Large deviations using non-uniform specification properties

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The talk in one slide

Setting: $X \subset \mathcal{A}^{\mathbb{N}}$ a shift space on a finite alphabet

Theorem (Known results)

Suppose X has specification. Then

1. *bounded distortion $\Rightarrow$ unique equilibrium state + Gibbs*
2. *Gibbs $\Rightarrow$ large deviations principle*

Goal: Same results with non-uniform versions of above properties

Key idea:

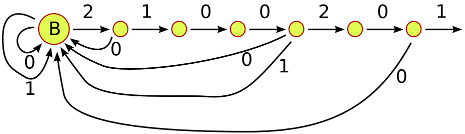
- $\mathcal{L}$ the language of X (space of finite orbit segments)
- Only require properties for $\mathcal{G} \subset \mathcal{L}$
- Get results if $\mathcal{G}$ is "big enough"

Shift spaces, languages, and sets of words

Shift space: closed, shift-invariant set $X \subset \mathcal{A}^{\mathbb{N}}$ ($\mathcal{A}$ finite: alphabet)

- Finite word $w \in \mathcal{A}^* = \bigcup_{n\geq 0} \mathcal{A}^n \rightsquigarrow$ cylinder $[w]$
- Language of X is $\mathcal{L} = \{w \in \mathcal{A}^* \mid [w] \neq \emptyset\}$.

Example: $\beta > 1 \rightsquigarrow X = \Sigma_\beta$ is coding space for $x \mapsto \beta x \pmod 1$



Sequence determined by $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$

$\mathcal{L} = \{$labels of paths starting at **B**$\}$

Consider subsets $\mathcal{D} \subset \mathcal{L}$ (points + times) / (orbit segments)

- $\mathcal{G} = \{$labels for paths starting and ending at **B**$\}$
- $\mathcal{C}^s = \{$labels for paths that never return to **B**$\}$

Specification

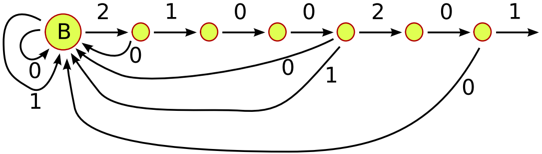
Various transitivity/mixing properties for (X, σ):

(irreducible) Markov/sofic $\Rightarrow$ (weak) specification $\Rightarrow$ transitive

Definition: $\mathcal{D} \subset \mathcal{L}$ has specification if $\exists \tau$ (gluing time) s.t. words from $\mathcal{D}$ can be glued together with connecting words of length $\leq \tau$

- $\forall w^1, \dots, w^k \in \mathcal{D}$ there exist $v^1, \dots, v^k \in \mathcal{L}$ such that $w^i v^i w^{i+1} v^{i+1} \dots w^{j-1} v^{j-1} w^j \in \mathcal{D}$ for all $1 \leq i < j \leq k$

Example: For the β-shifts, $\mathcal{G}$ has specification, but $\mathcal{L}$ does not



Large deviations

$\mathcal{M}(X) = \{\text{Borel prob. measures on } X\}$ $\qquad \mathcal{E}_n(x)(\varphi) = \frac{1}{n} S_n\varphi(x)$

- Empirical measures: $\mathcal{E}_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\sigma^k x}$

Fix a reference measure $m \in \mathcal{M}(X)$

- Assume m is σ-invariant and ergodic
- Birkhoff ergodic theorem: $\mathcal{E}_n(x) \to m$ for m-a.e. x

Large deviations: Given $U \subset \mathcal{M}(X)$, study $m\{x \mid \mathcal{E}_n(x) \in U\}$

- Goes to 0 if $m \notin U$. Exponentially? Polynomially?

Example: $m\{x \mid |\frac{1}{n} S_n\varphi(x) - \int \varphi \, dm| > \epsilon\}$

Thermodynamics

Pressure of φ on $\mathcal{D} \subset \mathcal{L}$ is $P(\mathcal{D}, \varphi) = \lim \frac{1}{n} \log \left(\sum_{\mathcal{D}_n} e^{\varphi_n(w)} \right)$

- $\mathcal{D}_n = \{w \in \mathcal{D} \mid |w| = n\}$ $\qquad \varphi_n(w) = \sup_{x \in [w]} S_n\varphi(x)$

Variational principle: $P(\varphi) = \sup\{h(\mu) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_\sigma(X)\}$

- $\mathcal{M}_\sigma(X) = \{\mu \in \mathcal{M}(X) \mid \mu \text{ is } \sigma\text{-invariant}\}$
- Supremum achieved by equilibrium states

Uniqueness of equilibrium state related to statistical properties

Classical (uniform) results

Bowen (1974): If (X, σ) has specification and φ is Hölder, then:

- φ has a unique equilibrium state $\mu \in \mathcal{M}_\sigma(X)$
- μ is Gibbs: $K \leq \frac{\mu[w]}{e^{-nP(\varphi)+S_n\varphi(x)}} \leq K'$ for all $x \in [w]$, $w \in \mathcal{L}_n$

Young (1990): If (X, σ) has specification and m is Gibbs for φ, then we have a large deviations principle with reference measure m:

$$U \subset \mathcal{M}(X) \text{ open } \Rightarrow \varliminf_{n\to\infty} \frac{1}{n} \log m\{x \mid \mathcal{E}_n(x) \in U\} \geq \sup_{\mu\in U} q(\mu)$$

$$F \subset \mathcal{M}(X) \text{ closed } \Rightarrow \varlimsup_{n\to\infty} \frac{1}{n} \log m\{x \mid \mathcal{E}_n(x) \in F\} \leq \sup_{\mu\in F} q(\mu)$$

Rate function $q(\mu) = \begin{cases} h(\mu) + \int \varphi \, d\mu - P(\varphi) & \mu \in \mathcal{M}_\sigma(X) \\ -\infty & \mu \notin \mathcal{M}_\sigma(X) \end{cases}$

Motivating idea

Similar theorems in non-uniform setting given following condition:

- "$\mathcal{G} \subset \mathcal{L}$ has good properties, and every word in $\mathcal{L}$ can be transformed into a word in $\mathcal{G}$ without too much fuss"

For **uniqueness**, this means every $\mathcal{G}^M$ has specification, and

- Transform $w \in \mathcal{L}$ to $v \in \mathcal{G}$ by removing "bad bits" from ends (Decompose as $w = u^p v u^s$)
- u^p, u^s come from a list $\mathcal{C} \subset \mathcal{L}$ of "obstructions", and list is "thermodynamically small" $(P(\mathcal{C}, \varphi) < P(\varphi))$

For **large deviations**, this means $\mathcal{G}$ has spec, m Gibbs on φ, and

- $\mathcal{L} \rightsquigarrow \mathcal{G}$ by making edits (insertions, deletions, changes)
- Number of edits $\leq g(|w|)$, where $\frac{g(n)}{n} \to 0$

Decompositions and uniqueness

Decomposition of $\mathcal{L}$: sets $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ such that $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$.

$$\mathcal{G}^M = \{uvw \in \mathcal{L} \mid u \in \mathcal{C}^p, v \in \mathcal{G}, w \in \mathcal{C}^s, |u|, |w| \leq M\}$$

Theorem (C.–Thompson, 2012)

Suppose $\mathcal{L}$ has a decomposition such that

1. *φ has bounded distortion on $\mathcal{G}$*
2. *$\mathcal{G}^M$ has specification for every M*
3. $P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(\varphi)$

Then φ has a unique equilibrium state μ. It is Gibbs on each $\mathcal{G}^M$.

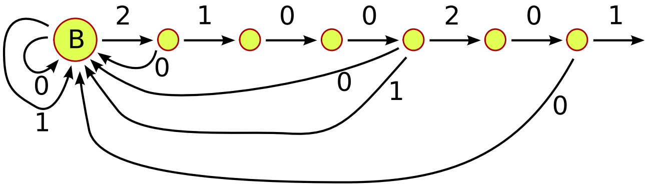
Strong spec. for $\mathcal{G}^M \Rightarrow (X, \sigma, \mu)$ is Kolmogorov (Ledrappier)

Example: β-shift

$\mathcal{C}^p = \emptyset$

$\mathcal{G} = \{$words (paths) starting and ending at $B\}$

$\mathcal{C}^s = \{$words (paths) starting at B and never returning$\}$



- $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$
- $\mathcal{G}^M = \{$paths ending in first M vertices$\}$ has spec. for each M
- $h(\mathcal{C}) = 0$, where $\mathcal{C} = \mathcal{C}^p \cup \mathcal{C}^s$
- In fact, $P(\mathcal{C}, \varphi) < P(\varphi)$ for every Hölder φ

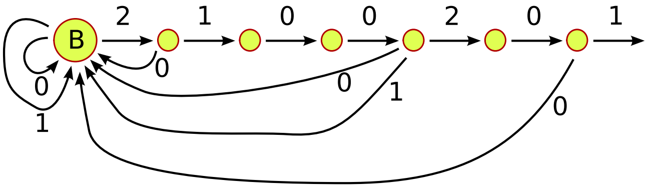
Statistical specification properties

Large deviations results have been obtained for β-shift and other systems by using statistical specification properties.

- Pfister, Sullivan (2005)
- Yamamoto (2009)
- Varandas (2012)

All reflect idea that the gluing procedure can be weakened in a way that does not interfere too much with Birkhoff averages.

β-shifts



Given any $v \in \mathcal{L}$, can transform v into a word $u \in \mathcal{G}$ by making a single change. (Change last non-zero symbol to 0).

Thus given any $v, w \in \mathcal{L}$, the word vw may not be in $\mathcal{L}$, but can be transformed into a word in $\mathcal{L}$ by making a single change.

General method for getting a word that concatenates statistical properties of v and w, as long as $\frac{\text{number of changes}}{\text{length of word}} \to 0$.

Edit metric

Goal: Define a metric on $\mathcal{A}^*$ (set of all finite words) that controls how much Birkhoff sums can vary.

An edit of a word w is any of the following:

- Substition: $w = uav \mapsto w' = ubv$ $\quad u, v \in \mathcal{A}^*, \ a, b \in \mathcal{A}$
- Insertion: $w = uv \mapsto w' = ubv$ $\quad u, v \in \mathcal{A}^*, \ b \in \mathcal{A}$
- Deletion: $w = uav \mapsto w' = uv$ $\quad u, v \in \mathcal{A}^*, \ a \in \mathcal{A}$

$\hat{d}(v, w) =$ minimum number of edits required to go from v to w.

- Induces a metric on $X \times \mathbb{N}$ via $(x, n) \mapsto x_1 \cdots x_n$

Key property: $\mathcal{E} \colon (X \times \mathbb{N}, \hat{d}) \to (\mathcal{M}(X), \text{weak*})$ is continuous

- $\mathcal{E}$ assigns to each (x, n) the empirical measure $\mathcal{E}_n(x)$

Edit approachability

mistake function: a non-decreasing sub-linear function $g\colon \mathbb{N} \to \mathbb{N}$.
$(\frac{g(n)}{n} \to 0)$

$\mathcal{L}$ is edit approachable by $\mathcal{G} \subset \mathcal{L}$ if there exists a mistake function g such that for every $v \in \mathcal{L}$, there is $w \in \mathcal{G}$ with $\hat{d}(v, w) < g(|v|)$.

Theorem (C.–Thompson–Yamamoto, 2013)

X a shift space on a finite alphabet, $\mathcal{L}$ its language. Suppose

1. *$\mathcal{L}$ is edit approachable by $\mathcal{G}$,*
2. *$\mathcal{G}$ has specification (with good concatenations),*
3. *$m \in \mathcal{M}(X)$ is Gibbs for φ on $\mathcal{G}$.*

Then X satisfies a LDP with reference measure m and rate q^φ

In particular, every Hölder continuous φ on a β-shift

Recap

Moral of the story:

Many good consequences of specification (and other properties) can still be obtained as long as properties hold on a "large enough" set of words (orbit segments)

"Large enough" means the ability to get from $\mathcal{L}$ to $\mathcal{G}$ with some "small" tinkering, where meaning of "small" depends on context

- Unique equilibrium state: only need to remove a prefix and a suffix from the word in $\mathcal{L}$, and these come from "small" lists
- Large deviations: only need to make a small number of edits

Coded systems

Present shift as paths on graph with edge labels from $\mathcal{A}$

- Finite graph $\rightsquigarrow$ sofic shift
- Countable graph $\rightsquigarrow$ coded shift

Decomposition in terms of graph presentation

- F a finite set of vertices, $\mathcal{G} =$ paths starting and ending in F
- $\mathcal{C}^p =$ paths only entering F on last step, or never
- $\mathcal{C}^s =$ paths starting in F and never returning

Presentation and decomposition in terms of generators

- $G \subset \mathcal{A}^*$ a set of generators, $\mathcal{G} = G^* = \{w^1 \cdots w^n \mid w^j \in \mathcal{G}\}$
- $\mathcal{C}^p =$ suffixes of generators, $\mathcal{C}^s =$ prefixes of generators

S-gap shifts

Fix $S \subset \mathbb{N}$, take generators $G = \{0^n 1 \mid n \in S\}$

- $\mathcal{L} = \{0^k 1 0^{n_1} 1 0^{n_2} 1 \cdots 0^{n_j} 1 0^\ell \mid n_i \in S\}$

Natural decomposition with $h(\mathcal{C}) = 0$ and edit approachability:

- $\mathcal{G} = \{0^{n_1} 1 \cdots 0^{n_j} 1 \mid n_i \in S\}$
- $\mathcal{C}^p = \{0^k 1 \mid k \in \mathbb{N}\}, \quad \mathcal{C}^s = \{0^\ell \mid \ell \in \mathbb{N}\}$

Def'n: (X, φ) is *hyperbolic* if $P(\varphi) > \sup_\mu \int \varphi \, d\mu$ $\quad (h(ES) > 0)$

Theorem

$(h(\mathcal{C}) = 0)$ + (hyperbolic) $\Rightarrow P(\varphi) > P(\mathcal{C}, \varphi)$

(S-gap, Hölder) $\Rightarrow$ hyperbolic $\Rightarrow$ unique ES, Gibbs on $\mathcal{G}^M$, LDP

Open questions

Transitive piecewise monotonic interval maps have coded codings

- $h(\mathcal{C})$ can be made arbitrarily small $\Rightarrow$ unique MME
- Edit approachable by specification? Hölder $\Rightarrow$ hyperbolic?

General conditions for Hölder to imply hyperbolic

- True for β-shift, S-gap shift
- Is it true whenever $\mathcal{L}$ edit approachable by specification?

Examples where Hölder does not imply hyperbolic

- Candidate: context-free shift $G = \{01^n2^n \mid n \in \mathbb{N}\}$
- Not edit approachable by specification (Scott Conrad)
- Non-hyperbolic Hölder potential?