Venn Diagrams, Necklaces, and Chain Decompositions of Posets

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Symmetric Venn diagrams for n sets have been considered for years by several researchers, including Henderson, Grünbaum, Ruskey, Edwards, Hamburger, and Wagon. The existence of such diagrams for n sets is possible only for primes n, and they were initially constructed for primes $n \leq 7$. A breakthrough was made when Hamburger devised a construction for $n = 11$ in 1999. For his inspiration he credited the Greene-Kleitman bracketing construction of a symmetric chain decomposition (SCD) of the Boolean lattice $\mathcal{B}_n$ of all subsets of $[n] := \{1, \ldots, n\}$, ordered by inclusion. In my study, it became apparent that one might be able to apply the bracketing construction to produce Venn diagrams for all primes n, a project which was successfully completed (G.-Killian-Savage 2004): The key ingredient of the proof is the construction of a SCD of the "Necklace Poset" N_n, n prime, in which each element consists of a subset of $[n]$ and its cyclic rotations. That is, N_n is the quotient poset B_n/Z_n, consisting of orbits of the Boolean lattice B_n under the action of the cyclic group Z_n. Researchers were convinced that the Necklace Poset should actually have a SCD for all n, prime or composite, but the previous method worked only for primes n. Finally, my student Kelly Kross Jordan (Ph.D., 2008) devised an insightful new method to prove that N_n does indeed have a SCD for general n. Tantalizing challenges remain open both on SCD's for general quotient posets B_n/G and on constructing "simple" symmetric Venn diagrams.