Decidability questions for Cuntz-Krieger algebras and their underlying dynamics

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2. Graph $C^*$-algebras
3. Systematic approach
4. Moves
Outline

1. Shifts of finite type
2. Graph $C^*$-algebras
3. Systematic approach
4. Moves
To a finite graph $E = (E_0, E_1, r, s)$ such as

\[ \begin{array}{c}
\bullet & \xrightarrow{\rightarrow} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xleftarrow{\leftarrow} & \bullet \\
\bullet & \xrightarrow{\rightarrow} & \bullet \\
\bullet & \xleftarrow{\leftarrow} & \bullet \\
\bullet & \xrightarrow{\rightarrow} & \bullet \\
\end{array} \]

we associate $X_E$ defined as

$$X_E = \{(e_n) \in (E_0)^\mathbb{Z} \mid r(e_n) = s(e_{n+1})\}$$

Note that $X_E$ is closed in the topology of $(E_0)^\mathbb{Z}$ and comes equipped with a shift map $\sigma : X_E \to X_E$ which is a homeomorphism. We call $X_E$ a **shift space** (of finite type) over the alphabet $E_0$. 
Definition

The **suspension flow** $SX$ of a shift space $X$ is $X \times \mathbb{R}/\sim$ with

$$(x, t) \sim (\sigma(x), t - 1)$$

Note that $SX$ has a canonical $\mathbb{R}$-action.

Definitions

Let $X$ and $Y$ be shift spaces.

- $X$ is conjugate to $Y$ (written $X \simeq Y$) if there is a shift-invariant homeomorphism $\varphi : X \to Y$.
- $X$ is flow equivalent to $Y$ (written $X \sim_{FE} Y$) if there is an orientation-preserving homeomorphism $\psi : SX \to SY$

Question

Are these notions decidable for shifts of finite type?
Question

Are these notions decidable for shifts of finite type?

Theorem (Boyle-Steinberg)

*Flow equivalence is decidable among shifts of finite type.*
**Definition**

Let $A \in M_n(\mathbb{Z}_+)\text{ and } B \in M_m(\mathbb{Z}_+)$ be given. We say that $A$ is **elementary equivalent** to $B$ if there exist $D \in M_{n \times m}(\mathbb{Z}_+)$ and $E \in M_{m \times n}(\mathbb{Z}_+)$ so that

$$A = DE \quad B = ED.$$ 

The smallest equivalence relation on $\bigcup_{n \geq 1} M_n(\mathbb{Z}_+)$ is called **strong shift equivalence**.

Let $G_A$ be the graph with adjacency matrix $A$. We abbreviate $X_A = X_{G_A}$.

**Theorem (Williams)**

$X_A \simeq X_B$ if and only if $A$ is strong shift equivalent to $B$. 
**Definition**

We say that that $A$ and $B$ are **shift equivalent** of lag $\ell$ when there exist $D \in M_{n \times m}(\mathbb{Z}_+)$ and $E \in M_{m \times n}(\mathbb{Z}_+)$ so that

\[
A^\ell = DE \quad B^\ell = ED \quad AD = DB \quad EA = BE.
\]

Strong shift equivalence implies shift equivalence.

**Theorem (Kim-Roush)**

*Shift equivalence is decidable.*

It took decades to disprove

**William’s conjecture**

*Shift equivalence coincides with strong shift equivalence.*

and indeed it is a prominent open question if conjugacy is decidable for shifts of finite type.
Outline

1. Shifts of finite type
2. Graph $C^*$-algebras
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Singular and regular vertices

**Definitions**

Let \( E \) be a graph and \( v \in E^0 \).

- \( v \) is a **sink** if \( |s^{-1}(\{v\})| = 0 \)
- \( v \) is an **infinite emitter** if \( |s^{-1}(\{v\})| = \infty \)

**Definition**

\( v \) is **singular** if \( v \) is a sink or an infinite emitter. \( v \) is **regular** if it is not singular.
Graph algebras

Definition

The graph $C^*$-algebra $C^*(E)$ is given as the universal $C^*$-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz-Krieger relations

1. $s_e^*s_e = p_{r(e)}$
2. $s_es_e^* \leq p_{s(e)}$
3. $p_v = \sum_{s(e) = v} s_es_e^*$ for every regular $v$

$C^*(E)$ is unital precisely when $E$ has finitely many vertices.
Observation

\[ \gamma_z(p_v) = p_v \quad \gamma_z(s_e) = zs_e \]

induces a **gauge action** \( \mathbb{T} \mapsto \text{Aut}(C^*(E)) \)

Definition

\[ \mathcal{D}_E = \text{span}\{s_\alpha s_\alpha^* \mid \alpha \text{ path of } E\} \]

Note that \( \mathcal{D}_E \) is commutative and that

\[ \mathcal{D}_E \subseteq \mathcal{F}_E = \{a \in C^*(E) \mid \forall z \in \mathbb{T} : \gamma_z(a) = a\} \]

\( \mathcal{D}_E \) has spectrum \( X_A \) when \( E = E_A \) arises from an essential and finite matrix \( A \). This fundamental case was studied by Cuntz and Krieger, using the notation \( \mathcal{O}_A = C^*(E_A) \).
Theorem (E-Restorff-Ruiz-Sørensen)

*-isomorphism and stable *-isomorphism of unital graph $C^*$-algebras is decidable.

Theorem (Carlsen-E-Ortega-Restorff, Matsumoto-Matui)

$$(C^*(E_A) \otimes K, \mathcal{O} \otimes c_0) \simeq (C^*(E_B) \otimes K, \mathcal{O} \otimes c_0) \iff X_A \sim_{FE} X_B$$

Theorem (Carlsen-Rout, Matsumoto)

$$(C^*(E_A) \otimes K, \mathcal{O} \otimes c_0, \gamma \otimes \text{Id}) \simeq (C^*(E_B) \otimes K, \mathcal{O} \otimes c_0, \gamma \otimes \text{Id})$$

$$\iff$$

$$X_A \simeq X_B$$
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Definition

With $x, y, z \in \{0, 1\}$ we write

$$E \overset{xyz}{\longrightarrow} F$$

when there exists a $\ast$-isomorphism $\varphi : C^*(E) \otimes K \to C^*(F) \otimes K$ with additionally satisfies

- $\varphi(1_{C^*(E)} \otimes e_{11}) = 1_{C^*(F)} \otimes e_{11}$ when $x = 1$
- $\varphi \circ (\gamma \otimes \text{Id}) = (\gamma \otimes \text{Id}) \circ \varphi$ when $y = 1$
- $\varphi(\mathcal{D}_E \otimes c_0) = \mathcal{D}_F \otimes c_0$ when $z = 1$. 
Theorem (E-Restorff-Ruiz-Sørensen)

\[ E \xrightarrow{x0z} F \text{ is decidable.} \]

Theorem (Carlsen-E-Ortega-Restorff, Matsumoto-Matui)

\[ E_A \xrightarrow{001} E_B \iff X_A \sim_{FE} X_B \]

Theorem (Carlsen-Rout, Matsumoto)

\[ E_A \xrightarrow{011} E_B \iff X_A \simeq X_B \]
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**Shifts of finite type**

**Graph \( C^* \)-algebras**

**Systematic approach**

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**Moves**

**Move (S)**

Remove a regular source, as

\[
\begin{array}{c}
\star \rightarrow \bullet \leftrightarrow \circ \rightsquigarrow \bullet \leftrightarrow \circ \\
\end{array}
\]

**Move (R)**

Reduce a configuration with a transitional regular vertex, as

\[
\begin{array}{c}
\bullet \leftrightarrow \star \rightarrow \bullet \rightsquigarrow \bullet \leftrightarrow \bullet \\
\end{array}
\]

or

\[
\begin{array}{c}
\circ \rightarrow \star \rightarrow \bullet \rightsquigarrow \circ \rightarrow \bullet \\
\end{array}
\]
Moves

**Move (S)**
Remove a regular source, as

\[
\begin{array}{c}
\ast \rightarrow \bullet \\
\end{array}
\begin{array}{c}
\Leftrightarrow \\
\Rightarrow \\
\Leftarrow \\
\end{array}
\begin{array}{c}
\circ \sim \rightarrow \bullet \\
\Leftrightarrow \\
\Rightarrow \\
\end{array}
\begin{array}{c}
\Rightarrow \\
\Leftarrow \\
\end{array}
\circ
\]

**Move (R)**
Reduce a configuration with a transitional regular vertex, as

\[
\begin{array}{c}
\bullet \Leftrightarrow \ast \rightarrow \bullet \\
\Leftrightarrow \\
\Rightarrow \\
\end{array}
\begin{array}{c}
\sim \rightarrow \bullet \\
\Leftrightarrow \\
\Rightarrow \\
\end{array}
\begin{array}{c}
\bullet \Leftrightarrow \\
\Rightarrow \\
\end{array}
\bullet
\]

or

\[
\begin{array}{c}
\circ \Leftrightarrow \ast \rightarrow \bullet \\
\sim \\
\bullet \Leftrightarrow \\
\end{array}
\begin{array}{c}
\circ \Leftrightarrow \\
\Rightarrow \\
\end{array}
\bullet
\]
Shifts of finite type

Graph $C^*$-algebras

Systematic approach

Moves

**Move (I)**

Insplit at regular vertex

![Diagram of Move (I)](image1)

**Move (O)**

Outsplit at any vertex (at most one group of edges infinite)

![Diagram of Move (O)](image2)
**Moves**

**Move (I)**

Insplit at regular vertex

\[ \begin{array}{c}
\bullet \\
\downarrow \\
\star \\
\downarrow \\
\bullet \\
\end{array} \quad \sim \quad \begin{array}{c}
\bullet \\
\rightarrow \\
\star \\
\rightarrow \\
\bullet \\
\end{array} \]

**Move (O)**

Outsplit at any vertex (at most one group of edges infinite)

\[ \begin{array}{c}
\bullet \\
\downarrow \\
\star \\
\downarrow \\
\bullet \\
\end{array} \quad \sim \quad \begin{array}{c}
\bullet \\
\rightarrow \\
\star \\
\rightarrow \\
\bullet \\
\end{array} \]
Move (C)

“Cuntz splice” on a vertex supporting two cycles
Move (P)

“Butterfly move” on a vertex supporting a single cycle emitting only singly to vertices supporting two cycles
Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) There is a finite sequence of moves of type $(S),(R),(O),(I),(C),(P)$ and their inverses, leading from $E$ to $F$. 