Stochastic behavior in deterministic dynamics

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The talk in one slide

PHENOMENON	Deterministic systems can exhibit stochastic behaviour
MECHANISM	Driven by expansion + recurrence in phase space
IDEA	Treat as stochastic process; observations are not independent, but correlations might decay quickly enough
CHALLENGE	Mechanisms driving stochasticity may not be uniform

Predictions in dynamical systems

Key objects:

- X = phase space for a dynamical system.
  Points in X correspond to configurations of the system.
- $f : X \circlearrowleft$ describes evolution of the state of the system over a single time step. $f^n = f \circ \cdots \circ f$ *(n times)*

Standing assumptions: X is a compact metric space, f is continuous

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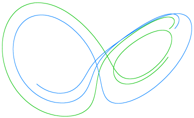
Common phenomenon: $\operatorname{diam} f^n(U)$ becomes large relatively quickly no matter how small U is. Stronger phenomenon:

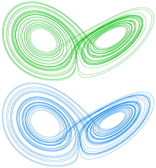
- iterates $f^n(U)$ become dense in X ← *mechanism for rigorous results*

Examples

Lorenz equations (1963) – atmospheric dynamics

$$\dot{x} = \sigma(y - x) \qquad \sigma = 10$$
$$\dot{y} = x(\rho - z) - y \qquad \rho = 28$$
$$\dot{z} = xy - \beta z \qquad \beta = 8/3$$





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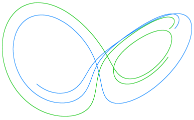

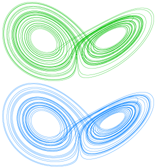

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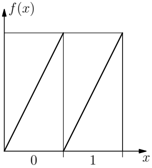

Lorenz system is non-uniformly hyperbolic

- *Expansion occurs at some times but not at others*

Simpler (less realistic) situation: uniform hyperbolicity

Doubling map $f\colon S^1 = \mathbb{R}/\mathbb{Z} \circlearrowleft$, $x \mapsto 2x \pmod 1$

Observations as random variables

$\varphi\colon X \to \mathbb{R}$	observation made at the present time
$\varphi \circ f^n\colon X \to \mathbb{R}$	the same observation made at time n

Main idea:

1. Expansion of $f^n(U)$ makes $\varphi \circ f^n$ difficult to predict
2. Instead, view $(X, \varphi \circ f^n)$ as a stochastic process
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To make this precise, need to specify a reference measure μ on X

- For doubling map, natural to choose Lebesgue measure on $S^1 = \mathbb{R}/\mathbb{Z}$
- Invariant:

$$\int \varphi \circ f\, dx = \int_{[0,\frac{1}{2}]} \varphi(2x)\, dx + \int_{[\frac{1}{2},1]} \varphi(2x-1)\, dx = 2\int_{[0,1]} \varphi(y)\, \frac{dy}{2} = \int \varphi\, dx$$

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- Lorenz system is dissipative: volume on $\mathbb{R}^3$ is *not* invariant. Is this a problem? Should we use a different measure instead? Which one?

Limit laws for i.i.d. sequences

Recall results from probability theory: let $Z_n\colon(\Omega,\mathcal{F},\mathbb{P})\to\mathbb{R}$ be a sequence of **i**ndependent **i**dentically **d**istributed random variables.

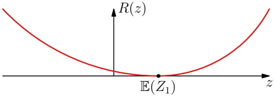
- Independent: $m\neq n \Rightarrow \mathbb{P}(Z_n\leq z|Z_m=w)=\mathbb{P}(Z_n\leq z)\ \forall z,w\in\mathbb{R}$

  ↳ Implies that *correlations* vanish: $C_{m,n}=\mathbb{E}(Z_nZ_m)-\mathbb{E}(Z_n)\mathbb{E}(Z_m)=0$.
- Identically distributed: $\mathbb{P}(Z_n\leq z)=\mathbb{P}(Z_m\leq z)\ \forall m,n\in\mathbb{N},z\in\mathbb{R}$

Then the following limit laws hold. *(We assume all moments exist)*

1. Strong law of large numbers: $\frac{1}{n}\sum_{k=1}^n Z_k\to\mathbb{E}(Z_1)$ with probability 1
2. Central limit theorem: $\mathbb{P}(\frac{1}{\sqrt{n}}\sum_{k=1}^n(Z_k-\mathbb{E}(Z_1))\leq z)\to\mathcal{N}(0,\sigma^2)(z)$
3. Large deviations:

$$\mathbb{P}(\tfrac{1}{n}\sum_{k=1}^n Z_k\in[a,b])\approx e^{-n\cdot\inf\{R(z):z\in[a,b]\}}$$



What about (non-i.i.d.) sequences of observations?

Back to a dynamical system $f : X \circlearrowleft$ and an observable $\varphi : X \to \mathbb{R}$.

- If μ is f-invariant, then $(X, \mu, \varphi \circ f^n)$ is identically distributed.

$$\text{invariance} \Rightarrow \mu(f^{-1}A) = \int \mathbf{1}_A(fx)\, d\mu = \int \mathbf{1}_A(x)\, d\mu = \mu(A)$$

$$\mathbb{P}(Z_n \le z) = \mu\{x \in X \mid \varphi(f^n x) \le x\} = \mu\{x \in X \mid \varphi(x) \le x\} = \mathbb{P}(Z_0 \le z).$$

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We say that μ is ergodic if it is not a convex combination of other invariant measures. *Equivalently, $f^{-1}A = A$ implies that $\mu(A) = 0$ or 1.*

Theorem (G.D. Birkhoff, 1931)

If μ is ergodic then $\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k(x) \to \int \varphi \, d\mu$ for μ-a.e. x

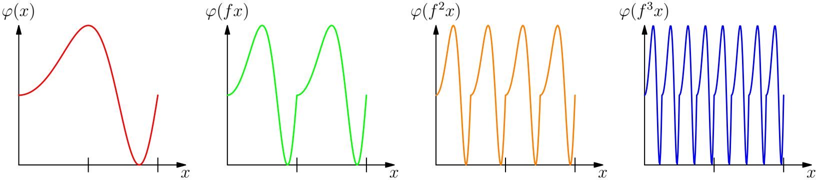
This gives SLLN as long as μ is ergodic. What about other limit laws?

- Goal: recover limit laws in dynamics using decay of correlations.

Expansion is the mechanism for decay of correlations

- Circle rotation $f(x) = x + \theta \pmod 1$ has no expansion and no decay: $n\theta \approx m \in \mathbb{Z} \Rightarrow \varphi \circ f^n \approx \varphi$, so $\mathbb{E}[\varphi \cdot (\varphi \circ f^n)] \approx \mathbb{E}[\varphi^2] \not\to (\mathbb{E}[\varphi])^2$

Doubling map: oscillations of $\varphi \circ f^n$ happen more quickly for large n



$$\mathbb{E}[\varphi \cdot (\varphi \circ f^n)] = \int \varphi(x)\varphi(f^n x)\,dx = \sum_{k=1}^{2^n} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} \varphi(x)\varphi(f^n x)\,dx$$

$$\approx \sum_{k=1}^{2^n} \varphi\left(\tfrac{k}{2^n}\right) \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} \varphi(f^n x)\,dx = \sum_{k=1}^{2^n} \varphi\left(\tfrac{k}{2^n}\right) \frac{\int \varphi}{2^n} \to \left(\int \varphi\right)^2 = (\mathbb{E}[\varphi])^2$$

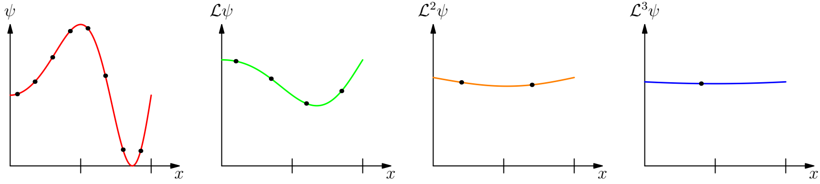
Ruelle–Perron–Frobenius transfer operator

To make this a little more rigorous, fix $\varphi \in L^1$ and $\psi \in L^\infty$.

$$\int \varphi \cdot (\psi \circ f) = \int_0^{\frac{1}{2}} \varphi(y)\psi(2y)\,dy + \int_{\frac{1}{2}}^1 \varphi(y)\psi(2y-1)\,dy$$

$$= \int_0^1 \left(\frac{\varphi(y_1(x)) + \varphi(y_2(x))}{2}\right)\psi(x)\,dx = \int (\mathcal{L}\varphi)(x)\psi(x)\,dx,$$

where $\mathcal{L}\varphi(x) = \sum_{y \in f^{-1}(x)} \frac{\varphi(y)}{|f'(y)|}$ defines an operator $\mathcal{L}\colon L^1 \circlearrowleft$.

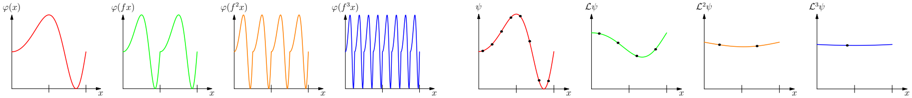


Note that $\mathcal{L}$ is the dual of the Koopman operator $U\colon \psi \mapsto \psi \circ f$ on L^∞.

Functional analysis and decay of correlations

Koopman operator $U\colon L^\infty \circlearrowleft$ treats φ as a measurement

- $U^n\varphi = \varphi \circ f^n$ means "make a measurement at time n"



Transfer operator $\mathcal{L}\colon L^1 \circlearrowleft$ treats ψ as a density

- If ψ represents the density of points in phase space at time 0, then $\mathcal{L}^n\psi$ gives the density at time n *(recall video of Lorenz system)*

Duality relationship: $\int \psi \cdot (\varphi \circ f^n) = \int \psi \cdot (U^n\varphi) = \int (\mathcal{L}^n\psi) \cdot \varphi$

- For doubling map, $\mathcal{L}^n\psi \to \int \psi$ exponentially quickly if ψ is Lipschitz.
- This gives exponential decay of correlations; also get CLT, LDP.

Complications for less simplistic models

All of this was just for the doubling map. What about something more realistic, like the Lorenz system? We can't have expansion in all directions, because the flow is volume contracting!

In fact, for (almost) every $x \in \mathbb{R}^3$ there is a splitting $\mathbb{R}^3 = E^u_x \oplus E^s_x \oplus E^0_x$ and Lyapunov exponents $\lambda^s(x) < 0 < \lambda^u(x)$ such that for large times t, we have

$$\|Df^t_x(v^u)\| \approx e^{\lambda^u(x)t}\|v^u\| \qquad \text{(expansion along } E^u\text{)}$$

$$\|Df^t_x(v^s)\| \approx e^{\lambda^s(x)t}\|v^s\| \qquad \text{(contraction along } E^s\text{)}$$

$$\|Df^t_x(v^0)\| = \|v^0\| \qquad \text{(isometry in the flow direction)}$$

Now there are (at least) two issues to deal with.

1. $\mathcal{L}$ doesn't "smooth things out" in the non-expanding directions
2. Expansion along E^u might take a long time to appear

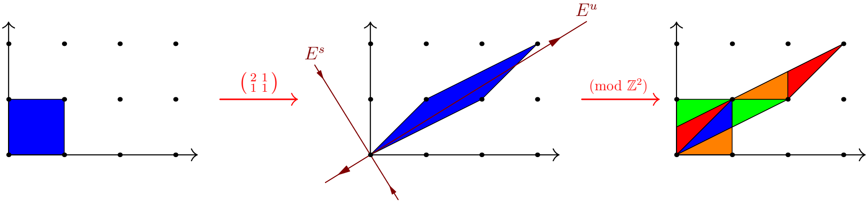
Anosov diffeomorphisms

Let M be a compact smooth Riemannian manifold. A diffeomorphism $f\colon M \circlearrowleft$ is Anosov if there are $\lambda^s < 0 < \lambda^u$ and an invariant splitting $T_x M = E^u_x \oplus E^s_x$ for every $x \in M$ such that for every $t \geq 0$,

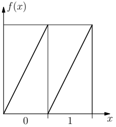
$$\|Df^t_x(v^u)\| \geq e^{\lambda^u t}\|v^u\| \qquad \text{(uniform expansion along } E^u\text{)}$$

$$\|Df^t_x(v^s)\| \leq e^{\lambda^s t}\|v^s\| \qquad \text{(uniform contraction along } E^s\text{)}$$

Example: $f\colon \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft$ given by $f(x) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)x \pmod{\mathbb{Z}^2}$



Symbolic codings



The doubling map admits a symbolic coding in terms of the full shift $\{0,1\}^{\mathbb{N}}$, $\sigma\colon y_0 y_1 y_2 \cdots \mapsto y_1 y_2 y_3 \ldots$

General procedure for symbolic description of dynamics:

1. Partition X as a disjoint union $A_1 \cup \cdots \cup A_d$
2. $f^n(x) \in A_{y_n}$ defines $y = \pi(x) \in \{1, \ldots, d\}^{\mathbb{N} \text{ or } \mathbb{Z}}$
3. $\pi\colon X \to \{1, \ldots, d\}^{\mathbb{N} \text{ or } \mathbb{Z}}$ is the coding map
4. $\Sigma = \overline{\pi(X)}$ is the coding space

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

For doubling map, $\Sigma = \{0,1\}^{\mathbb{N}}$, and $\pi(x)$ is the binary expansion of x.

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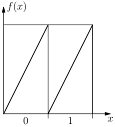

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For doubling map, $\Sigma = \{0,1\}^{\mathbb{N}}$, and $\pi(x)$ is the binary expansion of x.

- In general, there are many "forbidden" sequences (consider a rotation)
- Coding space is closed and σ-invariant: $\sigma(\Sigma) \subset \Sigma$

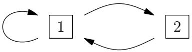
Markov shifts

Call $A = \{1, \dots, d\}$ the alphabet. A finite word $y_1 \cdots y_n \in A^n$ is legal for $\Sigma \subset A^{\mathbb{N} \text{ or } \mathbb{Z}}$ if it appears somewhere (anywhere!) in some element of Σ. *(That is, it codes a trajectory of the system.)* Say Σ is a Markov shift if

$$(y_{-k} \cdots y_{-1} y_0 \text{ legal and } y_0 y_1 \cdots y_\ell \text{ legal}) \Rightarrow (y_{-k} \cdots y_\ell \text{ legal})$$

Define a $d \times d$ transition matrix T by $T_{ij} = 1$ if ij is legal, and 0 otherwise.

- $(T^n)_{ij}$ = number of legal words starting with i and ending in j
  Can view T as a linear operator; special case of transfer operator $\mathcal{L}$
- Y = set of walks on a directed graph with vertices labeled $1, \dots, d$; draw an edge from i to j iff $T_{ij} = 1$.

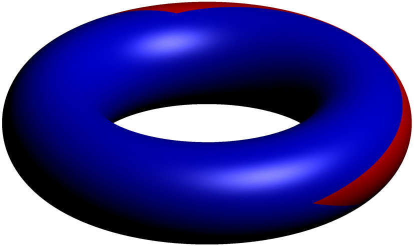


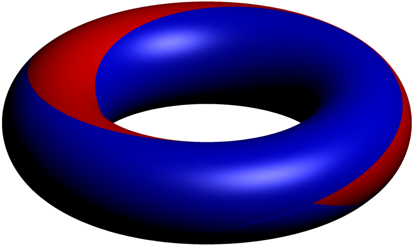
Y = {words on $\{1, 2\}$ such that 2 never follows 2}

Markov partitions for Anosov systems

Theorem (Sinai 1968, Bowen 1970)

Anosov systems admit partitions s.t. the coding space is a Markov shift.





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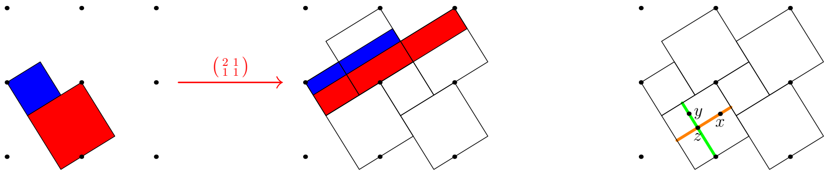


For any x, y in the same partition element, there is z such that

- the past coding of z agrees with that of x, and
- the future coding of z agrees with that of y.

Such a partition is called a Markov partition; elements are rectangles.

Decay of correlations for Markov shifts: Sinai's trick

Let $\Sigma \subset A^{\mathbb{Z}}$ be a Markov shift. Given $\varphi, \psi \in \mathrm{Lip}(\Sigma)$, questions about decay of correlations, CLT, LDP, etc., can be translated to the corresponding *one-sided* shift Σ^+ via Sinai's trick:

1. For each $a \in A$, choose $y^a \in \Sigma$ with $y_0^a = a$
2. Define $r : \Sigma \to \Sigma$ by $r(z)_k = z_k$ for $k \geq 0$, and $r(z)_k = y_k^{z_0}$ for $k \leq 0$
3. Let $u(z) = \sum_{j=0}^{\infty} (\varphi(\sigma^j z) - \varphi(\sigma^j r(z)))$.
4. $\varphi^+ := \varphi - u + u \circ \sigma \Rightarrow \boxed{\sum_{k=0}^{n-1} \varphi^+(\sigma^k z) = \left(\sum_{k=0}^{n-1} \varphi(\sigma^k z) \right) - u + u \circ \sigma^n}$
   and $\varphi^+(z)$ only depends on $z_0 z_1 z_2 \cdots$, not on $\cdots z_{-3} z_{-2} z_{-1}$.

This procedure can be thought of as "quotienting out the stable direction", since $\cdots z_{-3} z_{-2} z_{-1}$ encodes the position on E^s; we identify points on the same piece E^s, and only keep track of the expanding part.

Decay of correlations for Markov shifts: transfer operator

On a one-sided Markov shift Σ^+, we can write the transfer operator $\mathcal{L}$ as

$$\mathcal{L}\varphi(x) = \sum_{b \to x_0} g(bx)\varphi(bx).$$

Here $g \colon \Sigma^+ \to \mathbb{R}$ is given by the (inverse of) the expansion rate along E^u.

↪ Recall $g(x) = \frac{1}{2}$ for the doubling map

Theorem (Ruelle, Bowen)

If $g, \varphi \in \mathrm{Lip}$ and Σ^+ is mixing ($T^n > 0$), then $\mathcal{L}^n\varphi$ converges exponentially fast; the same is true for $\int \varphi \cdot (\psi \circ f^n) = \int (\mathcal{L}^n\varphi)\psi \to \int \varphi \int \psi$.

Putting it all together:

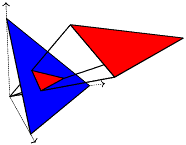
- An Anosov diffeo f can be coded by a Markov shift Σ.
- The Sinai trick lets us go from Σ to the one-sided shift Σ^+.
- EDC for Σ^+ leads to EDC for Σ and hence for f.
- One can go on to prove CLT and LDP for Anosov diffeomorphisms.

Exponential convergence of $\mathcal{L}^n$ using spectral gap

Idea behind this phenomenon: consider the following toy case.

- Replace Lip with space of locally constant functions $\varphi(x) = \varphi(x_0)$, take $g \equiv 1$; then $\mathcal{L}\varphi(x) = \sum_{b \to x_0} \varphi(b) = \sum_{b=1}^{d} T_{b,x_0} \varphi(b)$. In other words, the transfer operator $\mathcal{L}$ reduces to the transition matrix T.

Perron–Frobenius: T^n maps the positive cone in $\mathbb{R}^d$ strictly inside itself; everything converges exponentially to the eigenvector associated to the largest eigenvalue. This is because there is a gap between this and any smaller eigenvalues.



Ruelle's Perron–Frobenius theorem: replace $\mathbb{R}^d$ with Lip (infinite dim)

1. Mixing rules out eigenvalues with same abs. value (same as finite dim)
2. Lasota–Yorke inequality guarantees spectral gap (free in finite dim)

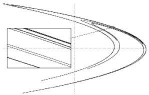
Non-uniformly hyperbolic systems

Uniform hyperbolicity is too restrictive to capture real-world phenomena; even the Lorenz model does not have expansion along E^u *all* the time.

Another example: the Hénon map (1976) $f(x, y) = (y + 1 - 1.4x^2, .3x)$

- $\mathbb{R}^2 = E^u_x \oplus E^s_x$, but $x \mapsto E^{u,s}_x$ is only measurable, and $\angle(E^u_x, E^s_x)$ can go to 0.
- Lebesgue measure (area) is not f-invariant.

↳ This issue appears for Anosov systems too.



Say μ is physical if $\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k x) \to \int \varphi\, d\mu$ for Leb-pos. set of x.

Theorem (Benedicks, Carleson, Young)

For suitable parameter values, the Hénon map has an invariant physical measure μ satisfying exponential decay of correlations and the CLT.

Hyperbolic measures

Let $f\colon M \circlearrowleft$ be a C^2 diffeomorphism and μ an ergodic f-invariant measure. Say that μ is hyperbolic if μ-a.e. x has a splitting $T_x M = E^u_x \oplus E^s_x$ such that

$$\varlimsup_{n\to\infty} \frac{1}{n} \log \|Df^n v^s\| < 0 \qquad \text{asymptotic forward contraction in } E^s$$

$$\varlimsup_{n\to\infty} \frac{1}{n} \log \|Df^{-n} v^u\| < 0 \qquad \text{asymptotic backward contraction in } E^u$$

For a given "non-uniformly hyperbolic" system, the goal is to

1. find a hyperbolic physical measure μ;
2. establish decay of correlations, CLT, etc., for μ.

In uniformly hyperbolic systems, these were both accomplished using Markov partitions. NUH systems do not have (finite) Markov partitions.

Towers

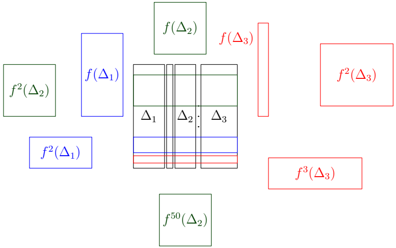
(Warning: The following imprecise summary jumbles up ideas of Hofbauer, Young, Alves, and many others, omitting most of the details.)

Idea: given $x \in M$, one iterate $f\colon U_x \to U_{f(x)}$ (where U_x is a nbhd of x) may not behave hyperbolically; E^u may contract, E^s may expand, etc. But *some* iterate $f^{n(x)}\colon U_x \to U_{f^{n(x)}(x)}$ behaves hyperbolically.

Only consider these "hyperbolic times"; can we get a Markov structure?

Tower: a region $\Delta \subset M$ with a countable partition $\Delta = \bigcup_k \Delta_k$ such that

1. each Δ_k is a dynamical rectangle;
2. $f^{n_k}(\Delta_k) \subset \Delta$, crossing all the other Δ_j completely;
3. $f^{n_k}\colon \Delta_k \to \Delta$ is unif. hyperbolic.



The tail of the tower

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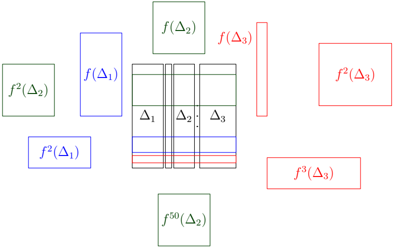

If $\mu(\Delta) > 0$, then statistics of (M, f, μ) governed by the tail of the tower.

- Return time function $R \colon \Delta \to \mathbb{N}$ given by $R|_{\Delta_k} = n_k$
- Tail is $\{x \in \Delta \mid R(x) \geq t\}$; points that take longer than t to return

The tower has exponential tails if $\mu\{x \in \Delta \mid R(x) \geq t\} \leq Ce^{-\alpha t}$ for some $\alpha > 0$. In this case, the methods from uniform hyperbolicity can be adapted to prove EDC, CLT, etc.

When do towers exist?

Until recently, no general results on existence of towers; all results were for specific classes of systems.

- Interval maps: Takahashi (1973), Hofbauer (1979), Jakobson (1981)
- Hénon maps: Benedicks, Carleson, Young (1990s)
- Partial hyperbolicity: Alves, Gouëzel, Li, Luzzatto, Pinheiro (2000s)

Theorem (Sarig 2013)

If $\dim M = 2$ *and* $f\colon M \circlearrowleft$ *is a* C^2 *diffeomorphism, then for every hyperbolic measure* μ *there is a tower* (Δ, R) *such that* $\mu(\Delta) > 0$*.*

Theorem (C.–Luzzatto–Pesin 2016)

Not only is there a tower, but there are verifiable conditions that will guarantee that it has exponential tails. (Or polynomial tails, etc.)