Motivating examples in dynamical systems

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University of Houston

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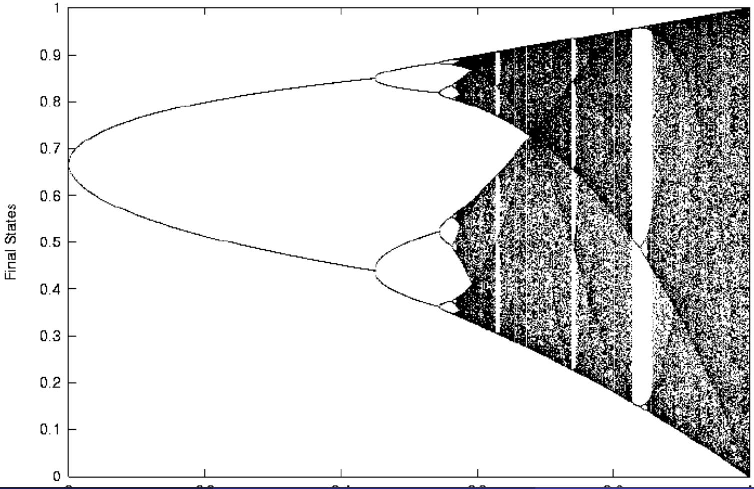
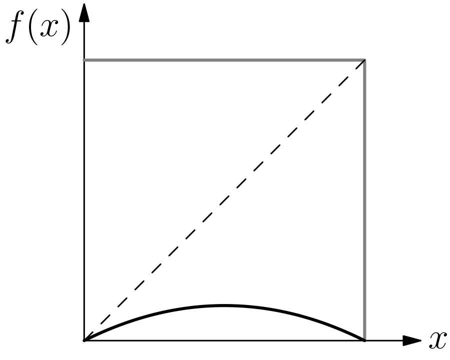
A pretty picture



The logistic map

Logistic map: $f(x) = \lambda x(1-x)$ $\qquad 0 \leq \lambda \leq 4$



$\lambda = 0.5$

- Maps the interval $[0,1]$ into itself
- Iterate over and over again: represents state of a dynamical system evolving in time

What is the long-term behaviour, and (how) does it depend on λ?

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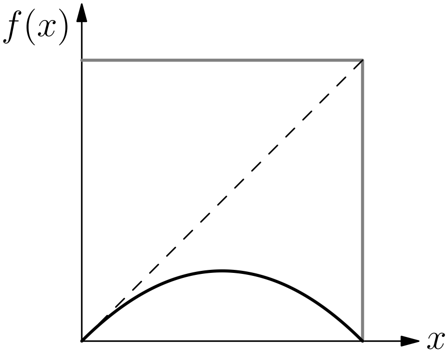

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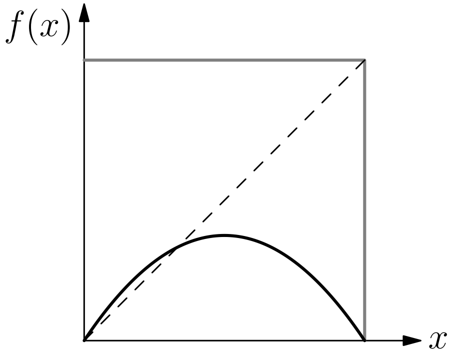

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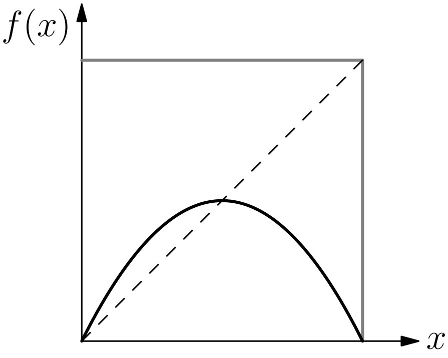

$\lambda = 2$

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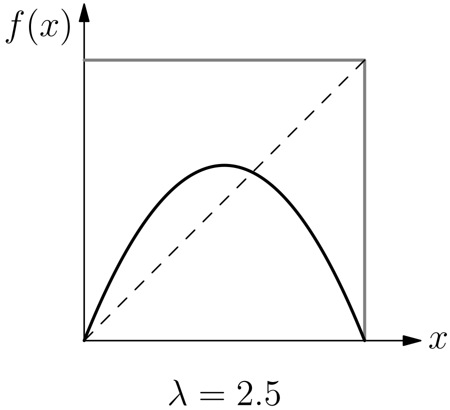

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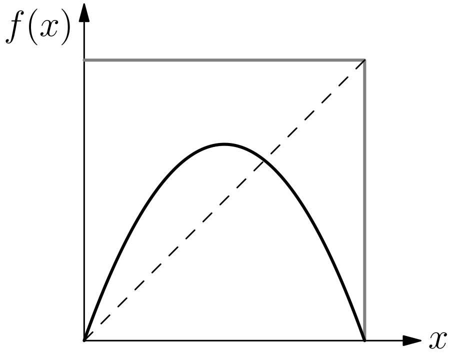

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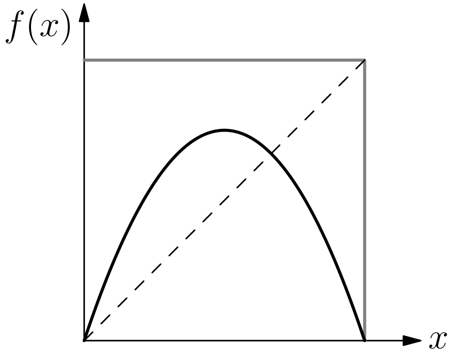

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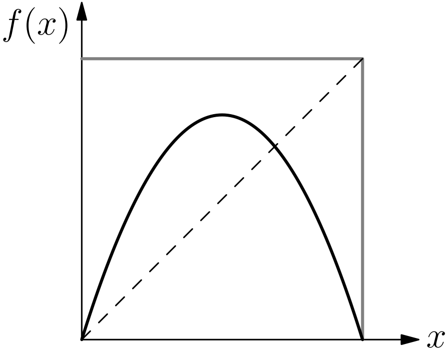

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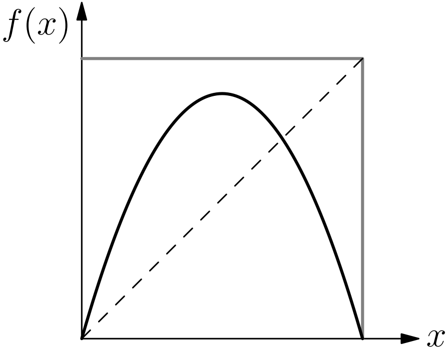

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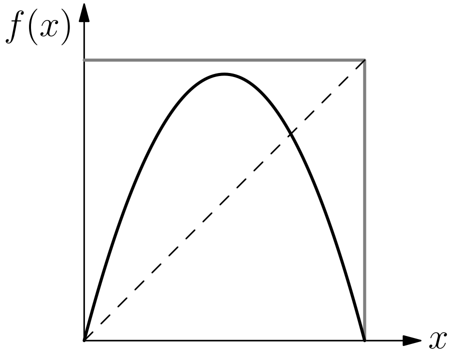

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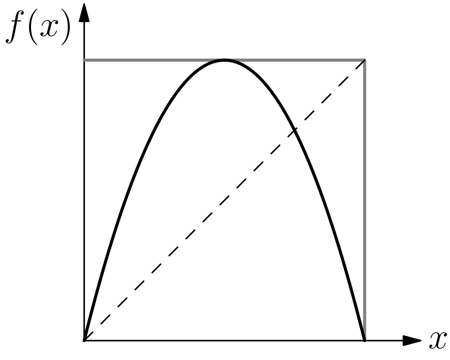

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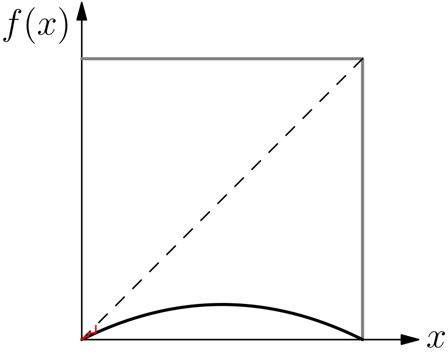

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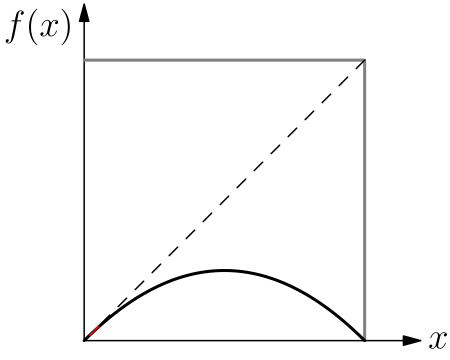

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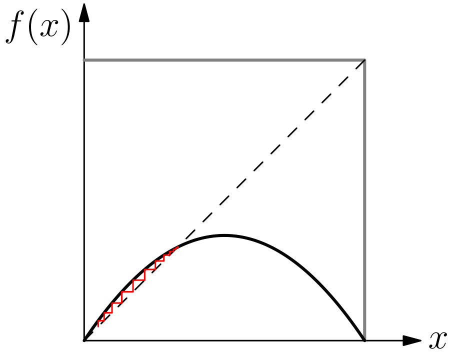

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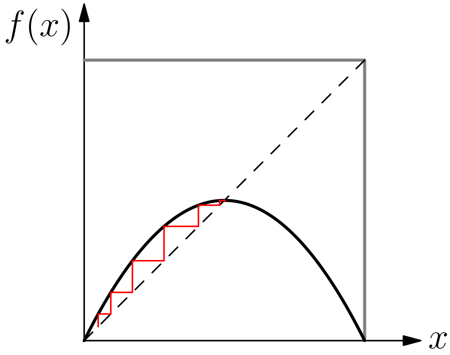

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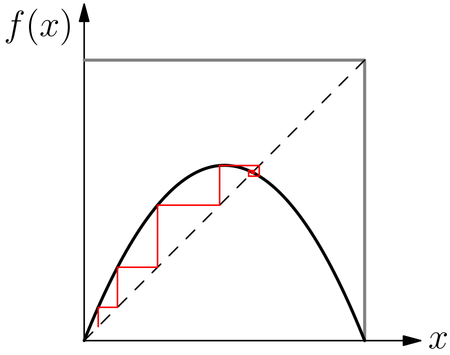

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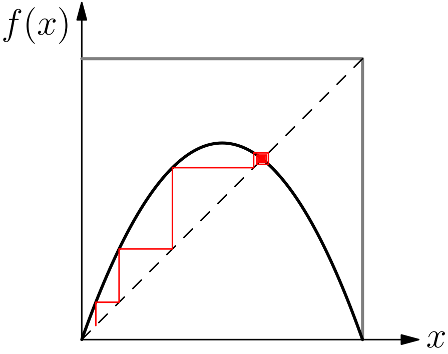

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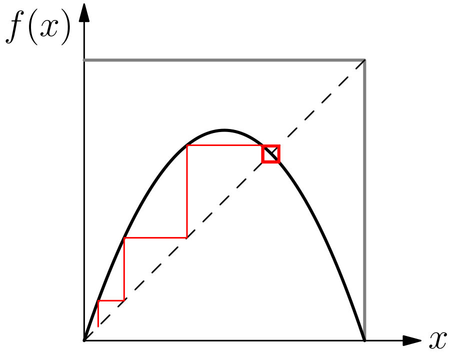

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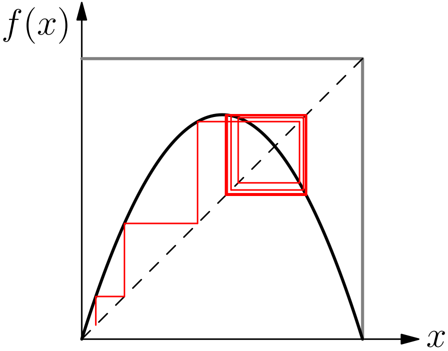

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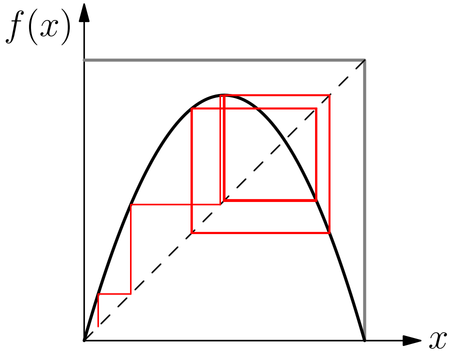

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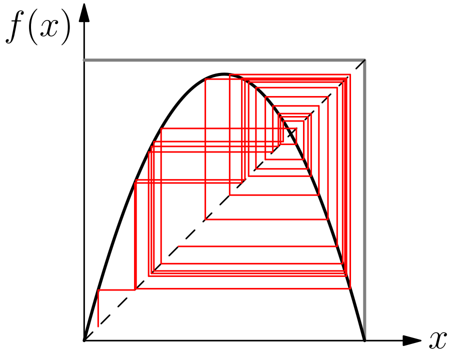

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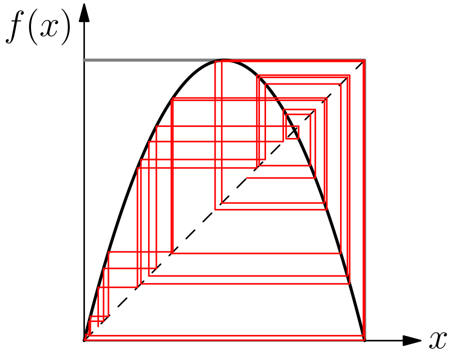

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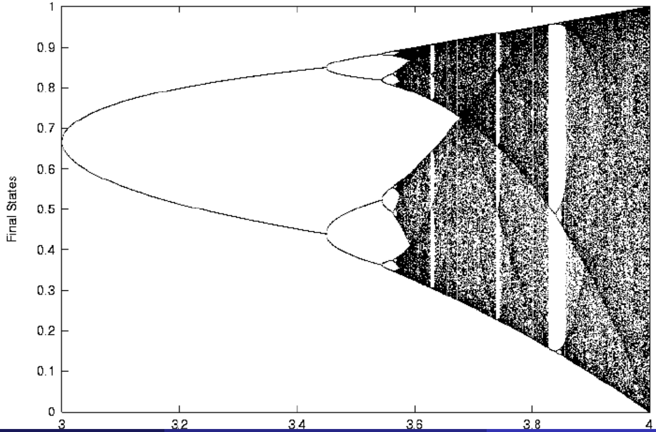
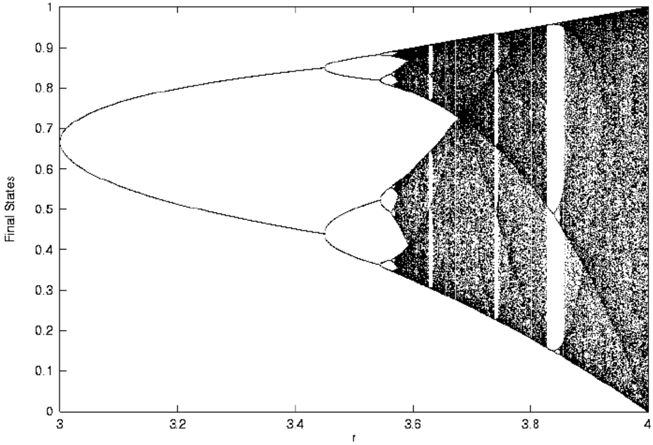
More than just a pretty picture

Bifurcation diagram. Horizontal = parameter, vertical = recurrent states



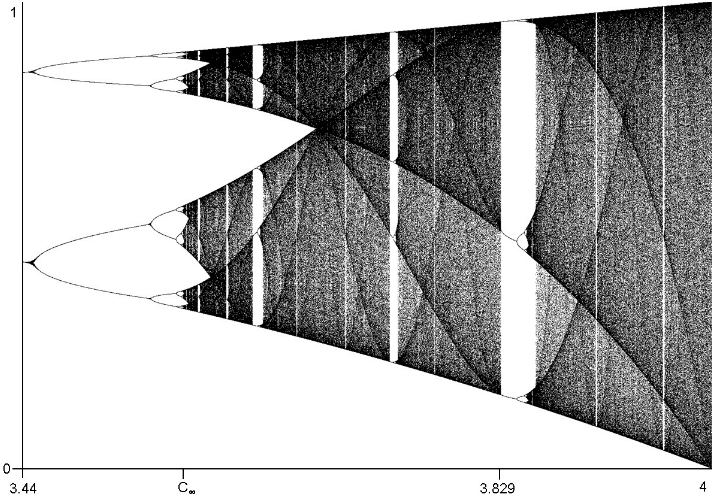
More than just a pretty picture

$\lambda \in [3, 3.57\ldots]$ ← period-doubling cascade



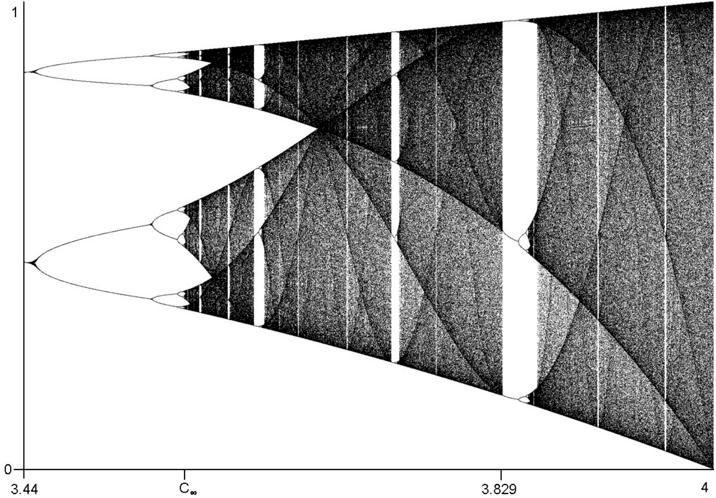
More than just a pretty picture

$\lambda \in [3.832, 3.857\ldots] \leftarrow$ window of stability

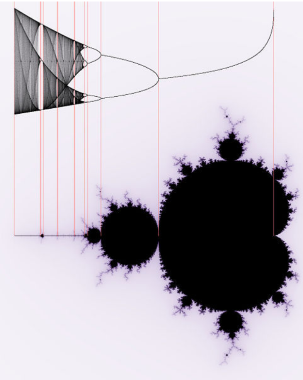


More than just a pretty picture

$\lambda = 4 \leftarrow$ chaos

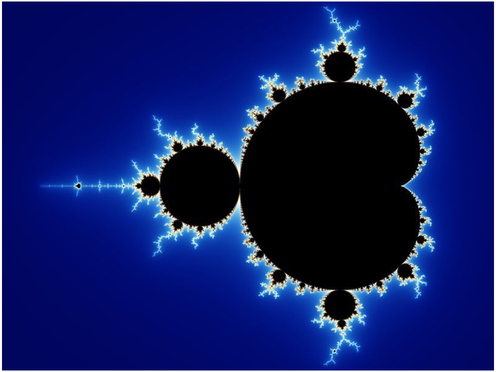


Aside: Mandelbrot set



Fix $c \in \mathbb{C}$: let $z_0 = 0$, $z_{n+1} = z_n^2 + c$.

Mandelbrot set $M = \{c \mid z_n \not\to \infty\}$



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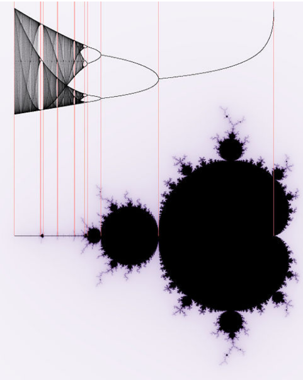

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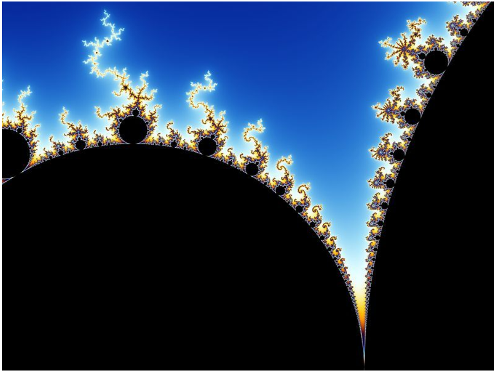

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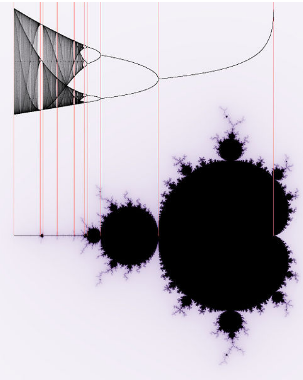

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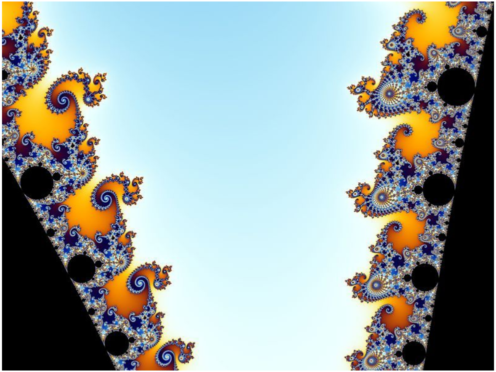

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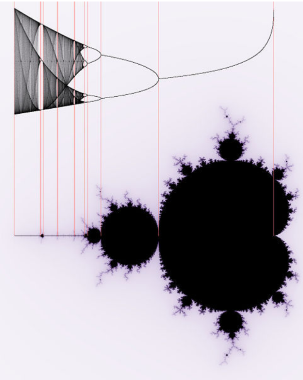

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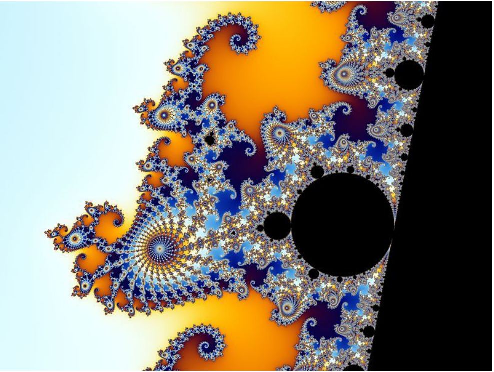

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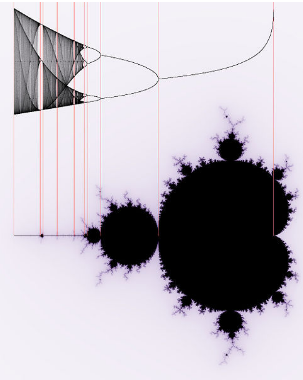

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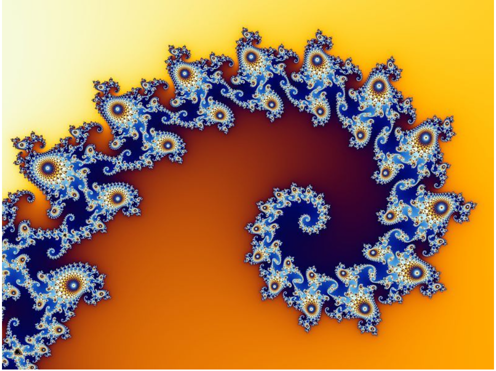

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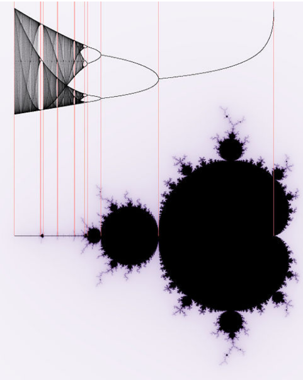

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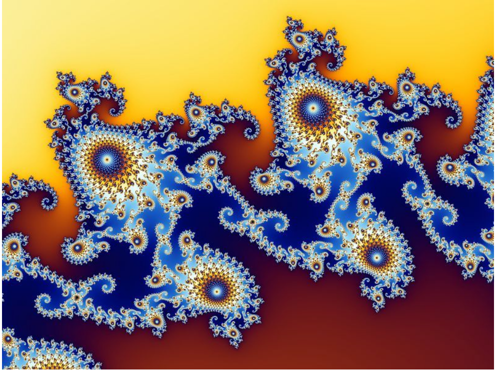

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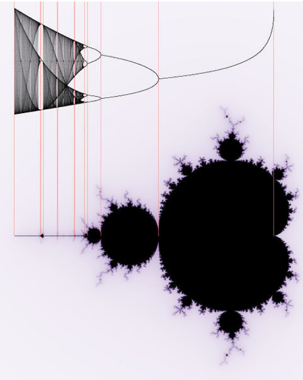

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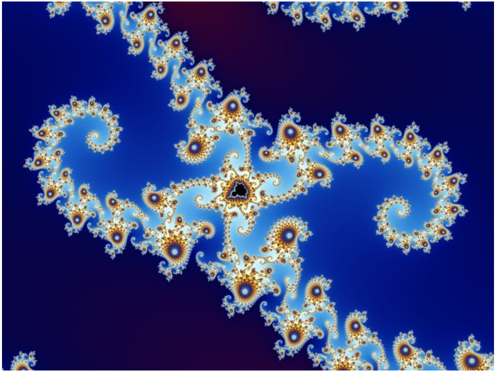

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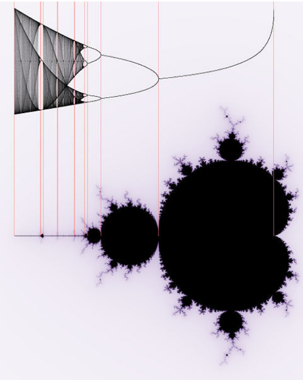

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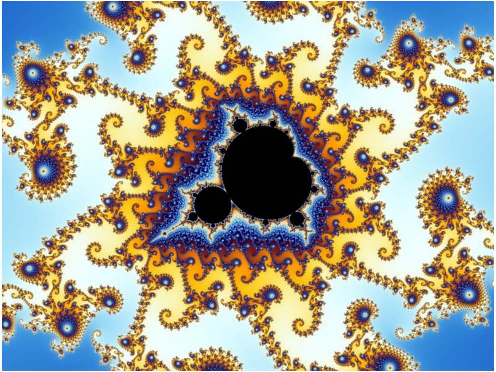

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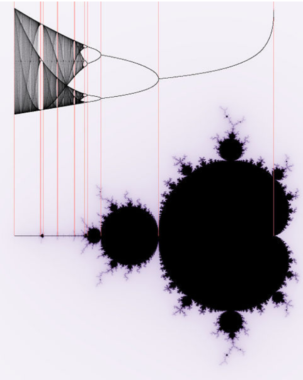

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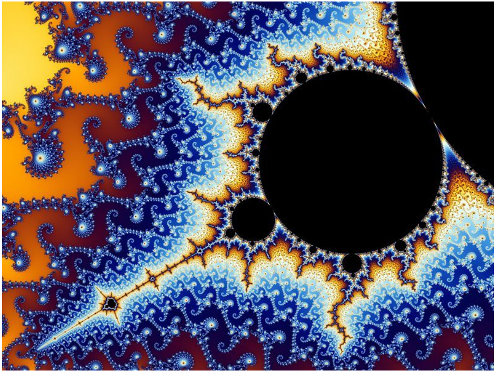

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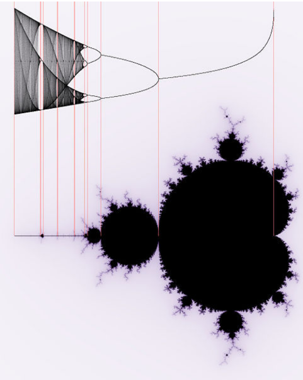

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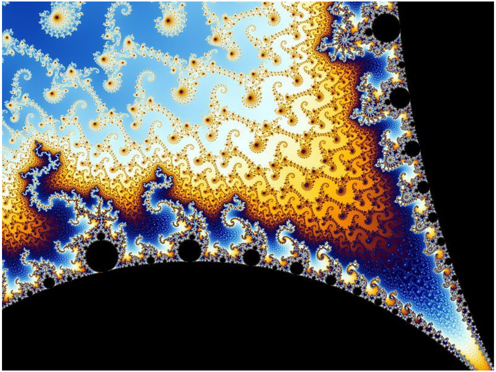

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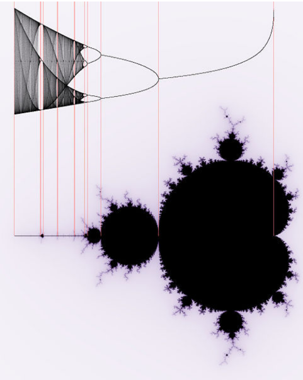

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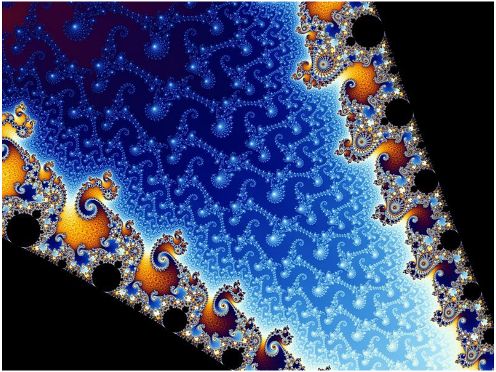

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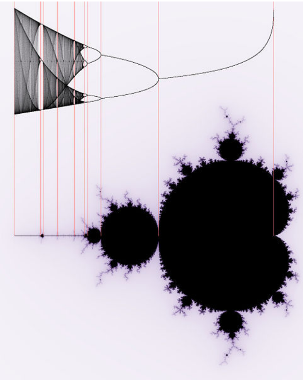
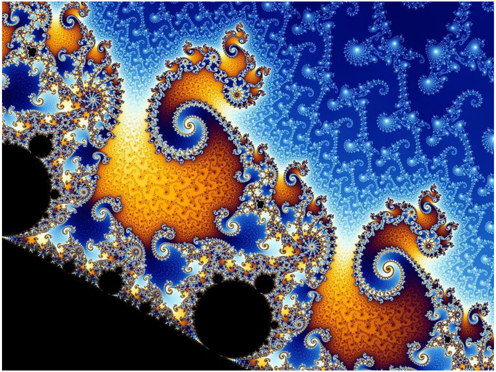


Fix $c \in \mathbb{C}$: let $z_0 = 0$, $z_{n+1} = z_n^2 + c$.

Mandelbrot set $M = \{c \mid z_n \not\to \infty\}$



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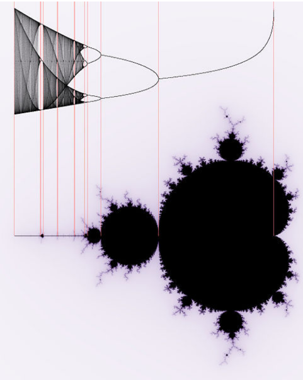

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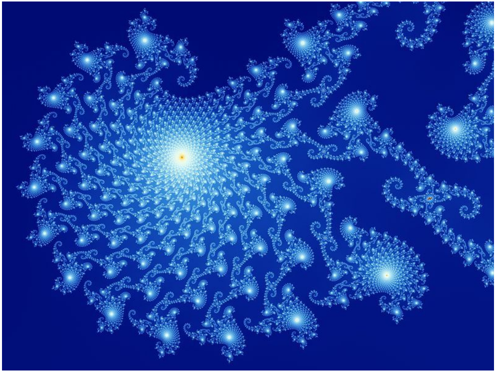

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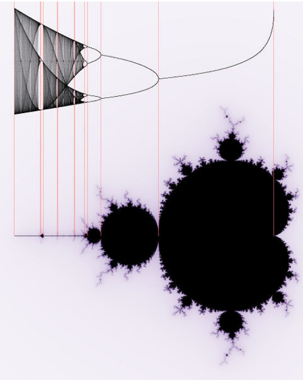

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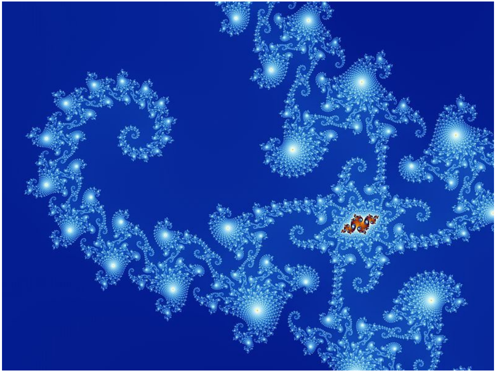

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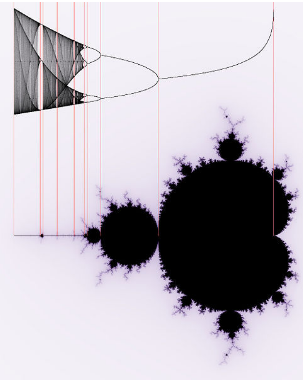

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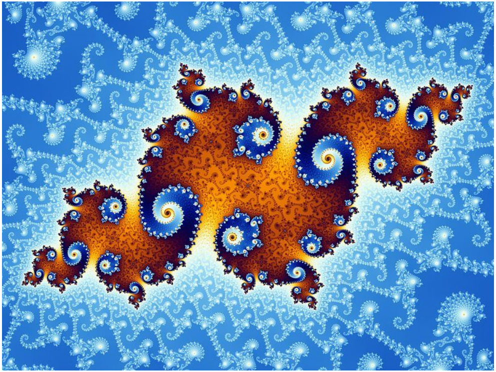

General questions

Numerical picture of bifurcation diagram for logistic maps raises questions:

1. Various phenomena are suggested by numerics: period-doubling cascades, windows of stability, self-similarity, chaos. Can their existence be proved rigorously?
2. Qualitative behaviour depends on parameter. How large are the parameter sets on which different behaviours occur?
3. Can consider other one-parameter families of interval maps $f_\lambda \colon [0, 1] \circlearrowleft$. Does the same story happen here?
4. What about higher dimensions ($f_\lambda \colon \mathbb{R}^d \circlearrowleft$) or manifolds ($f_\lambda \colon M \circlearrowleft$)?

General answers

1. Rigorous phenomena for logistic maps.
   Period-doubling, windows of stability, self-similarity, chaos: All can be rigorously proved to exist.

2. Size of parameter sets (prevalence of different behaviours).

3. Other interval maps.

4. Higher dimensions.

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   Numerics suggest a similar story, but proofs are much harder and most answers are still unknown.