diam} U_i \mid U_i \in \mathcal{U}\}$ is the diameter of the cover $\mathcal{U}$.

Given $\delta > 0$, let

$$\varepsilon(\delta) = \sup\{|\varphi(x) - \varphi(y)| \mid d(x,y) < \delta\},$$

and observe that since φ is continuous and X is compact, φ is in fact uniformly continuous, hence $\varepsilon(\delta)$ is finite, and $\lim_{\delta\to 0} \varepsilon(\delta) = 0$. Furthermore, given $x \in X$, $y \in B(x,n,\delta)$, we have

$$|S_n\varphi(x) - S_n\varphi(y)| < n\varepsilon(\delta).$$

Now for a fixed $\delta > 0$, we choose a cover $\mathcal{U}$ with $|\mathcal{U}| < \varepsilon(\delta)$. Let $\gamma(\mathcal{U})$ be the Lebesgue number of $\mathcal{U}$, and consider $\{(x_i,n_i)\} \in \mathcal{P}(Z,N,\gamma(\mathcal{U}))$. Then for each (x_i,n_i) there exists $\mathbf{U}_i \in \mathcal{S}_{n_i}(\mathcal{U})$ such that $B(x_i,n_i,\gamma(\mathcal{U})) \subset X(\mathbf{U}_i)$; let $\mathcal{G}' = \{\mathbf{U}_i\}$, and then

$$\begin{aligned} m'_P(Z,\varphi,\mathcal{U},s,N) &= \inf_{\mathcal{S}(Z,N,\delta)} \sum_{\mathbf{U}\in\mathcal{G}} \exp\left(-sm(\mathbf{U}) + \sup_{x\in X(\mathbf{U})} S_{m(\mathbf{U})}\varphi(x) \right) \\ &\le \sum_{\mathbf{U}_i\in\mathcal{G}'} \exp\left(-sm(\mathbf{U}_i) + \sup_{x\in X(\mathbf{U}_i)} S_{m(\mathbf{U}_i)}\varphi(x) \right) \\ &\le \sum_{(x_i,n_i)} \exp\left(-n_i(s-\varepsilon(\delta)) + S_{n_i}\varphi(x_i)\right). \end{aligned}$$

Since the collection $\{(x_i, n_i)\}$ was arbitrary, we have

$$m'_P(Z, \varphi, \mathcal{U}, s, N) \leq m_P(Z, s - \varepsilon(\delta), \varphi, N, \gamma(\mathcal{U})).$$

Taking the limit $N \to \infty$ yields

$$P'_Z(\varphi, \mathcal{U}) \leq P_Z(\varphi, \gamma(\mathcal{U})) - \varepsilon(\delta),$$

and as $\delta \to 0$ we obtain

$$P'_Z(\varphi) \leq P_Z(\varphi).$$

For the other inequality, fix a cover $\mathcal{U}$ of X, with $|\mathcal{U}| < \delta$. Given $\mathcal{G} \in \mathcal{S}(Z, \mathcal{U}, N)$, we may assume without loss of generality that for every $\mathbf{U} \in \mathcal{G}$, we have $X(\mathbf{U}) \cap Z \neq \emptyset$ (otherwise we may eliminate some sets from $\mathcal{G}$, which does not increase the sum in (A.2)). Thus for each such $\mathbf{U}$, we choose $x_{\mathbf{U}} \in X(\mathbf{U}) \cap Z$; we see that $X(\mathbf{U}) \subset B(x_{\mathbf{U}}, m(\mathbf{U}), \delta)$, and so

$$\begin{aligned} m'_P(Z, \varphi, \mathcal{U}, s, N) &= \inf_{\mathcal{S}(Z,\mathcal{U},N)} \sum_{\mathbf{U} \in \mathcal{G}} \exp\left(-sm(\mathbf{U}) + \sup_{x \in X(\mathbf{U})} S_{m(\mathbf{U})}\varphi(x) \right) \\ &\geq \inf_{\mathcal{P}(Z,N,\delta)} \sum_{(x_i,n_i)} \exp\left(-n_i s + S_{n_i}\varphi(x_i)\right) \\ &= m_P(Z, s, \varphi, N, \delta). \end{aligned}$$

Thus $P'_Z(\varphi, \mathcal{U}) \geq P_Z(\varphi, \delta)$, and taking the limit as $\delta \to 0$ gives

$$P'_Z(\varphi) \geq P_Z(\varphi),$$

which completes the proof. □

Appendix B

Local dimensional quantities

B.1 Estimating topological pressure from a weak Gibbs property

We now prove that a weak Gibbs measure as defined in (5.3) gives us bounds on the topological pressure.

Theorem B.1.1. *Let X be a compact metric space, $f\colon X \to X$ a continuous map, and $\varphi\colon X \to \mathbb{R}$ a continuous potential function. Given any subset $Z \subset X$ and $\mu \in \mathcal{M}(X)$, consider the following quantities:*

$$\overline{P} = \sup_{x\in Z} \overline{\lim_{\delta\to 0}}\, \overline{\lim_{n\to\infty}} \frac{1}{n}(-\log \mu(B(x,n,\delta)) + S_n\varphi(x)),$$
$$\underline{P} = \inf_{x\in Z} \underline{\lim_{\delta\to 0}}\, \underline{\lim_{n\to\infty}} \frac{1}{n}(-\log \mu(B(x,n,\delta)) + S_n\varphi(x)).$$

Then $P_Z(\varphi) \leq \overline{P}$. If in addition we have $\mu(Z) > 0$, then $P_Z(\varphi) \geq \underline{P}$.

Proof. First we prove the upper bound. Given $\varepsilon > 0$, $N \in \mathbb{N}$, and $\delta > 0$, consider the following set:

$$\overline{Z}^{\varepsilon}_{N,\delta} = \{x \in Z \mid \mu(B(x,n,\delta')) \geq e^{-n(\overline{P}+\varepsilon)+S_n\varphi(x)} \text{ for all } n \geq N \text{ and } 0 < \delta' < \delta\}.$$

Let $\overline{Z}^{\varepsilon}_{\delta} = \bigcup_N \overline{Z}_N$: it follows from the hypotheses of the theorem that for every $\varepsilon > 0$ we have $Z = \bigcup_{\delta>0} \overline{Z}^{\varepsilon}_{\delta}$.

Now for any (n,δ')-separated subset $E_n = \{x_i\} \subset \overline{Z}^{\varepsilon}_{N,\delta}$ with $n \geq N$ and $0 < \delta' < \delta$, we have

$$1 = \mu(X) \geq \sum_i \mu(B(x_i, n, \delta')) \geq \sum_i e^{-n(\overline{P}+\varepsilon)+S_n\varphi(x_i)},$$

and hence the nth partition function for φ on $\overline{Z}^{\varepsilon}_{N,\delta}$ is bounded as follows:

$$\sup_{E_n} \sum_i e^{S_n\varphi(x_i)} \leq e^{-n(\overline{P}+\varepsilon)}.$$

It follows that

$$P_{\overline{Z}_{N,\delta}}(\varphi, \delta') \leq \overline{CP}_{\overline{Z}^{\varepsilon}_{N,\delta}}(\varphi, \delta') \leq \overline{P} + \varepsilon,$$

and taking the limit as $\delta' \to 0$ gives

$$P_{\overline{Z}^{\varepsilon}_{N,\delta}}(\varphi) \leq \overline{P} + \varepsilon.$$

Now by countable stability, we get $P_{\overline{Z}^{\varepsilon}_{\delta}}(\varphi) \leq \overline{P} + \varepsilon$. Taking the union over $\delta_n = 1/n$ and using countable stability again gives $P_Z(\varphi) \leq \overline{P}$, and since $\varepsilon > 0$ was arbitrary, this completes the proof of the upper bound.

For the lower bound, we first consider the sets

$$\underline{Z}^{\varepsilon}_{N,\delta} = \{x \in Z \mid \mu(B(x,n,\delta)) \leq e^{-n(\underline{P}-\varepsilon)+S_n\varphi(x)} \text{ for all } n \geq N\}$$

and observe that since $\mu(Z) > 0$, for every $\varepsilon > 0$ there exists $\delta > 0$ and $N \in \mathbb{N}$ such that $\mu(\underline{Z}^{\varepsilon}_{N,\delta}) > 0$. Now for every cover $\{B(x_i, n_i, \delta)\}$ of $\underline{Z}^{\varepsilon}_{N,\delta}$ by Bowen balls of length $n_i \geq N$ with $x_i \in \underline{Z}^{\varepsilon}_{N,\delta}$, we have

$$\sum_i e^{-n_i(\underline{P}-\varepsilon)+S_{n_i}\varphi(x_i)} \geq \sum_i \mu(B(x_i, n_i, \delta)) \geq \mu(\underline{Z}^{\varepsilon}_{N,\delta}) > 0.$$

Thus $m_P(X^{\varepsilon}_{N,\delta}, \underline{P} - \varepsilon, \varphi, \delta) > 0$, and hence $P_X(\varphi) \geq P_{X^{\varepsilon}_{N,\delta}}(\varphi) \geq \underline{P} - \varepsilon$. This gives $P_Z(\varphi) \geq \underline{P} - \varepsilon$, and since $\varepsilon > 0$ was arbitrary, we have $P_Z(\varphi) \geq \underline{P}$. □

Note that Theorem B.1.1 does not require the measure μ to be invariant; however, it must be fully supported (that is, $\operatorname{supp} \mu = X$).

Note also that the lower bound in Theorem B.1.1 is stronger than it appears:

in fact, we have

$$P_Z(\varphi) \geq \sup\{P \in \mathbb{R} \mid \mu(\{x \in Z \mid \underline{CP}_\mu(x) \geq P\}) > 0\},$$

which follows easily by using the fact that $P_Z(\varphi) \geq P_{Z'}(\varphi)$ for every subset $Z' \subset Z$.

B.2 Existence of weak Gibbs measures

Now we show that (not necessarily fully supported) weak Gibbs measures actually exist in many cases, so that we can apply Theorem B.1.1 with $\underline{P} = \overline{P}$ to obtain an exact value for $P_Z(\varphi)$, where Z is the support of the measure.

Theorem B.2.1. *Let X be a compact metric space and $f\colon X \to X$ a continuous map with the property that for some $\delta > 0$, we have $d(f(x), f(y)) \geq d(x, y)$ whenever the latter quantity is less than δ. Then for any continuous potential $\varphi\colon X \to \mathbb{R}$, there exists measure μ that is weak Gibbs on its support—that is, a measure $\mu \in \mathcal{M}(X)$ such that $P_\mu(x)$ exists and is constant everywhere on* $\operatorname{supp}\mu$.

Proof. Let $\mathcal{L}_\varphi\colon C(X) \to C(X)$ be the Perron–Frobenius operator given by

$$(\mathcal{L}_\varphi h)(x) = \sum_{y \in f^{-1}(x)} e^{\varphi(y)} h(y),$$

and let $\mathcal{L}_\varphi^*\colon C(X)^* \to C(X)^*$ be its dual. Define a map $\mathcal{P}\colon \mathcal{M}(X) \to \mathcal{M}(X)$ by

$$\mathcal{P}(\mu) = \frac{\mathcal{L}_\varphi^*(\mu)}{(\mathcal{L}_\varphi^*(\mu))(\mathbf{1}_X)},$$

where $\mathbf{1}_Z$ denotes the characteristic function of the set Z. Then $\mathcal{P}$ is a continuous self-map of a compact convex subset of a locally convex topological vector space, and so the Schauder–Tychonoff fixed point theorem implies that there exists $\mu \in \mathcal{M}(X)$ such that $\mathcal{P}(\mu) = \mu$. In particular, there exists $P \in \mathbb{R}$ such that

$$\mathcal{L}_\varphi^*(\mu) = e^P \mu.$$

We claim that μ is exactly the weak Gibbs measure we are looking for (note that it need not be invariant) and that for the constant P, the weak Gibbs property (5.1) is satisfied at all $x \in X$.

Because f is nowhere contracting, we have $f(B(x,n,\delta)) = B(f(x), n-1, \delta)$ for all $x \in X$ and $n \geq 1$ once δ is sufficiently small. Let $\varepsilon = \varepsilon(\delta) > 0$ be the modulus of continuity for φ—that is, $d(x,y) < \delta$ implies $|\varphi(x) - \varphi(y)| < \varepsilon$. Then

$$\begin{aligned}
\mu(B(x,n,\delta)) &= \int \mathbf{1}_{B(x,n,\delta)}\, d\mu \\
&= e^{-P} \int \mathbf{1}_{B(x,n,\delta)}\, d(\mathcal{L}_\varphi^* \mu) \\
&= e^{-P} \int (\mathcal{L}_\varphi \mathbf{1}_{B(x,n,\delta)})(y)\, d\mu(y) \\
&= e^{-P} \int \sum_{z \in f^{-1}(y)} e^{\varphi(z)} \mathbf{1}_{B(x,n,\delta)}(z)\, d\mu(y) \\
&\approx e^{-P} e^{\varphi(x)} \mu(B(f(x), n-1, \delta)),
\end{aligned}$$

where the precise error bounds are given as follows:

$$e^{-\varepsilon} \leq \frac{\mu(B(x,n,\delta))}{e^{\varphi(x)-P}\mu(B(f(x),n-1,\delta))} \leq e^{\varepsilon}. \tag{B.1}$$

Iterating (B.1), we obtain

$$e^{-n\varepsilon}\mu(B(f^n(x),\delta)) \leq \frac{\mu(B(x,n,\delta))}{e^{S_n\varphi(x)-nP}} \leq e^{n\varepsilon}\mu(B(f^n(x),\delta)). \tag{B.2}$$

Lemma B.2.2. *For every $\delta > 0$ there exists $\gamma > 0$ such that $\mu(B(y,\delta)) > \gamma$ for all $y \in \operatorname{supp}\mu$.*

Proof. $\operatorname{supp}\mu$ is compact, so we can cover it with a finite number of balls $B(y_i, \delta/2)$. Let $\gamma = \min\{\mu(B(y_i,\delta/2))\}$, then for every $y \in \operatorname{supp}\mu$ we have $y \in B(y_i,\delta/2)$ for some i, and hence $B(y,\delta) \supset B(y_i,\delta/2)$, which suffices. □

Using the lemma, we see that (B.2) implies

$$\gamma e^{-n\varepsilon} \leq \frac{\mu(B(x,n,\delta))}{e^{S_n\varphi(x)-nP}} \leq e^{n\varepsilon},$$

which completes the proof. □

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Vita

Vaughn Alan Climenhaga

Vaughn Climenhaga was born in Lancaster, Pennsylvania in 1982, and grew up in Pennsylvania, Kentucky, Zambia, Zimbabwe, Indiana, and Manitoba before setting out to make his own way in the world. He was introduced to higher mathematics at the University of Waterloo, which he attended from 2000 to 2005, and where he remembers with special fondness the courses in calculus and analysis taught by Ken Davidson, Brian Forrest, and Laurent Marcoux, as well as the community he found at Conrad Grebel College, where he lived, laughed, loved, danced, sang, and all those other wonderful things.

During this time Vaughn spent a memorable year abroad, returning to Africa with his family for four months and then spending a term studying mathematics in Budapest. After one final year at Waterloo, he received his B.Math. degree, was awarded the Alumni Association Gold Medal by the Faculty of Mathematics, and moved south to pursue his Ph.D. at the Pennsylvania State University.

Shortly after arriving at Penn State in 2005, Vaughn fell under the sway of the research group in dynamical systems, particularly Yakov Pesin (who would become his dissertation advisor) and the inimitable Anatole Katok. Along the way, he spent three years as a TA for the MASS program, where he worked with many remarkable undergraduate students and discovered a predilection for expository writing, leading to his appearance as a co-author on three books:

- *Lectures on surfaces: (almost) everything you wanted to know about them*, with A. Katok;
- *Lectures on fractal geometry and dynamical systems*, with Ya. Pesin;
- *Lectures on groups and their connections to geometry*, with A. Katok.

While completing his thesis research, Vaughn was awarded the Pritchard Dissertation Fellowship by the Department of Mathematics; he has also received departmental and university-wide teaching awards.

Throughout his time at Penn State, Vaughn has remained active in and drawn sustenance from the community at University Mennonite Church and its student groups, with whom he has been privileged to journey for these past five years.

For the time being, Vaughn's research interests are in ergodic theory, dimension theory, thermodynamic formalism, and multifractal analysis, especially in the close interrelationships between these fields. The vast majority of the results in this thesis come from the following two research papers:

- Bowen's equation in the non-uniform setting. To appear in *Ergodic Theory and Dynamical Systems*. arXiv:0908.4126.
- Multifractal formalism derived from thermodynamics. arXiv:1002.0789.